Hydrodynamics of periodic breathers

A. Chabchoub¹, B. Kibler², J. M. Dudley³ and N. Akhmediev⁴

¹Centre for Ocean Engineering Science and Technology, Swinburne University of Technology, Hawthorn, Victoria 3122, Australia
²Laboratoire Interdisciplinaire Carnot de Bourgogne, UMR 6303 CNRS, Université de Bourgogne, Dijon, France
³Institut FEMTO-ST, UMR 6174 CNRS-Université de Franche-Comté, 25030 Besançon, France
⁴Optical Sciences Group, Research School of Physics and Engineering, The Australian National University, Canberra, Australian Capital Territory 0200, Australia

We report the first experimental observation of periodic breathers in water waves. One of them is Kuznetsov–Ma soliton and another one is Akhmediev breather. Each of them is a localized solution of the nonlinear Schrödinger equation (NLS) on a constant background. The difference is in localization which is either in time or in space. The experiments conducted in a water wave flume show results that are in good agreement with the NLS theory. Basic features of the breathers that include the maximal amplitudes and spectra are consistent with the theoretical predictions.

1. Introduction

Rogue waves (RWs) from the object of myths, legends and seafarers tales are converted now into the subject of scientific studies [1]. Scientific measurements, collected from buoys, satellites and oil- and gas-platforms after the first measurement of the Draupner wave [2], confirmed the existence of extreme wave events in the world oceans. RWs are presently studied both theoretically and experimentally. The linear theories provide one possible explanation for appearance of RWs [3]. However, the nonlinear nature of ocean waves is now well established [4,5]. The nonlinearity is especially important in the description of high amplitude waves which is the main feature of RWs.
Modelling wave dynamics with the nonlinear Schrödinger equation (NLS) is an approach that takes into account the nonlinearity and dispersion at the lowest order \[6,7\] thus providing the nonlinear description of these waves at the fundamental level. Governing the wave motion in dispersive nonlinear media, it gives an explanation and description of extreme waves, which appear from nowhere and disappear without a trace \[8\]. In particular, the NLS describes the Benjamin–Feir or modulational instability \[9,10\], which is the main mechanism, leading to the generation of RWs in the ocean \[1,5,11\] and in other nonlinear dispersive media \[12\].

The integrability of the NLS \[13\] enables us to write exact breather solutions on finite background in analytical form \[14\]. These solutions describe localized carrier perturbations with increase of the amplitude of the background wave by a factor of higher than two \[15\]. As a result, such breather solutions can explain the generation of RWs.

Recently, exact breather solutions have also attracted attention. Specifically, the single-peak Peregrine solution \[16\] has been observed in optics \[17\], in water waves \[18\] and in plasma \[19\]. These experiments confirmed deep analogies between diverse nonlinear dispersive media, where the NLS approach is used. There are several types of breathers. Among them, we can mention the time-periodic Kuznetsov–Ma soliton (KM-soliton) \[20,21\] and the space-periodic Akhmediev breather (AB) \[22,23\]. The Peregrine breather can be considered as the infinite-period limiting case of either of them. These three solutions belong to the class of first-order breather solutions of the NLS \[24\]. The significance of the AB solution is that its initial stage describes a well-studied process of modulational instability \[23\]. Moreover, the full growth-return cycle of the AB provides a solution for the Fermi–Pasta–Ulam-recurrence (FPU) paradox \[25\]. Thus, their experimental observation can be considered as a major step in physics of nonlinear dispersive media.

So far, exact fundamental ABs and KM-solitons have been observed only in optics \[26–28\]. In this work, we demonstrate, for the first time, the existence of periodic breather solutions in water waves. These observations confirm once again the strong analogy between nonlinear wave dynamics as well as phenomena in optics and hydrodynamics. Our work can stimulate similar experiments in other nonlinear dispersive media. It may possibly take the research a further step closer to prediction of extreme waves in treacherous marine conditions \[29\].

### 2. Mathematical preliminaries

The scaled form of the NLS \[4\]
\[
iψ_T + ψ_{XX} + 2|ψ|^2ψ = 0, \quad (2.1)
\]

admits a family of stationary solutions and pulsating solutions also referred to as breathers \[30\]. These solutions can be derived using several mathematical integration techniques \[13,14,23,31,32\].

The family of KM-soliton solutions \[20,21\] can be expressed in terms of a real parameter \(ϕ ∈ \mathbb{R}\):

\[
ψ_{KM}(X, T) = \frac{\cos(ΩT - 2iϕ) - \cosh(ϕ) \cosh(pX)}{\cos(ΩT) - \cosh(ϕ) \cosh(pX)} \exp(2iT), \quad (2.2)
\]

where \(Ω = 2 \sinh(2ϕ)\) and \(p = 2 \sinh(ϕ)\).

The solution is periodic in time \(T\) and localized in space \(X\). Both, period in \(T\) and the degree of localization in \(X\) are defined by the parameter \(ϕ\). The solution is shown in figure 1a for a particular value of \(ϕ = 0.3\). The amplitude of the background here is 1. It can be rescaled to any value by adding the scaling parameter into the solution \[14\]. This additional parameter does not change the ratio of the maximum soliton amplitude \(\max(ψ_{KM})\) to the background which is given by the expression \(1 + 2 \cosh(ϕ)\). Clearly, this ratio is always higher than 3. If the background is 1, then \(\max(ψ_{KM}) > 3\).

The KM-soliton solution describes periodic beating of the ordinary soliton with the background field. This can be seen from the simple geometrical construction first presented in \[33\] and shown here in figure 1b. The soliton with the amplitude \(2 \cosh(ϕ)\) represented by the line OA rotates around the point O\((-1, 0)\) of the complex plane. This rotation results in oscillations
Figure 1. (a) Time-periodic KM soliton for $\varphi = 0.3$ and for a scaled carrier background $|\psi(X \to \pm \infty)| = 1$. (b) Trajectory of the centre point of the KM soliton with $\varphi = 0.3$ on the complex plane. The trajectory follows the solution along the line $X = 0$, while $T$ varies. The trivial phase factor $\exp(2iT)$ is omitted. The soliton has the amplitude and the phase represented by the continuously rotating point A around the point $O(-1,0)$. Maximum amplitude of the solution appears when the point A passes the point B. (Online version in colour.)

of the total amplitude of the KM soliton between the values $-1 + 2\cosh(\varphi)$ and $-1 - 2\cosh(\varphi)$. These oscillations are clearly seen in figure 1a. They can be considered as beating between the two complex fields. Such oscillatory soliton has recently been observed in optics [27].

The solution (2.2) belongs to the class of first-order solutions of the NLS [24]. Here, it is presented in terms of trigonometric and hyperbolic functions used in [30]. The total family contains three independent parameters controlling the form of the solution. Thus, this is the rich family that covers several physically important types of solutions. A simple plot, showing the correspondence between the solutions and the space of parameters can be found in [14]. Here, we are concerned about two types of periodic solutions.

The second family of breathers which also belongs to the class of first-order solutions is the following:

$$
\psi_{AB}(X, T) = \frac{\cosh(\Omega T - 2i\phi) - \cos(\phi) \cos(pX)}{\cosh(\Omega T) - \cos(\phi) \cos(pX)} \exp(2iT),
$$

(2.3)

where $\Omega = 2\sin(2\phi)$ and $p = 2\sin(\phi)$. It is presently known as AB [30]. Writing it in the form similar to the form of (2.2) shows that each of the two solutions is a part of a larger family. Although parameters $\varphi$ and $\phi$ must be real, formally, the two families can be transformed into each other using the continuation into imaginary axis $\varphi \to i\phi$. Solution (2.3) depends periodically on the parameter $\phi$. Without restricting generality, we can consider it only within the interval $0 < \phi < \pi/2$. The amplitude profile of the AB solution on the $(X, T)$-plane is shown in
Figure 2. (a) Space-periodic AB for $\phi = 0.7$ and with scaled carrier amplitude of $|\psi(T \to \pm \infty)| = 1$. (b) Trajectory of the AB with $\phi = \pi/4$ (maximal modulation instability growth rate) on the complex plane. The trajectory follows the solution on the line $X = 0$, while $T$ varies. The trivial phase factor $\exp(2iT)$ is omitted. The point A(0, 1) of the complex plane corresponds to the initial plane wave with the unit amplitude and the phase $\pi/2$. The point B corresponds to the maximum amplitude of the AB at $X = 0$ and $T = 0$. The point C(0, $-1$) corresponds to the plane wave with the phase $3\pi/2$. Thus, the AB imposes the nonlinear shift $\pi$ on the phase of the plane wave. (Online version in colour.)

This solution describes full growth-return cycle that starts with modulation instability at $T \to -\infty$. The maximum growth rate of modulation instability occurs when $\phi = \pi/4$. The growth rate decays to zero at each end of the interval $0 < \phi < \pi/2$. The maximal amplitude of the solution reached at the peaks of periodic structure is given by $\max(|\psi_{AB}|) = 2\cos(\phi) + 1$. This function has maximum of 3 at $\phi = 0$. Thus, if we consider the growth of instability as amplification of the background, the total amplification provided by the AB solution is limited by the factor of 3. Therefore, the maximal amplitude of each peak of the solution cannot be higher than three times the amplitude of the background $1 < \max(|\psi_{AB}|) < 3$. Experimentally, this solution has been observed and studied in nonlinear optics [17,26,28].

The AB solution describes heteroclinic trajectory in the phase space of infinite-dimensional dynamical system. This can be seen from the geometric construction presented earlier in [23] and shown here in figure 2b. The point A in this plot corresponds to the plane wave at $T \to -\infty$. It is modulationally unstable and in terms of the theory of dynamical systems, this is a saddle. When $T$ increases, the point moves along the trajectory according to the AB solution. It reaches the point B at $T = 0$ when the amplitude of the breather is maximal. Further evolution corresponds to decay of the amplitude when $T \to +\infty$. The trajectory ends up in another saddle point C. The latter corresponds to the plane wave at $T \to +\infty$. The plain wave gains an additional phase $\pi$.
due to the action of AB. The trajectory is part of a circle if we choose \( X = 0 \). It becomes a part of an ellipse for any other fixed \( X \). However, the qualitative features of the trajectory are the same as described above. If we consider the evolution of the spectrum of the AB [23], it starts with a single frequency at \( T \to -\infty \), spreads to a wide discrete spectrum at \( T = 0 \) and returns to a single frequency at \( T \to +\infty \). Thus, this solution can be considered as a solution of an FPU paradox for this dynamical spectrum.

Each of the above solutions is a one-parameter family. In the limit \( \varphi \to 0 \) or \( \phi \to 0 \), period of the solutions tends to infinity leaving only a single bump out of periodic sequence. This common limit can be described by a simple first-order rational solution, known as the Peregrine breather [16]

\[
\psi_{PB}(X, T) = \left[ -1 + 4 \frac{1 + 4iT}{1 + 16T^2 + 4X^2} \right] \exp(2iT).
\]

This is a solution localized both in \( X \) and \( T \) with the maximum amplitude three times the background level. Experimental observations of the Peregrine solution received considerable attention recently [15,17–19,34,35], mainly because it serves as the basic prototype of RWS. More generally, it can be considered as an elementary building block in more complicated families of higher order RW solutions of the NLS [36,37]. Its presence in various physical media confirmed the validity of the NLS in describing nonlinear phenomena and strong localizations simultaneously in time and in space in these media.

### 3. Experimental results

Experiments related to the elementary periodic KMs and ABs, described above, have been conducted in a deep-water wave facility. The flume has a length of 15 m, its width is of 1.5 m, the depth of the water of 1 m, whereas the sensitivity of the wave gauge is of 1.06 V/cm, as described earlier in [38]. The surface gravity waves are generated by a single flap-type wave maker, which is driven by a computer-controlled hydraulic cylinder. It has been shown that it is sufficient to assume a linear transfer function to the wave maker in order to generate soliton-type waves [39]. In order to generate the solution on the surface of water, it is necessary to write it in dimensional units. The dimensional form of deep-water NLS is [7]

\[
-\frac{i}{c_g} (\dot{\psi}_l + c_g \dot{\psi}_x) + \alpha \psi_{xx} + \beta |\psi|^2 \psi = 0,
\]

where \( c_g = \omega/2k \), \( \alpha = \omega/8k^2 \), \( \beta = \omega k^2/2 \), \( \omega \) is the wave frequency and \( k \) is the wave number. The latter are connected through the linear dispersion relation for deep-water waves \( \omega = \sqrt{gk} \) [40]. Equation (3.1) provides a weakly nonlinear approach for the description of narrow-banded water wave field dynamics. For a given water depth \( h \), the NLS (3.1) is valid in approximation of an ideal, i.e. incompressible, inviscid and irrotational fluid when \( kh \gg 1 \) [40]. It is obtained from equation (2.1) by transforming it to the frame moving with the group velocity

\[
X = x - c_gt, \quad T = \beta t \quad \text{and} \quad \psi = \sqrt{\frac{\beta}{2\alpha}} \Psi
\]

To second-order in steepness, the bound Stokes surface elevation is given by [41]

\[
\eta(x, t) = \Re(\Psi(x, t) \exp[i\vartheta]) + \frac{1}{2}k\Psi^2(x, t) \exp[2i\vartheta],
\]

where \( \vartheta = kx - \omega t \) is the phase of the carrier wave. Equation (3.3) is used in experiment to determine the initial conditions as well as to provide the theoretical predictions for water elevation at any specific position \( x^* \). The boundary conditions calculated from an exact solution are then applied to the flap, thus, generating the solution of interest. We translated the solution in space in order to observe the maximal breather compression at the distance of 9 m from the flap. As mentioned in [42], two other important parameters in the experiment are the amplitude of the background \( a \) and the steepness \( \varepsilon := ak \) of the carrier. The value of the latter should be chosen in the way to avoid wave breaking and to ensure ideal breather dynamics in the wave flume for a
given parameter $\varphi$ or $\phi$. It is important to mention that periodic breather dynamics in water waves will not be exactly the same as shown in figure 2a or b. After the transformation to the moving frame (3.2) each breather experiences continuous translation in the laboratory frame. Thus, the observations of periodic breathers in water waves, satisfying equation (3.1), differ from the optical case.

We started the set of experiments by generating a KM soliton with the parameter $\varphi = 1.0$ for the carrier parameter $a = 0.5$ cm and the steepness $\epsilon = 0.08$. The breather evolution with the chosen parameters is shown in figure 3a. Being started with nearly flat background wave slightly phase and amplitude modulated at the wave maker position, the soliton increases its amplitude when moving towards the other end of the water tank. As explained above, the KM-soliton position moves in space when reaching each successive gauge. The soliton reaches its maximum amplitude at the ninth gauge as programmed by the initial conditions. The wave profile at the point of maximum amplitude shown in figure 3b is close to the profile predicted by the theory except for the asymmetry. The latter is caused by higher-order dispersion and Stokes contributions that have not been accounted in the theory.

One more example of the KM soliton evolution is shown in figure 4. Here, we reduced the soliton parameter to $\varphi = 0.8$. Doing so, we decreased the amplitude amplification of the KM soliton. Simultaneously, we increased the steepness value to $\epsilon = 0.09$, keeping the amplitude of the background at the level $a = 0.5$ cm, without observing any breaking of the waves. The latter would strongly affect the breather dynamics during propagation. Figure 4 demonstrates the dynamics similar to the previous case although the maximum amplitude of the soliton is lower.

Small initial modulation seen in the lowest blue curve in figures 3a and 4a is focused by the KM soliton during the wave evolution in the flume. Modulation reaches its maximal compression at the gauge 9. This corresponds to the point B in figure 1b. Experimental wave profiles are in an excellent agreement with the theoretical predictions as can be seen in figures 3b and 4b. Periodic wave dynamics over a large propagation distance has been recently observed in [43]. This shows that observation of several oscillations of periodically evolving KM-solitons is, in principle, possible although the initial conditions in this work have not been inspired by the NLS.

Next, we turn our attention to the evolution of ABs in the water wave flume. This solution, describes the FPU recurrence for the NLS. Its growth-return cycle also describes the growth and decay of RWs. Thus, its observation is indeed important. First, we have chosen the parameters in the experiment to be $a = 0.5$ cm and $\epsilon = 0.09$, while the breather parameter is $\phi = 0.3$. Figure 5a shows the evolution of this solution in the wave tank. Comparison of the experimental curve at the position of maximal amplification (blue curve) with the theoretical one (red curve) is shown in figure 5b. We took into account the transformation (3.2) in plotting the theoretical curve. The variation of the amplitudes in the successive periods of the breather which is related to the group velocity shifts is clearly seen. The highest amplitude of the breather appears on the right-hand side of the curves in figure 5b. It is consistent with the theoretical prediction.

In the second experiment, we have chosen the parameters of the AB to be: $\phi = 0.1$, $a = 0.5$ cm and $\epsilon = 0.08$. The results of this experiment are displayed in figure 6. In this case, the $\varphi$-value is closer to zero. This means that the modulation period is higher and the peaks of the solution are well separated. The shape of each peak is close to the Peregrine breather profile. The amplitude amplification in this case is close to three. Similar AB-like dynamics, referred to as solitons on finite background, has been previously observed in water waves [44]. In addition to the effects related to the group velocity shifts, we can also note an asymmetry of the surface wave profiles due to higher order dispersion effects and to the mean flow. These are not taken into account in the NLS wave dynamics [45].

The amplitude amplifications predicted in theory and reached in the experiments above related to both KM solitons and ABs, are summarized in the table 1. Generally, there is better agreement with the NLS solutions with smaller wave steepness and smaller nonlinearity [46]. We can see that the maximal experimental values for the surface elevations are indeed a slightly higher than the ones predicted in theory. Again, this is due to the fact that the surface elevation described in terms of NLS solution on finite background can be considered as a free surface.
Figure 3. (a) Evolution of a KM soliton for $\varphi = 1.0$, $a = 0.5$ cm and $\varepsilon = 0.08$. The nine blue curves are measured by the set of equidistantly separated gauges. The soliton position in time shifts relative to the position at the previous gauge. (b) Comparison of the wave profile with maximum amplitude, measured 9 m from the wave maker (blue top curve), with the predicted theoretical second-order Stokes wave profile (red bottom curve), evaluated at the point of maximum soliton amplitude. (Online version in colour.)

wave. The measured waves are, in fact, bound waves [41]. They include, additionally, higher order Stokes harmonics. Depending on the steepness of the waves, these harmonics can be of the order of three and even higher. Such waves have higher crests and flatter troughs than a purely
Figure 4. (a) Evolution of a KM soliton for $\phi = 0.8$, $a = 0.5$ cm and $\varepsilon = 0.09$. The nine blue curves are measured by the set of equidistantly separated gauges. The soliton position in time shifts relative to the position at the previous gauge. (b) Comparison of the wave profile with maximum amplitude, measured 9 m from the wave maker (blue top curve), with the predicted theoretical second-order Stokes wave profile (red bottom curve), evaluated at the point of maximum soliton amplitude. (Online version in colour.)

linear and sinusoidal wave train. The amount of deviation depends on the nonlinearity, i.e. the steepness of the background waves.

Finally, we compare the semi-log angular frequency spectra of the observed KM soliton and AB at the points of maximal compression with the theoretical predictions. The experimental
Figure 5. (a) Evolution of an AB for $\phi = 0.3$, $a = 0.5$ cm and $\varepsilon = 0.09$. (b) Comparison of maximal wave profile, measured 9 m from the wave maker (blue top curve), with the predicted and theoretical second-order Stokes wave profiles (red bottom curve), evaluated at $x^* = 0$. (Online version in colour.)

spectra calculated from the data in figures 3b and 5b, using the discrete Fourier transform as described in [2], are shown in figure 7a, b, respectively. They are in agreement with the theoretical spectra presented in figure 7c, d, respectively. These curves are in qualitative agreement with the spectra, observed in optics [26,27] stressing the existing analogies between hydrodynamics and optics. For example, the central peak in figure 7a represents the soliton background while the
Figure 6. (a) Evolution of an AB for $\phi = 0.1$, $a = 0.5$ cm and $\varepsilon = 0.08$. (b) Comparison of maximal wave profile, measured 9 m from the wave maker (blue top curve), with the predicted and theoretical second-order Stokes wave profiles (red bottom curve), evaluated at $x^* = 0$. (Online version in colour.)

The quasi-discreteness of the spectra in figure 7b corresponds to the spatial periodicity of the AB. Slight asymmetry of the spectra are naturally explained by the asymmetry of the surface wave profiles [47], already mentioned. These spectra are the first of its kind in observations of nonlinear water waves. They may happen to be useful in practice as suggested in [48].
Figure 7. (a) The spectrum of the KM-soliton at the point of maximal surface elevation $x^* = 0$. The experimental data are the same as in figure 3b. Parameters are: $\varphi = 1.0$, $a = 0.5$ cm and $\varepsilon = 0.08$. (b) The spectrum of the AB at the point of maximal surface elevation $x^* = 0$. The experimental data are the same as in figure 5b. Parameters are: $\varphi = 0.3$, $a = 0.5$ cm and $\varepsilon = 0.09$. (c) Theoretical KM-soliton spectrum corresponding to (a). (d) Theoretical AB spectrum corresponding to (b). Theoretical spectra are computed with the same resolution, as the experimental data. (Online version in colour.)

Table 1. NLS theoretical and experimental maximal amplitude amplifications reached, which correspond to the parameters of the above-described observations of KMs and ABs.

<table>
<thead>
<tr>
<th>breather</th>
<th>$\varphi$</th>
<th>$\phi$-value</th>
<th>carrier amplitude $a$ (cm)</th>
<th>steepness $\varepsilon$</th>
<th>$\max(\eta_{\text{NLS}})$ (cm)</th>
<th>$\max(\eta_{\text{experiment}})$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>KM</td>
<td>1.0</td>
<td>0.5</td>
<td>0.08</td>
<td>4.08</td>
<td>4.77</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>0.5</td>
<td>0.09</td>
<td>3.67</td>
<td>3.71</td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>0.3</td>
<td>0.5</td>
<td>0.09</td>
<td>2.91</td>
<td>3.23</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.5</td>
<td>0.08</td>
<td>2.99</td>
<td>3.34</td>
<td></td>
</tr>
</tbody>
</table>

It would be indeed interesting to analyse experimental spatial spectra of periodic breathers as well. However, significantly more time-measurements along the water tank with very small interval-distance would be required to reconstruct the spatial dynamics of breather solutions.

4. Conclusion

In this work, we experimentally confirmed the existence of the families of time-periodic as well as space-periodic first-order breather solutions in water. These experiments complement our
previous observation of the Peregrine breather which is a limiting case of these families when each of the periods goes to infinity. They confirm, once again, the validity of modelling the water waves using the NLS equation. Accurate limits of periodic breather existence in water waves should be determined in framework of fully nonlinear evolution equations [49,50]. Our results may motivate similar experiments in plasmas, Bose–Einstein condensates and other media where the NLS plays the role of the governing equation.

Acknowledgements. A.C. would like to thank Odin Gramstad for interesting and fruitful discussions.

Funding statement. A.C. acknowledges support from the Region Bourgogne and the Isaac Newton Institute for Mathematical Sciences. B.K. acknowledges support from the French National Research Agency (ANR-12-B504-0011 OPTIROC). J.M.D. is supported by project ERC-2011-AdG-290562 MULTIWAVE. N.A. acknowledges the support of the Volkswagen Stiftung and partial support of the Australian Research Council (Discovery Project no. DP140100265) and also support from the Alexander von Humboldt Foundation.

References


39. Trulsen K, Dysthe KB. 1996 A modified nonlinear Schrödinger equation for broader
bandwidth gravity waves on deep water. Wave Motion 24, 281–289. (doi:10.1016/S0165-2125(96)00020-0)


