The applications and implications of two recently addressed asymptotic descriptions of exact coherent structures in shear flows are discussed. The first type of asymptotic framework to be discussed was introduced in a series of papers by Hall & Smith in the 1990s and was referred to as vortex–wave interaction theory (VWI). New results are given here for the canonical VWI problem in an infinite region; the results confirm and extend the results for the infinite problem inferred the recent VWI computation of plane Couette flow. The results given define for the first time exact coherent structures in unbounded flows. The second type of canonical structure described here is that recently found for asymptomatic suction boundary layer and corresponds to freestream coherent structures (FCS), in boundary layer flows. Here, it is shown that the FCS can also occur in flows such as Burgers vortex sheet. It is concluded that both canonical problems can be locally embedded in general shear flows and thus have widespread applicability.

1. Introduction

Our concern is with nonlinear equilibrium solutions of the Navier–Stokes equations thought to underpin turbulent flows. Almost all the work in this area has been restricted to numerical investigations of the discretized governing equations for shear flows. The discretized system is a high-dimensional dynamical system and many nonlinear equilibrium solutions have been identified as fixed points or periodic orbits of the system (see [1–4] for plane Couette flow, [5,6] for pipe flow, [7,8] for plane Poiseuille flow). In the computational community, these structures are referred to as exact coherent structures.
Figure 1. The variation of the wall drag $\Delta$ along the equilibrium solution branch for plane Couette flow. The wall speed and half channel height are used as typical velocity and length scales. The horizontal axis represents the non-dimensional streamwise wavenumber $\alpha$. The non-dimensional spanwise wavenumber is fixed as 2. The solid line and open circles correspond to the limiting VWI curve and results at a Reynolds number 16 000 obtained by Deguchi & Hall [19].

The initial computations of exact coherent structures were described by Nagata [1] though it was Waleffe and co-workers, who subsequently uncovered the physics behind the numerical results and coined the ‘self-sustained process’ label (e.g. [9]).

The ‘lower branch’ and ‘upper branch’ terminology used within the numerical community refers to solutions on the lower or upper part of the amplitude-Reynolds number curve following the birth of the equilibrium solutions in a saddle-node bifurcation. For typical shear flows, the sequence of saddle-node bifurcations starts at quite low Reynolds numbers, where the basic flow is linearly stable. Direct numerical simulations suggest that in channel flows exact coherent structures play at least two fundamental roles. Firstly, the lower branch solutions act as edge states separating disturbances which evolve into turbulence or return to the laminar state [10,11]. Secondly, the upper branch solutions eventually become associated with attractors for turbulent flows [12,13].

For many years, the Reynolds number dependence of turbulence and transition phenomena has been an area of major interest in fluid dynamics. Recently, asymptotic analyses have proved to be extremely useful for describing high Reynolds number exact coherent structures. Some years before the ‘self-sustained process’ view of exact coherent structures became common, a formal asymptotic framework describing the crucial tripartite interaction between rolls, streaks and waves forming the basis of the states was given by Hall & Smith [14–16] and named vortex–wave interaction theory (VWI). Subsequently, Hall & Sherwin [17] conclusively showed that, in the context of plane Couette flow, the lower branch solutions were simply finite Reynolds number forms of VWI states. For a full discussion of the relationship between the approaches, see Deguchi et al. [18]. Subsequently, Deguchi & Hall [19] suggested that all equilibrium solutions in a periodic box, lower or upper branches, become of VWI type as the Reynolds number increases, thereby pointing to VWI as the fundamental building block for exact coherent structures in high Reynolds number flows (figure 1). Until recently, exact coherent structures in external flows have received little attention even though the original VWI formulation of Hall & Smith [16] was for growing boundary layers. The VWI description of exact coherent structures in the asymptotic suction boundary layer (ASBL) flow was reported in Deguchi & Hall [20] and related to the time-periodic exact coherent structures found by Kreilos et al. [21]. The VWI-type exact coherent structures must appear in the near-wall boundary layer since the theory requires $O(1)$ basic flow shear at the interaction position.

A quite different type of exact coherent structure in ASBL was recently described in Deguchi & Hall [20]. Based on numerical computations of the equilibrium solutions from the full Navier–Stokes equations, an asymptotic description of the solutions, totally distinct from VWI, was given. The new structure was referred to as a freestream coherent structure (FCS)
and involves a wave, roll and streak interacting in a layer located in the freestream. As its name indicated, the structure is convected by almost freestream speed and located at a large distance from the wall. The distance to the interaction position was found to be scaled by some function of Reynolds number as quite opposed to VWI. The FCS interaction, which is inherently associated with the boundary-layer nature of the unperturbed basic state and which therefore has no counterpart in wall-bounded shear flows, generates a large amplitude streak that grows towards the wall. Therefore, the fundamental signal of the interaction is a large-amplitude wall streak located well away from the nonlinear interaction in freestream, which generates it.

The asymptotic theories for exact coherent structures have been considered in simple shear flows such as plane Couette flow or ASBL flow, so one might conclude that the theories are flow configuration dependent. However, in this paper, we shall give results for VWI and FCS states in more general situations, thus underlining the apparent universality of the theories. One of the key studies is Blackburn et al. [22], who showed that when a sufficiently small spanwise wavelength is chosen the VWI solutions in plane Couette flow become unaware of the presence of walls. Moreover, it was shown that at small wavelengths the appropriate lengthscale in the VWI formulation is the spanwise wavelength which therefore becomes scaled out of the problem so that a universal set of solutions emerges and is valid for all short wavelengths. Interestingly, it was shown by Blackburn et al. [22] that the energy density and dissipation of the wave of the small wavelength VWI state satisfy the Kolmogorov 5/3 law.

Furthermore, we note that the FCS described in Deguchi & Hall [20] is an exact coherent structure existing in an unbounded flow. Both the VWI and the FCS problems in unbounded domains are Reynolds number independent and describe some of the possible asymptotic forms of exact coherent structures. Once we obtain solutions of these canonical unbounded problems, we can apply the results to quite arbitrary shear flows by embedding the solutions locally in such flows. Of course, outside the local structure, we need to construct the passive asymptotic solutions matching onto the canonical problems. Here, we shall demonstrate how to obtain solutions of such canonical unbounded problems numerically, and how to embed them in a range of quite general fluid flows. In §2, we will give a brief survey of VWI theory and present new results suggesting its relevance to high-Reynolds-number exact coherent structures in arbitrary shear flows. In §3, we investigate the Burgers vortex sheet problem to show an application of FCS theory to more general boundary-layer flows. Finally in §4, we draw some conclusions and discuss future extensions of the work.

2. Vortex–wave interaction theory for shear flows

In the first instance, we give the VWI equations for fully developed unidirectional shear flows in channels or pipes of arbitrary cross section driven by a streamwise pressure gradient and/or the motion of the walls. Consider the non-dimensional Navier–Stokes equations for a viscous fluid in the Cartesian form

\[
\begin{align*}
(\partial_t + \mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + R^{-1}\nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

(2.1)

where \(x, y, z\) are the streamwise, normal and spanwise directions, respectively, \(R\) is the Reynolds number and the equations have been made dimensionless using a typical fluid speed in the streamwise direction and a typical lengthscale in the cross section of the pipe or channel. In addition, we have defined \(\nabla = (\partial_x, \partial_y, \partial_z)\), \(\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2\). The Navier–Stokes equations allow an exact solution of the form \(\mathbf{u} = (U, R^{-1}V, R^{-1}W)\), \(p = Qx + R^{-2}P\) where \(U, V, W, P\) are functions only of \(y\) and \(z\) and \(Q\) is the pressure gradient in the \(x\)-direction driving the flow, as \(u_B\) is not defined here. It has become conventional to refer to \((V, W)\) and the associated pressure as the roll flow and \(U\) as the streak flow. In the absence of any forcing through for example wall curvature or a wave system propagating in the fluid \(\mathbf{u} = u_B = (u_B(y, z), 0, 0)\) is the unique solution of the system.
Now suppose that an inviscid wave propagating downstream with speed $c$ and wavenumber $\alpha$ rides on top of the roll-streak flow. Following the work of Hall & Horseyman [23], it is well known that the leading-order problem for $\tilde{p}(y,z)$, the amplitude of the wave pressure, satisfies the pressure form of the generalization of Rayleigh’s equation appropriate to the unidirectional flow $(U(y,z), 0, 0)$:

$$
\left(\partial_{yy}^2 + \partial_{zz}^2 - \alpha^2 - 2 \frac{\partial y U}{U - c} \partial_y - 2 \frac{\partial_z U}{U - c} \partial_z\right) \tilde{p} = 0.
$$

(2.2)

Here, the wave will have a viscous critical layer of thickness $O(R^{-1/3})$, where $U = c$ and within that layer the Reynolds stresses associated with the wave drive a roll flow. Note that the wave forcing is dominant in the critical layer because the wave increases in size by a factor $R^{1/3}$ there. The forcing decays to zero at the edge of the critical layer, but, in a manner analogous to steady streaming problems, it produces finite jumps in the roll stress across the layer (see Hall & Sherwin [17] for details of that process). The upshot then is that the outer roll flow of size $O(R^{-1})$ is driven by that jump. The crucial property of the interaction is that tiny waves can drive this small outer roll flow which is itself coupled to an $O(1)$ streak flow. That streak flow is linearly unstable to the wave and the tripartite interaction is established. Moreover, the form of the jump condition is generic to all channel and pipe flows in containers of arbitrary cross section and also to boundary layers. The discussion in Hall & Sherwin [17] shows that the wave size needed to drive the outer roll flow is $O(R^{-7/6})$ so that away from the critical layer the velocity field for a VWI in a shear flow expands as

$$
\mathbf{u} = (U, R^{-1}V, R^{-1}W) + R^{-7/6}[\tilde{\mathbf{u}}E + \text{c.c.}] + ..,
$$

$$
p = R^{-2}P + R^{-7/6}[\tilde{p}E + \text{c.c.}] + ..,
$$

where $E = \exp(\alpha(x - ct))$ and $U, V, W, P \in \mathbb{R}$ and the wave components $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p} \in \mathbb{C}$ are functions of $y, z$. Substituting into the Navier–Stokes equations we find that, away from the critical layer, the equations to determine the roll and streak are

$$
(U \cdot \nabla_{2D})\mathbf{U} = (Q, 0, 0) - \nabla_{2D}P + \nabla_{2D}^2\mathbf{U}
$$

(2.3)

and

$$
\nabla_{2D} \cdot \mathbf{U} = 0,
$$

(2.4)

where $\nabla_{2D} = (0, \partial_y, \partial_z)$, $\nabla_{2D}^2 = \partial_{yy}^2 + \partial_{zz}^2$ and the wave pressure $\tilde{p}$ satisfies (2.2). The roll and wave equations must be considered within the critical layer in the manner described in the study of Hall & Sherwin [17] and Hall & Smith [16]. The analysis in Hall & Sherwin [17] is perhaps the most accessible and uses a coordinate system based on the critical layer position and is quite similar to classical steady streaming analyses. It is found that across the layer the following jumps for the outer roll flow are generated

$$
[\partial_n V^s]^{+} = n_0 \alpha^{-5/3} \partial_n (\mu^{-5/3} \partial_n \tilde{p})^2
$$

(2.5)

and

$$
[P]^{+} = -n_0 (\alpha \mu)^{-5/3} \Lambda_0 |\partial_s \tilde{p}|^2.
$$

(2.6)

Here, $V^s(n,s)$ is the roll velocity tangential to the critical layer in the $y$-$z$ plane, $n, s$ are normal and tangential coordinates along the critical layer, $\Lambda_0$ is the layer curvature, $\mu = \partial_n U$ and $n_0 = 2\pi(2/3)^{2/3}(-2/3)!$. Note that the form of the roll, streak and wave equations and the stress jump are generic to all unidirectional fully developed shear flows. The differences between flows arise only from the boundary conditions to be applied. The obvious cases of interest are as follows.
(a) Spanwise periodic vortex–wave interaction states in channels

Here, the boundary conditions for VWI states periodic in the spanwise direction z with wavenumber \( \beta \) are

\[
[U, \tilde{p}](x, y, z + \frac{2\pi}{\beta}) = [U, \tilde{p}](x, y, z), \quad (2.7)
\]

\[
[U, \tilde{p}](x + \frac{2\pi}{\alpha}, y, z) = [U, \tilde{p}](x, y, z) \quad (2.8)
\]

and

\[
(U, V, W, \tilde{p}_y) = (u_B, 0, 0, 0), \quad y = \pm 1. \quad (2.9)
\]

For plane Couette flow, the driving pressure gradient \( Q \) is taken to be zero and \( u_B(y) = y \); that is the case studied in \([17–19, 22]\). For plane Poiseuille flow, we take \( Q = -2 \) and \( u_B(y) = 1 - y^2 \).

If we seek spanwise-localized solutions, then the periodicity in the spanwise direction can be replaced by \( (U, V, W, \tilde{p}) \to (u_B, 0, 0, 0) \) as \( |z| \to \infty \) so that a long way from \( z = 0 \) the flow returns to its unperturbed state. Limiting solutions of this problem have been suggested in Deguchi et al. \([18]\) by taking \( \beta \ll 1 \) in the calculations; in that case, the flow localizes in the spanwise directions with a local \( O(1) \) wavenumber. No direct solutions for the localized problem have yet been calculated.

(b) Vortex–wave interaction states in pipes of arbitrary cross section

Consider a fluid flow through a pipe of arbitrary cross section defined by \( y = F(z) \). Here, the appropriate conditions become

\[
[U, \tilde{p}](x + \frac{2\pi}{\alpha}, y, z) = [U, \tilde{p}](x, y, z) \quad (2.10)
\]

and

\[
U = 0, \quad \frac{\partial \tilde{p}}{\partial N} = 0, \quad y = F(z), \quad (11.11)
\]

where \( N \) is distance measured normal to the pipe wall. In particular, these equations apply to circular pipes for which numerous equilibrium solutions of the finite Reynolds number versions of the states have been found. As yet, no VWI solutions have been found directly from the asymptotically reduced problem and so it remains to be seen how well they compare to the full Navier–Stokes solutions.

Now we shall concentrate on the plane Couette flow case for stationary waves so that \( c = 0 \). For plane Couette flow, system (2.2)–(2.9) with \( Q = 0 \) defines a nonlinear eigenvalue problem for the wave amplitude in terms of the wavenumbers \( \alpha, \beta \). The simplification of the problem from the full Navier–Stokes equations is huge since solving the VWI problem requires only the solution of the steady two-dimensional Navier–Stokes equations being a steady advection diffusion equation and the pressure wave equation. The latter equation, i.e. (2.2), is a partial differential equation eigenvalue problem for the real wavespeed \( c \). Little is known about the general properties of this equation; it is straightforward to extend Howard’s semicircle theorem to the equation but no analogue of Rayleigh’s criterion has yet been found.

The first numerical solutions of the VWI equations were given by Hall & Sherwin \([17]\) and concerned stationary solutions given by Nagata \([1]\) and Wang et al. \([2]\); these are characterized by a sinuous streaky flow. We refer to this type of solution as the sinuous mode, which is labelled EQ1/EQ2 in the terminology of Gibson et al. \([3]\). The conclusion of Hall & Sherwin \([17]\) was that, for plane Couette flow, VWI predicted with remarkable accuracy the sinuous mode lower branch solutions. Recently, Deguchi & Hall \([19]\) followed the development with Reynolds number of the full Navier–Stokes sinuous mode solution branch in order to track its approach to the curve of Hall & Sherwin \([17]\). At quite modest values of \( R \), convergence to the VWI lower branch curve was found for \( \alpha, \beta \sim O(1) \). In addition, a continuation of the VWI curve beyond the turning point was achieved and convergence to an upper branch of VWI solutions was established by increasing \( R \) significantly. Thus, there is significant evidence to conjecture that for \( O(1) \) streamwise and...
spanwise wavenumbers all equilibrium solutions in plane Couette flow deform into VWI states as the Reynolds number increases. For a fixed spanwise wavelength, the upper limit of \( \alpha \) below which solutions exist correlates remarkably well with the minimum box size needed for sustained turbulence to exist [24].

At low values of \( \alpha \) VWI breaks down and all streamwise harmonics come into play. Here, the distinguished limit is \( \alpha \approx R^{-1} \) and the interaction is described by the boundary region equations. Solutions of those equations were given by Deguchi et al. [18] and matched onto the VWI results at large scaled wavenumbers \( \alpha R \). A full account of the structures which emerge at low \( \alpha R \) was given by Deguchi & Hall [19] using the full Navier–Stokes equations so we simply quote the most important results here. Firstly, the lower branch VWI results of Hall & Sherwin [17] were shown to connect initially with the long-wavelength results of Deguchi et al. [18]. For increasing \( R \) that connection holds for increasingly small \( \alpha \), but ultimately for any fixed \( R \), at sufficiently small \( \alpha \), the solutions become entangled and the corresponding solution branches vary rapidly when the parameters are changed. In that regime, the rapidly varying nature of the solution branch corresponds to the flow adopting a VWI structure locally in the long-wavelength regime and the solutions take on many similarities with turbulent spots. On the upper branch the situation is similar except that, for sufficiently small \( \alpha \), the rapid variation of the solution branch appears to be associated with a long-wavelength mode of instability producing vertical jet-like structures in the flow.

While plane Couette flow is a much-loved prototype problem for understanding nonlinear structures, it is not a flow of much practical importance. However, as discussed earlier, the VWI formulation applies to flow in a channel of arbitrary cross section and so applies to circular pipes. Also the original VWI formulation was for growing boundary layers which can be described by fixing the wave frequency and allowing the wavenumber to vary slowly in the flow direction.

Further generic VWI results were given by Blackburn et al. [22], who considered plane Couette flow VWI states at large values of \( \beta \). Their results suggested that for large enough \( \beta \) the walls play no role in the interaction, and thus the states define exact coherent structures in unbounded homogeneous shear flows. In that case the spanwise and streamwise wavelengths become the only lengths available. If the spanwise wavelength is taken as a typical lengthscale, and a typical velocity is taken to be the local shear multiplied by the spanwise wavelength, then a ‘local’ Reynolds number based on that velocity, the spanwise wavelength and kinematic viscosity can be defined. The VWI problem can then be reformulated in a layer of thickness comparable to the spanwise wavelength by letting the local Reynolds number tend to infinity. Thus, as we will verify here from Navier–Stokes solutions, this new VWI state defines a new class of exact coherent structure which exists locally in any shear flow. Now let us state the VWI problem for ‘localized’ states in unbounded shear flows and show how they can be extracted from Navier–Stokes solutions for plane Couette flow.

We shall denote the variables of the local canonical problem with an asterisk and note that (2.2)–(2.6), now written in terms of asterisk variables, still apply. However in the spanwise direction, the wavelength is now \( 2\pi \) and the conditions at \( \pm 1 \) must be replaced by conditions at \( \pm \infty \). Thus, the required conditions for the canonical problem are

\[
[U^*, \tilde{p}^*](x^*, y^*, z^* + 2\pi) = [U^*, \tilde{p}^*](x^*, y^*, z^*),
\]

\[
[U^*, \tilde{p}^*] \left( x^* + \frac{2\pi}{\alpha^*}, y^*, z^* \right) = [U^*, \tilde{p}^*](x^*, y^*, z^*)
\]

and

\[
(u^*, v^*, w^*, \tilde{p}_{y^*}) \rightarrow (y^* \mp d_0, 0, 0, 0) \quad \text{as} \quad y^* \rightarrow \pm \infty.
\]
and then the canonical VWI problem would then involve only the scaled spanwise wavenumber. Henceforth, we will refer to the VWI problem specified by (2.2)–(2.6) and (2.12)–(2.14) as the unbounded VWI problem.

The solution of the unbounded VWI problem can be computed, as suggested in Blackburn et al. [22], by using the large-spanwise wavenumber limit of VWI applied to plane Couette flow. For large $\beta$, Blackburn et al. [22] showed that the VWI structure for large $\beta$ shrinks at the same rate in the $x, y, z$ directions as

$$[x, y, z] = \beta^{-1}[x^*, y^*, z^*]$$

and this does not change the VWI formulation. Therefore, we can take $\alpha^* = \alpha/\beta$. In order to embed the solution of the local canonical VWI problem in plane Couette flow, we need to apply the scaling

$$[U, V, W, P, \tilde{p}] = \left[ \left( \frac{S}{\beta} \right)^{1/3} U^*, \beta V^*, \beta W^*, \beta^{2/3} P^*, \beta^{1/3} S^{5/6} \tilde{p}^* \right].$$

Note that the scaling factor for the streak component is modified from that given by Blackburn et al. [22] by a factor of $S$, which denotes homogeneous mean flow slope outside the interaction region in terms of non-asterisk variables. For stationary plane Couette flow solutions, $S$ is equal to the wall drag $\Delta$ and the no-slip boundary conditions then give $\Delta = \beta/(\beta - d_0)$, i.e. $d_0 = \beta(\Delta - 1)/\Delta$. Note that here the wall drag is defined by a $y$ derivative of $x-z$ averaged streamwise velocity $u$ at $y = 1$.

Since a given large $R$ the unbounded VWI states should emerge from equilibrium solutions, we calculate from the full Navier–Stokes equations for plane Couette flow when $\beta, R \to \infty$ with $\alpha/\beta$ fixed. Figure 2 shows calculations we have carried out in order to verify that conclusion from a Navier–Stokes standpoint for $\alpha/\beta = 0.5$. In figure 2, we first compute the upper curve by continuation from the lower branch sinuous mode solution for $(\alpha, \beta) = (1, 2)$ shown in figure 1. The numerical scheme used in the continuation is outlined in Deguchi et al. [18]. Interestingly in this configuration, we find a bifurcation point of the sinuous mode from the mirror-symmetric solutions. We shall henceforth refer to the latter solutions, which correspond to EQ7/EQ8 in Gibson et al. [3] and HVS in Itano & Generalis [4], as the mirror symmetric mode. This bifurcation hierarchy of the sinuous and mirror-symmetric modes is indirectly suggested by results for heated channel flow by Itano & Generalis [4] or sliding Couette flow by Deguchi & Nagata [25], but has not been reported in plane Couette flow so far. The continuation of the mirror-symmetric mode solution branch results in the lower curve in the figure. These curves are already converged to VWI states since the bifurcation curves at $R = 16000$ and 30 000 have almost collapsed onto a single curve. If the results are rescaled, then we anticipate that $d_0$, defined earlier.

Figure 2. Bifurcation diagram of the lower branch equilibrium solutions in plane Couette flow for $\alpha/\beta = 0.5$. (a) Wall drag $\Delta$ for $R = 16000$ (thin line) and $R = 30000$ (thick line). (b) The displacement velocity $d_0$ for $R = 30000$. The sinuous mode solution branch connects to the mirror-symmetric solution branch at the bifurcation point indicated by the open circle.
for the unbounded VWI problem, should approach a constant with increasing $\beta$. That result is confirmed in our calculations for $R = 30\,000$ as shown in figure 2b, thus substantiating the claim that the unbounded VWI state exists in plane Couette flow when the spanwise wavelength shrinks to zero.

Figure 3 shows the flow field associated with the development of the structures with increasing $\beta$ along the solution branches in figure 2b and the birth of canonical coherent structures for both of the modes. As predicted by Blackburn et al. [22], for large $\beta$ the flow structure visualized by the streamwise vorticity of the roll component detaches from the wall and shrinks into a layer of depth $\beta^{-1}$ around the critical layer. The scaling (2.15)–(2.16) is confirmed in figure 4, where we show the mean flow $u_M(y)$, i.e. the spanwise averaged streak, of the sinuous mode at $R = 30\,000$ for increasing values of $\beta$. We can see in figure 4a that the mean flow for $\beta = 8, 10, 12$ has uniform slope except for the interaction layer at the channel centre. There is a jump in the mean flow across the layer of depth $O(\beta^{-1})$ where the VWI is active and this jump corresponds to the displacement velocity $d_0$. These results are scaled to the canonical VWI form and then plotted in figure 4b. The figure clearly shows that at large $\beta$ the VWI solution in plane Couette flow defines exact coherent structures for unbounded shear flow. The thin dotted lines are $u_M^* = y^* \mp d_0$ with converged $d_0 = 3.3$ in figure 2b. Thus, we have demonstrated using full numerical solutions of the equations of motion that what we refer to as the unbounded VWI state exists locally in a flow of constant shear.
The plane Couette flow solutions computed have symmetry with respect to $y$ and this symmetry pins the position to the nonlinear interaction at the channel centre. However, in general, the interaction can occur at any location $y = g$. In this case, if $\beta \gg 1$, the mean flow correction $u_m(y) = u_M(y) - u_B(y)$ can be written as

$$u_m(y) = \begin{cases} u_{m+}(y) = \Delta_m (y - 1) & \text{if } y \in (g, 1] \\ u_{m-}(y) = \Delta_m (y + 1) & \text{if } y \in [-1, g) \end{cases}$$

(2.17)

where $\Delta_m$ is the wall drag correction $(d_y u_m)|_{y=1}$. Note that generally the upper and lower wall drag corrections must be in balance. The displacement velocity is determined by the relationship $2d_0 = u_{m+}^{(0)} - u_{m-}^{(0)}$ since $d_0 = \beta \Delta_m / S$. The scaling needed in order to embed the unbounded VWI solution is the same as (2.15) and (2.16), except that now $y^* = \beta (y - g)$ and the wave now propagates with the speed of the fluid at $y = g$.

For a more general shear flow, we expand the profile locally at any point to deduce that the unbounded VWI state can exist there too at any point where the shear is non-zero. Let us discuss how to embed the unbounded VWI states in plane Poiseuille flow as an example. The basic flow is $u_B(y) = (1 - y^2)$, which is not a linear profile. If we now write the velocity field in terms of $y^*$ defined above, we obtain $u_B = (\beta (1 - g^2) - 2gy^* - y^{*2} / \beta) \beta$, and if the second term is larger than the third term, i.e. $2g\beta \gg 1$, we can use the same argument given earlier. The best approximation is expected for $\beta \gg 1$, while the structure will fail near the channel centre. The wave propagates with speed $\beta (1 - g^2)$ and the slope used in the scaling is $S = (d_y u_B)|_{y=g} + \Delta_m = -2g + \Delta_m$. The unbounded VWI state is now properly embedded in plane Poiseuille flow with mean flow given by (2.17). Thus, the unbounded VWI state can be embedded around any location in an arbitrary shear flow whenever the mean flow can be locally approximated by a linear profile and the local Reynolds number as defined above is large.

The result that an arbitrary shear flow can support the unbounded VWI structure locally at any point in the flow where the shear is non-zero can be extended with interesting consequence to more general flows. Consider a background flow $(u^+(y^+, z^+), 0, 0)$, where $a^+$ represents a dimensional quantity; we focus attention on a curve $C^+$ defined by $y^+ = G^+(z^+)$ on which $u^+ = c^+$. The curve can be closed or open depending on the flow under consideration. Let $D^+$ be a typical length along the curve, and $\lambda^+$ is a wavelength taken to be small compared with $D^+$.

\[ u_m(y) = \begin{cases} u_{m+}(y) = \Delta_m (y - 1) & \text{if } y \in (g, 1] \\ u_{m-}(y) = \Delta_m (y + 1) & \text{if } y \in [-1, g) \end{cases} \]
We measure distance with respect to some given point on the curve by $s$, scale it with respect to $D^+$ and take $x, n$ to be distance in the streamwise direction and distance normal to the curve, both scaled on $\lambda^+$. Noting that the dimensional distance in the normal direction is $n^+ = \lambda^+ n$, we can expand near the contour $C^+$ to give

$$u^+ = c^+ + \frac{\partial u^+}{\partial n^+} n^+ + \ldots = c^+ + \left[ \lambda^+ \frac{\partial u^+}{\partial n^+} \right]_{s=n=0} \Gamma(s)n + \ldots,$$

and the quantity inside the square brackets is taken as a typical flow velocity used with $\lambda^+$ to define a Reynolds number within the VWI expansion procedure. The function $\Gamma(s)$ is the local normal dimensionless shear along the contour and varies with $s$. Next, we introduce a phase variable $\Phi$ defined by $\Phi = (D^+ \lambda^+) \int_0^{s'} \beta(s') ds'$ so that a derivative along the curve becomes $(\dot{\beta}/\lambda^+) \partial \Phi$. Now suppose we consider a wave of wavenumber $\alpha_0$ propagating in the $x$-direction with speed $c^+$. The interaction takes place locally at each position $s$ along the curve but along the curve the ‘spanwise’ dependence is in terms of $\Phi$. At each position, the interaction takes on the unbounded VWI canonical form with $(x, y, z)$ replaced by $(x, n, \Phi)$, etc., but now with the condition that the mean flow tends to $\Gamma n$ as $n \to \pm \infty$ and the local wavenumber $\beta$ premultiplies all $\Phi$ derivatives. If we now fix $\beta = \Gamma(s)$, then the equations can be rescaled back to the standard unbounded VWI form but with the streamwise wavenumber appearing in the equations now as $\alpha^* = \alpha_0 / \Gamma(s)$. The quantity $\alpha_0$ is independent of $s$ so at each position along the contour the solution is obtained from the unbounded VWI solution with local wavenumber $\alpha^*$. The solution will be well defined so long as $\alpha^*$ is in the interval for which the unbounded VWI problem has a solution. Note that the Reynolds number in the expansion must be adjusted to account for the rescaling at each $s$. For the flow in a finite container, the contour will be closed and single-valuedness must be achieved; that causes a quantization condition on allowable $\lambda^+$ and requires that $(D^+ \lambda^+) \int_{C^+} \beta(s') ds' = 2N\pi$ for some large integer $N$. This approach can be used to express the large azimuthal wavenumber circular pipe exact coherent structure solutions in terms of the unbounded VWI solutions. Moreover, the condition that the flow is unidirectional can be relaxed so long as the other velocity components are small, so the analysis would apply when $u^+(y^+, z^+)$ is itself the streak of an $O(1)$ wavelength interaction. Also the streak part of the flow produced after the interaction can itself have a small wavelength state introduced and so on. Thus, the mechanism leads to a way to generate successively smaller scales, the cascade ending when the local Reynolds number becomes $O(1)$.

Finally in this section, we remark on the properties of the mirror-symmetric mode at high $R$. Our belief that all equilibrium solutions deform into VWI states at high $R$ to laminar flow at infinite $R$ was apparently made based on the conclusion was apparently made based on the formal VWI expansion procedure since the results eventually converge to an $O(1)$ quantity $U_{m0}$. We see in figure 5b that after rescaling $(u_m - U_{m0})$ by $R^{1/3}$ all of the solutions collapse and the results show excellent agreement...
Figure 5. The mean-flow correction $u_m(y)$ for the mirror-symmetric solutions. Red dashed, green dotted and blue dash-dotted curves are $R = 10^5, 5 \times 10^5$ and $10^6$, respectively. Thick grey curves are (a) $U_m(y)$ and (b) $U_m(y)$ which gives the least-squares approximation $u_m = U_{m0} + R^{-1/3} U_m$ for $R \in [10^5, 10^6]$. 320 Chebyshev modes were used in the wall-normal direction while 3 and 50 Fourier modes were used for the streamwise and spanwise directions, respectively. (Online version in colour.)

to the next order correction $U_m$. In the view of these results, we conclude Itano et al. [26] came to the wrong conclusion because of the smallness of the leading order term for the mirror-symmetric mode mean-flow correction.

3. A theory of freestream coherent structures

It is well known that in turbulent boundary layers coherent structures can simultaneously exist in the main part of the boundary layer and in the freestream. The dependence of these structures on each other has been the subject of many numerical and experimental investigations. Rao et al. [27] suggest that the structures are coupled while Jiménez & Pinelli [28] argue that they are in fact independent. Certainly, the VWI states found in plane Couette flow have counterparts in boundary layers, and Deguchi & Hall [20] presented results for ASBL having scales typical of VWI. In addition to the VWI mode, Deguchi & Hall [20] also uncovered a new exact coherent structure propagating at the edge of the boundary layer with almost the freestream speed. Within a layer at the edge of the boundary layer, waves, rolls and streaks interact to sustain an equilibrium state, but a separation of those quantities is not possible and the interaction is not of the VWI type.

Henceforth, we will refer to this layer as the production layer (PL). The PL is located where the basic flow differs from its freestream value by $O(R^{-1})$. For ASBL, which has the basic flow $(u, v, w) = (1 - e^{-y}, -R^{-1}, 0)$, the PL is of depth comparable to the unperturbed boundary-layer depth and is located a distance $O(\ln R)$ from the wall. This is one of the remarkable distinctions from VWI since the higher $R$, the more PL disturbance can penetrate into freestream in consistent with experimental observation; recall that the critical layer where the VWI occurs keeps a constant distance from the wall. Another crucial distinction is that in the PL all disturbance amplitude is $O(R^{-1})$, whereas VWI is driven by $O(1)$ streak. Therefore, the PL structure is ‘closer’ to the basic flow compared with VWI states.

Deguchi & Hall [20] refer to this new type of flow structure as the FCS, and it turns out that it is determined by the solution of a nonlinear eigenvalue problem associated with the three-dimensional Navier–Stokes equations at unit Reynolds number in a region unbounded in both upper and lower sides of the layer. In other words, the wave is viscous everywhere and thus there is no critical layer singularity.
The asymptotic theory presented by Deguchi & Hall [20] showed that the streak disturbance is generated within the PL and continues to grow below that layer in negative wall normal direction before ultimately being reduced to zero at the wall. The growth of the streak disturbance is driven by advection with the crucial interaction taking place between the decaying roll and exponentially growing mean flow as the PL is exited downwards. The outcome then is a small exact coherent structure which is generated in the freestream, yet its signature is most easily seen in the main part of the boundary layer. As such, it can therefore be viewed as a coherent structure which couples the interaction in the freestream and the main part of the boundary layer.

ASBL is of course not typical of boundary layers in general since it is of constant thickness so that non-parallel effects are absent. However, the nonlinear interaction in the PL is generic to all boundary layers which approach their freestream speed through an exponentially small correction. We shall here demonstrate its relevance to a wide range of high-Reynolds-number boundary layer flows. In order to show the essential features of the FCS, we consider Burgers vortex sheet since, as well as being a relatively simple flow, it is of boundary layer type yet relevant to vortex sheet dynamics. This flow is a planar version of Burgers vortex which is an exact Navier–Stokes solution and is thought to play a fundamental role in the dynamics of turbulence (e.g. [29–32]). Interest has primarily focused on the linear instability of the vortex though its instability to a general finite-amplitude three-dimensional perturbation remains uncertain.

With respect to Cartesian coordinates, we take the dimensionless equations of motion for a viscous fluid as given in (2.1). Note that this means that the lengthscale now used is the one associated with the strain rate of the vortex while the typical speed used is the freestream $x$ velocity. The basic flow of Burgers vortex sheet is given by

$$u = 1 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\psi^2/2} d\psi, \quad v = -\frac{y}{R}, \quad w = \frac{z}{R} \quad \text{and} \quad p = -\left(\frac{y^2 + z^2}{2R^2}\right).$$

This is an exact solution of the Navier–Stokes equations, and we will concentrate on the high Reynolds number limit. We seek a nonlinear FCS in a layer centred somewhere in the upper freestream, i.e. $y \gg 1$, propagating downstream with almost unit speed. A similar FCS can occur in a PL in the lower freestream but does not couple to the one in the upper PL, we will comment further on this point later. We take the PL in the upper freestream to be located a distance of $O(K)$ with $K \gg 1$ from the wall. The quantity $K$ remains to be found but writing $y = K + Y/K$ gives $[u - 1] \approx -\sqrt{2/\pi} K^{-1} e^{-K^2/2} e^{-Y}$ so if we want this quantity to be comparable with the FCS velocity components, we must take the layer to be of depth $K^{-1}$. Within the PL, we seek a nonlinear wave structure with the difference of the velocity field from the equilibrium solution having velocity components of the same size and operating on the same $O(K^{-1})$ lengthscale in the $x, y, z$ directions. If we restrict any streamwise dependence to be in the form of a wave moving downstream with the freestream speed, then the nonlinear and viscous terms in the Navier–Stokes equations will balance within the PL if $K^{-1} e^{-K^2/2} = KR^{-1}$ so that we take

$$K = \sqrt{2 \ln R} - \frac{2 \ln \left(\sqrt{2 \ln R}\right)}{\sqrt{2 \ln R}} + \cdots \quad (3.1)$$

and define

$$X = K(x - ct), \quad Z = Kz. \quad (3.2)$$

If we write $Y = K(y - K)$, then for $Y = O(1)$ the basic streamwise velocity $\approx 1 - KR^{-1} \sqrt{2/\pi} e^{-Y}$. In order to produce the same scaled PL problem as found by Deguchi & Hall [20] for ASBL, we shift the origin in $Y$ by writing $Y = K(y - K) - \ln \left(\sqrt{2/\pi}\right)$ so that the basic state is now $\approx 1 - KR^{-1} e^{-Y}$. The wavespeed $c$ is taken to be unity plus a correction term the same size as the decaying part of the basic flow in the PL; we therefore write

$$c = 1 - \frac{K}{R} c_1 + \cdots \quad (3.3)$$
Finally, we expand the velocity components in the form

\[ u = 1 + \frac{K}{R} U(X,Y,Z) + \cdots, \]  
\[ v = \frac{K}{R} V(X,Y,Z) + \cdots, \]  
\[ w = \frac{Z}{KR} + \frac{K}{R} W(X,Y,Z) + \cdots \]

and

\[ p = -\frac{K^2}{R^2} + \frac{K^2}{R^2} P(X,Y,Z) + \cdots, \]  

where the equilibrium pressure is now written in terms of the PL variables. Note also that the unperturbed flow is \((U, V, W) = (-e^{-Y}, -1, 0)\). If we substitute into (2.1) and retain the leading order terms, then we find that \(U, V, W, P\) satisfy

\[(U - c_1 i \cdot \nabla) U = -\nabla P + \nabla^2 U \]

and

\[ \nabla \cdot U = 0, \]

where \(\nabla = (\partial_X, \partial_Y, \partial_Z)\), \(\nabla^2 = \partial_X^2 + \partial_Y^2 + \partial_Z^2\). This is exactly the PL problem for ASBL and any boundary layer approaching its freestream value exponentially, see Deguchi & Hall [20]. The FCS is assumed periodic in \(X, Z\) with wavenumbers \(\alpha, \beta\), respectively, while above and beneath the PL we require that the FCS approaches the unperturbed flow to leading order. Thus for the streamwise velocity component beneath the PL, we require \(U e^Y \rightarrow -1\) as \(Y \rightarrow -\infty\) and this allows for the \(z\)-dependent part of \(U\) to grow exponentially in large negative \(Y\) so long as it is not as quickly as \(e^{-Y}\). Thus, the PL equations are to be solved subjected to

\[ U \left( X, Y, Z + \frac{2\pi}{\beta} \right) = U(X,Y,Z), \]  
\[ U \left( X + \frac{2\pi}{\alpha}, Y, Z \right) = U(X,Y,Z), \]

\(U \rightarrow 0, V \rightarrow -1, W \rightarrow 0\) as \(Y \rightarrow \infty\),

and

\[ U e^Y \rightarrow -1, V \rightarrow -1, W \rightarrow 0\]  
\(\) as \(Y \rightarrow -\infty\).

The PL equations together with the above conditions constitute a nonlinear eigenvalue problem for \(c_1 = c_1(\alpha, \beta)\). The solution of this problem can be extracted from the high-Reynolds-number results for ASBL; details of the eigenrelation and that extraction are given in Deguchi & Hall [20]. The solutions found in that paper have, as required, the roll, streak and wave all tending to zero exponentially when \(Y \rightarrow \infty\). For large negative \(Y\), the wave and roll disturbance part of the flow decay to zero. However, for the range of values of \(\beta\) where a solution of the nonlinear eigenvalue was found, the streamwise velocity component has the asymptotic form

\[ U \sim -e^{-Y} + \sum_{n=1}^{\infty} I_n e^{(\omega_n - 1)Y} \cos(2n\beta Z) + \cdots, \]

where \(\omega_n - 1 = (\sqrt{1 + 16n^2 \beta^2} - 3)/2\), and \(I_n\) is a constant fixed by the nonlinear eigenvalue problem for \(c_1\). The first part of the above expression is simply the deviation of the streamwise component of laminar ASBL flow from a uniform stream, and the second term grows exponentially in large negative \(Y\) whenever \(\beta < 1/\sqrt{2}\). Thus, the mean part of \(U\) grows exponentially when \(Y \rightarrow -\infty\) while also does the \(z\)-dependent part but at a slower rate. Figure 6a,b shows the disturbance to the streak and roll field of the PL solution for \((\alpha, \beta) = (0.2, 0.4)\). We can see the growing streak disturbance beneath the PL and the roll disturbance...
trapped in the PL. The visualization of the PL solution in terms of the so-called $\lambda_2$ criterion is shown in figure 7. This criterion was first introduced by Jeong & Hussain [33] and is widely used to identify the vortex core. Note that negative $\lambda_2$ implies the fluid experiences strong rotation. In the figure, we can see that the flow field is characterized by $A$ shape vortices.

Beneath the PL there is no $X$ dependence. As the PL is exited downwards, the flow returns to the unperturbed Burgers vortex sheet plus exponentially growing terms proportional to $\cos(2n\beta Z)$ for $n\beta < 1/\sqrt{2}$. If such terms exist, then the disturbance will be eventually dominated by the $n = 1$ term and so we concentrate on that case. Since the PL is located a distance $O(K)$ from the wall, we anticipate that the growth of the $z$-dependent part of $U$ will be arrested in the region $y \sim O(K)$. Now let us indicate the nature of that process in the region $\zeta = y/K = O(1)$ beneath the PL. Here, the unperurbed flow can be written as

\[
  u = 1 - \sqrt{\frac{2}{\pi K\zeta}} e^{-K^2(\zeta^2/2)}, \quad v = -\frac{K\zeta}{R} \quad \text{and} \quad w = \frac{Z}{KR}.
\]

(3.15)
If we assume that for \( \zeta \sim O(1) \) the normal velocity component is written as \( \nu = -K\zeta/R + \hat{V}(\zeta) \cos(2\beta z) \), where \( \hat{V} \) is small enough for linearization around the unperturbed state to be valid, then the equation for \( \hat{V}(\zeta, X, Z) \) can be approximated by

\[
\left[ \frac{1}{K^2} \frac{\partial^2}{\partial \zeta^2} + \xi \frac{\partial}{\partial \zeta} - 4\beta^2 K^2 \right] \hat{V} = 0.
\]

Note that the term associated with spanwise diffusion is proportional to \( K^2 \), because the spanwise variable was originally scaled on \( K^{-1} \). The solution of the above equation can be found by a WKB approach taking \( \hat{V} = \exp(k^2 \int \xi \Theta(\chi) \, d\chi) [\hat{V}_0(\zeta) + \cdots] \). After some analysis, we find that

\[
\frac{d\hat{V}_0}{\hat{V}_0} = -\Theta \, d\Theta \left[ \frac{1}{(4\beta^2 + \Theta^2)} + \frac{2}{(\Theta^2 - 4\beta^2)} \right].
\]  

The required root of the eikonal equation (3.16) has the least rate of decay and is given by

\[
2\sqrt{\xi^2 + 16\beta^2} = \sqrt{\xi^2 + 16\beta^2} \quad \text{and} \quad \hat{V}_0 = \frac{M(K)}{\left[4\beta^2 + \Theta^2 \right]^{1/2} \left(\Theta^2 - 4\beta^2\right)}
\]

where \( M \) is to be fixed by matching with the PL solution. This roll solution then forces the streak equation which is again solved by the WKB method. The dominant contribution of the roll to the streak equation comes from the term forced by the roll field multiplied by the normal derivative of the exponentially growing part of the mean flow for large negative \( Y \). After some analysis, we find that the streakwise velocity in that region matching onto the PL solution is

\[
u \sim 1 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{y} e^{-\psi^2/2} \, d\psi + \frac{\sqrt{2 \ln R}}{R} \frac{A(1)}{A(\Theta)} \gamma_1 \cos(2\beta z) \exp \left[ k^2 \int_{1}^{\zeta} [-\zeta' + \Theta(\zeta')] \, d\zeta' \right] + \cdots.
\]  

Here the amplitude function \( A \) is given by

\[
A = (\Theta^2 - \xi^2 - 4\beta^2)(\Theta^2 - 4\beta^2)(\Theta^2 + 4\beta^2)^{1/2},\tag{3.18}
\]

while

\[
\gamma_1 = \left[ \frac{2}{\pi} \right] \left( \sqrt{1 + 16\beta^2} - 3 \right)^{1/4}.
\]

The integrand in (3.17) can be evaluated analytically but it is rather lengthy so we leave it in its present form. It is easy to show that when \( \beta < 1/\sqrt{2} \) the integrand is negative between \( \zeta = 1 \) and \( \zeta = \sqrt{2}\beta \) so that the term proportional to \( \cos(2\beta z) \) has its maximum value at \( \zeta = \sqrt{2}\beta \). Thus, the streak generated within the PL grows exponentially below that layer until it reaches a maximum at \( y = 2\sqrt{\beta \ln R} \). Below this location, the integrand becomes positive and the term proportional to \( \cos(2\beta z) \) in (3.17) decreases exponentially to zero. Note that for small \( \zeta \), \( A(\Theta) \) tends to zero as a WKB turning point is approached and a new layer is needed. However, that layer is passive and connects with a solution continuing to decay exponentially all the way to \( y = -\infty \). Thus, the nonlinear structure in the upper PL does not induce a similar structure in the lower PL; however, by symmetry we can construct a nonlinear structure in the lower PL which acts independently of the upper PL. It follows from the definition of \( K \) that at the position where the maximum disturbance size occurs the \( z \)-dependent part of \( u \) becomes of size \( R^{\Lambda - 1} \sqrt{2 \ln R} \) where the exponent \( \Lambda \) is given by

\[
\Lambda = \frac{3 - \sqrt{1 + 16\beta^2}}{2} + 8\beta^2 \ln \left( \frac{4\sqrt{2}\beta}{1 + \sqrt{1 + 16\beta^2}} \right).
\]  

This is quite distinct from the ASBL case where the maximum streak velocity is attained in the unperturbed boundary layer. Here, the nonlinear interaction involving rolls, waves and the streak produces a much larger streak flow which attains its maximum in the logarithmic layer \( \zeta \sim O(1) \). In order to illustrate the streak field there, we have plotted contours of constant \( u \) given by (3.17) in figure 8a. Here the solution is scaled to match to the PL solution for \( (\alpha, \beta) = (0.2, 0.4) \) which
Parameter have been chosen so that the solution matches to the PL solution shown in figures 6 and 7. (a) Total velocity given in (3.18), \( \text{Col}_{\text{max}} = 0.965 \) and (b) disturbance to the laminar flow, \( \text{Col}_{\text{max}} = 0.028 \). (Online version in colour.)

By symmetry of the system, we conclude then that FCS of the type found in Deguchi & Hall [20] can exist in either a thin layer in the upper or lower part of the freestream for Burgers vortex sheet. The manifestation of the nonlinear interaction is a much large spanwise periodic flow in a layer of depth \( O(\sqrt{\ln R}) \) which is of course outside the main part of the boundary layer.

4. Conclusion and discussion

We have shown how the formal VWI and FCS asymptotic frameworks can be generalized to produce canonical exact coherent structures for a range of shear flows. In these canonical problems, the nonlinear interaction takes place locally in an unbounded domain. The solutions for the canonical problems for both the VWI and FCS states can be deduced as done here by extrapolation from high-Reynolds-number full Navier–Stokes solutions or by custom numerical schemes exploiting the simplified nature of the interaction equations. The extracted solutions have been applied to simple example problems together with matched asymptotic solutions outside of the local interaction region.

(a) Vortex–wave interaction theory in unbounded domain

The structure of the unbounded VWI state consists of a critical layer at the centre of a VWI and a layer surrounding the critical layer. In the latter outer layer, the roll-streak flow induced by a
stress jump across the critical layer decays with distance from the critical layer. Away from the layer, the effects of the interaction is to produce a displacement effect on the streamwise velocity component. By an appropriate rescaling the canonical problem for an unbounded shear flow can be recovered from the plane Couette flow VWI problem of Hall & Sherwin [17]. Therefore, as pointed out in Blackburn et al. [22], the unbounded VWI problem can be solved by computing limiting equilibrium solutions in plane Couette flow. In this work, the converged VWI solutions were inferred by solving the full Navier–Stokes equations for the exact coherent structures at sufficiently high Reynolds numbers.

A bifurcation point of the sinuous mode and mirror-symmetric mode solutions, corresponding to EQ1/EQ2 and EQ7/EQ8 of Gibson et al. [3], respectively, was found as a by-product of that calculation. The lower branch for both of the modes was generated for a wide range of spanwise wavenumbers, and we confirmed that these solutions deform with increasing spanwise wavenumber into the canonical state first discussed by Blackburn et al. [22]. The results strongly support the assertion in Deguchi & Hall [20] that all exact coherent structures in plane Couette flow become VWI states for sufficiently high Reynolds numbers. We speculate that for each known plane Couette flow solution, for example that listed in Gibson et al. [3], a corresponding VWI solution of (2.1)–(2.6) exists.

The solution of the unbounded VWI problem extracted from plane Couette flow was shown to be applicable to arbitrary shear flows with the velocity displacement of the unbounded VWI solution creating a mean-flow jump across the local interaction region when the structure is embedded in a particular shear flow. When the unbounded VWI problem is applied to wall bounded shear flows the velocity jump associated with the nonlinear interaction, and boundary conditions determine the mean-flow distortion outside of the interaction region. In this work, the theory was applied to plane Poiseuille flow, and it was found that the solution can be properly embedded in the near wall region and an analytic expression for the mean-flow distortion was derived.

The unbounded VWI formulation breaks down in certain situations. First we note that for fixed finite $R$ an upper and lower branch exist and connect at a finite value of the streamwise or spanwise wavenumbers as shown in figure 1. In figure 2, we showed that the branches exhibit turning points when the spanwise wavenumber is increased beyond the range of $\beta$, where we have confirmed unbounded VWI states for the lower-branch solutions. This means that the new asymptotic problem emerges when the spanwise wavelength becomes very short. It was shown numerically that the turning point moves to smaller wavelength with increasing Reynolds numbers. Second in our discussion, we made no mention of what might happen to the unbounded VWI state as the scaled streamwise wavenumber tends to zero. Certainly for plane Couette flow, streamwise localization occurs in that limit and it would be of interest to investigate whether the unbounded VWI states for arbitrary shear flows have counterparts of the long-wavelength solutions of Deguchi et al. [18].

(b) Freestream coherent structure theory

In order to demonstrate the widespread applicability of the FCS first found for ASBL in Deguchi & Hall [20], we used Burgers vortex sheet as a model problem. The solution of the unbounded problem within the PL was obtained by rescaling the FCS solutions of ASBL computed in Deguchi & Hall [20]. The PL appears when the streamwise basic flow can be approximated by the freestream velocity plus an exponentially decaying function. The origin of the growing streak disturbance is the interaction of the decaying roll disturbance and the basic flow. For ASBL, the correction of the unperturbed velocity field from a uniform outer stream is an exponential function of the scaled distance from the wall. In that case, the streak induced in the PL is forced uniformly by the mean state all the way to the unperturbed boundary layer, where it takes on its maximum value. For more complex flows such as Burgers vortex sheet the decay to a uniform stream of the unperturbed flow is only locally an exponential function of the scaled distance from the centre of the PL. This means that the interaction beneath the PL is more subtle and must
be described using a WKB approach with the upshot that the streak disturbance now takes on its maximum value at a distance $O\left(\sqrt{2\ln R}\right)$ from the wall. Here we note that the spanwise component of the unperturbed straining field played no role to the order considered. At higher order, this will not be the case; alternatively, we point out that this analysis is valid only for $O(1)$ values of $z$. We further note that the extension of our analysis to the circular Burgers vortex must also account for centrifugal and Coriolis effects not present here.

(c) Future works

One of the most important practical applications of the canonical VWI and FCS theories is to growing boundary layers. When the development of the FCS in growing boundary layers is considered, the PL problem for such flows remains unchanged but with local wavenumbers appropriate to the local boundary layer thickness. Below the PL, non-parallel effects come into play and lead to flows quite distinct from the parallel case. Asymptotic and numerical investigation of the structure beneath the PL is currently underway.

The numerical solution of the original VWI problem given in Hall & Smith [16] for growing boundary layers has yet to be carried out. In that problem, the streak/roll field satisfy the boundary region equations with the generic stress jumps induced by the wave to be applied at the critical layer. The boundary region equations are non-parallel and so the wave propagates with constant frequency and a slowly varying streamwise wavenumber. The non-parallel nature of the required computational problem makes the computation more difficult and its solution remains a challenge.

Finally, we note that it is possible to write down a set of ‘viscous’ VWI equations involving Tollmien–Schlichting waves when the critical layer is located at the wall [16]. Those equations have a quite different character than those considered here, and the interaction of the critical layer and the main part of the flow is governed by triple-deck theory or its reduced double-deck form for internal flows. In fact, a clear distinction between internal and external flows arises for VWI states involving viscous waves. Thus for external flows, the forcing is in the wall layer, whereas for internal flows it is distributed across the channel. Progress with the latter case is more straightforward since the weakly nonlinear limit can be taken if the unperturbed flow is unstable to Tollmien–Schlichting waves. In view of the apparent close relationship of the VWI and FCS states with turbulent flows, it will be interesting to see what role is played by the viscous VWI modes numerically.

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References


