We investigate a control of the motion of localized structures (LSs) of light by means of delay feedback in the transverse section of a broad area nonlinear optical system. The delayed feedback is found to induce a spontaneous motion of a solitary LS that is stationary and stable in the absence of feedback. We focus our analysis on an experimentally relevant system, namely the vertical-cavity surface-emitting laser (VCSEL). We first present an experimental demonstration of the appearance of LSs in a 80 µm aperture VCSEL. Then, we theoretically investigate the self-mobility properties of the LSs in the presence of a time-delayed optical feedback and analyse the effect of the feedback phase and the carrier lifetime on the delay-induced spontaneous drift instability of these structures. We show that these two parameters affect strongly the space–time dynamics of two-dimensional LSs. We derive an analytical formula for the threshold associated with drift instability of LSs and a normal form equation describing the slow time evolution of the speed of the moving structure.
1. Introduction

Transverse localized structures (LSs), often called cavity solitons, were observed experimentally in vertical-cavity surface-emitting laser (VCSEL) [1,2]. Owing to the maturity of the semiconductor technology and the possible applications of LSs of light in all-optical delay lines [3] and logic gates [4], these structures have been the subject of active research in the field of nonlinear optics. Moreover, the fast response time of VCSELs makes them attractive devices for potential applications in all-optical control of light. LSs appear as solitary peaks or dips on a homogeneous background emitted by a nonlinear microresonator with a high Fresnel number. These structures consist of bright or dark pulses in the transverse plane orthogonal to the propagation axis. The spatial confinement of light was investigated more than two decades ago (for reviews, see [5–10]). When they are sufficiently far away from each other, localized peaks are independent and randomly distributed in space. However, when the distance between the peaks becomes small enough they start to interact via their oscillating, exponentially decaying tails. This interaction then leads to the formation of clusters [11–16]. The relative stability analysis of different LSs of closely packed localized peaks has been carried out in [17] near the optical bistability threshold. These stable LSs arise in a dissipative environment and belong to the class of dissipative structures found far from equilibrium. Transport processes such as diffraction, dispersion, or diffusion tend to restore spatial uniformity. On the contrary, nonlinearity has a tendency to amplify spatial inhomogeneities. The competition between the transport processes and nonlinearity leads in a dissipative environment to a self-organization phenomenon responsible for the formation of either extended or localized patterns. This is a universal phenomenon that was theoretically predicted first in the context of reaction–diffusion systems in the seminal papers of Turing [18] and Prigogine & Lefever [19].

Various mechanisms have proved to be responsible for the generation of LSs in VCSELs: coherent optical injection (holding beam, i.e. the part of the optical injection that is used to ensure bistability of the system) in combination with a narrow writing beam (the part of optical injection which is used to perform local switching between the lower branch of the bistability curve and the upper branch) [2,20–22], frequency selective feedback with [23] or without [24] a writing beam, saturable absorption [25], 'spatial translational coupling' introduced in [26,27], and others (see [28] for a review).

LSs are not necessarily stationary objects. They can start to drift spontaneously in the laser transverse section in the presence of saturable absorption [29]. In particular, in the case when the pump beam is axially symmetric LSs can move along the boundary on a circular trajectory [30]. It was shown that they can undergo a spontaneous motion due to thermal effects [31,32]. Delayed feedback control is a well-documented technique that has been applied to various spatially extended systems in optics, hydrodynamics, chemistry and biology. It has been demonstrated recently that a simple feedback loop provides a robust and controllable mechanism responsible for the motion of LSs and localized patterns [33–39]. These works demonstrated that, when the product of the delay time and the feedback rate exceeds some threshold value, LSs start to move in an arbitrary direction in the transverse section of the device. In these studies, the analysis was restricted to the specific case of nascent optical bistability described by the real Swift–Hohenberg equation with a real feedback term. More recently, analytical study supported by numerical simulations revealed the role of the phase of the delayed feedback and the carrier lifetime on the motion of cavity solitons in a broad-area VCSEL structure, driven by a coherent externally injected beam [39]. It was shown that for certain values of the feedback phase LSs can be destabilized via a drift bifurcation leading to a spontaneous motion of a solitary two-dimensional LS. Furthermore, the slower the carrier decay rate in the semiconductor medium, the higher the threshold associated with the motion of LSs.

The paper is organized as follows. In §2, we report on experimental evidence of the spontaneous formation of stationary LSs in a 80 μm diameter VCSEL biased above the lasing threshold and subjected to optical injection. Such LSs exhibit a bistability when the injected beam power and the VCSEL current are changed. The static LSs have been found in the absence...
of delayed feedback. In §3, we introduce a VCSEL model to study theoretically the effect of time-delayed optical feedback on these structures. In §4, we investigate the drift instability induced by delayed feedback. We conclude in §5.

2. Experimental observation of stationary localized structures in a medium-sized vertical-cavity surface-emitting laser

In recent years, a considerable amount of experimental work has been carried out on stationary LSs in VCSELs. They were observed in very broad (aperture $d > 100 \, \mu m$) [2], broad ($d \sim 80 \, \mu m$) [40] and medium-sized ($d \sim 40 \, \mu m$) [41] VCSELs. Here, we present experimental results obtained with a bottom-emitting InGaAs multiple quantum well VCSEL with $d = 80 \, \mu m$ and a threshold current of 42.5 mA at 20°C. The holding (injection) beam is provided by a commercial tunable semiconductor laser (Sacher Lasertechnik TEC100-0960-60 External Cavity Diode Laser), isolated from the rest of the set-up by an optical isolator (OFR IO5-TiS2-HP). The long-term electrical and temperature stability of this laser are less than 20 mA RMS and 0.05°C, respectively. A half-wave plate is used to adjust the linear polarization of the holding beam to be the same as that of the VCSEL. The injection beam power is tuned using a variable optical density filter. The detuning between the master laser and the VCSEL is defined as $\theta = \nu_{\text{inj}} - \nu_{\text{slave}}$, where $\nu_{\text{inj}}$ is the frequency of the injection beam and $\nu_{\text{slave}}$ is the frequency of the strongest peak in the spectrum of the stand-alone VCSEL. It is experimentally tuned by changing the wavelength of the injection beam. The beam waist $d_{\text{inj}}$ is defined as the diameter of the smallest circle in the plane of propagation of the injection beam containing half of the beam power when it encounters the VCSEL. The power of the source is monitored by a Newport 818-SL photodiode connected to a Newport 2832-C powermeter. The near field is recorded by imaging it on a CCD camera.

An example of stationary LSs is presented in figure 1 illustrating the process of spontaneous creation and annihilation of two-dimensional LSs. These experimental measurements have been performed when the VCSEL operated in an injection-locked regime. The injection beam waist and the detuning are fixed to $d_{\text{inj}} = 50 \, \mu m$ and $\theta = -146 \, GHz$. When increasing the injected beam power, a new LS appears at $P_{\text{inj}} = P_{\text{on}}^{\text{inj}}$ as shown in the insets of figure 1. This results in a slight jump of the total output power, as shown in the light-versus-current characteristics. The process of switching off is realized when decreasing the injection power: the recently created LS persists until $P_{\text{inj}} = P_{\text{off}}^{\text{inj}}$ with $P_{\text{off}}^{\text{inj}} < P_{\text{on}}^{\text{inj}}$, i.e. a hysteresis region exists with an additional LS turned either on or off. Figure 1b shows one-dimensional scans along the vertical lines indicated in the near-field images. Note that the line corresponding to the upper of these two scans intersects a pair of LSs, as can be seen from the respective near-field image.

Experimental investigation of the effect of the delayed feedback on the mobility properties of the LSs will be a subject of our future work. The delayed optical feedback will be implemented experimentally in a self-imaging external cavity configuration (e.g. fig. 1 in [33,34]). Using this configuration, the effect of diffraction in the external cavity on the feedback field will be minimized, which would allow implementation of the two-dimensional point-to-point optical feedback. As soon as we have provided experimental evidence of the existence of LSs in our 80 µm VCSELs (figure 1), adding such a type of delayed feedback to our experimental set-up is a straightforward task. However, in order to detect experimentally the spontaneous drift instability of LSs, it is very important to know how the feedback phase and the carrier relaxation rate affect the instability threshold. This problem is addressed theoretically in §3.

3. Model equations

The mean field model describing the space–time evolution of the electric field envelope $E$ and the carrier density $N$ in a VCSEL subjected to optical injection is given by the following set of
Figure 1. (a) Total output power as a function of injection power displaying bistability when a new LS appears. The insets show the near-field images of the upper and the lower branch of the hysteresis curve. (b) One-dimensional scans along the vertical lines indicated in the near-field images. (Online version in colour.)

dimensionless partial differential equations:
\[
\frac{\partial E}{\partial t} = -\left(\mu + i\theta\right)E + 2C(1 - i\alpha)(N - 1)E + E_i - \eta e^{i\phi}E(t - \tau) + i\nabla^2 E
\]  
(3.1)

and
\[
\frac{\partial N}{\partial t} = -\gamma [N - 1 + (N - 1)|E|^2 - d\nabla^2 N].
\]  
(3.2)

Here the parameter \(\alpha\) describes the linewidth enhancement factor, and \(\mu\) and \(\theta\) are the cavity decay rate and the cavity detuning parameter, respectively. The parameter \(E_i\) is the amplitude of the injected field, \(C\) is the bistability parameter, \(\gamma\) is the carrier decay rate, \(I\) is the injection current and \(d\) is the carrier diffusion coefficient. The diffraction of light and the diffusion of the carrier density are described by the terms \(i\nabla^2 E\) and \(d\nabla^2 N\), respectively, where \(\nabla^2\) is the Laplace operator acting in the transverse plane \((x, y)\). Below we consider the case when the laser is subjected to coherent delayed feedback from an external mirror. To minimize the effect of diffraction on the feedback field, we assume that the external cavity is self-imaging [33]. The feedback is characterized by the delay time \(\tau = 2L_{\text{ext}}/c\), the feedback rate \(\eta \geq 0\) and phase \(\phi\), where \(L_{\text{ext}}\) is the external cavity length, and \(c\) is the speed of light. The link between dimensionless and physical parameters is provided in [34]. Using the expression for the feedback rate \(\eta = (r^{1/2}(1 - R))/R^{1/2}\tau_{\text{in}}\) given in [42], where \(r\) (\(R\)) is the power reflectivity of the feedback (VCSEL top) mirror and \(\tau_{\text{in}}\) is the VCSEL cavity round trip time, we see that the necessary condition for the appearance of the soliton drift instability \(\eta \tau > 1\) [33] can be rewritten in the form \(r > R\tau_{\text{in}}^2/((1 - R)^2\tau^2)\). In particular, for \(R = 0.3\) and \(\tau = 20\tau_{\text{in}}\) the latter inequality becomes \(r > 1.5 \times 10^{-3}\).

4. Drift instability threshold

When the delayed feedback is absent, \(\eta = 0\), equations (3.1) and (3.2) are transformed into the well-known mean field model [43], which supports stable stationary patterns and LSs [2,41,44,45]. It was demonstrated recently that when the feedback rate \(\eta\) exceeds a certain threshold value, which is inversely proportional to the delay time \(\tau\), the LS starts to move in the transverse direction.
Examples of moving two-dimensional LSs are shown in figure 2. The single and the three moving peaks are obtained from numerical simulations of equations (3.1) and (3.2). The boundary conditions are periodic in both transverse dimensions.

In the case when the system is transversely isotropic, the velocity of the LS motion has an arbitrary direction. The self-induced motion of the LS is associated with a pitchfork bifurcation where the stationary LS loses stability and a branch of stable LSs uniformly moving with the velocity \( v = |v| \) bifurcates from the stationary LS branch. The bifurcation point can be obtained from the first-order expansion of the uniformly moving LS in power series of the small velocity \( v \).

Close to the pitchfork bifurcation point, this expansion reads as follows:

\[
E(x - vt, y) = E_0(x - vt, y) + vE_1(x - vt, y) + \cdots
\]

and

\[
N(x - vt, y) = N_0(x - vt, y) + vN_1(x - vt, y) + \cdots,
\]

where without loss of generality we assume that the LS moves along the \( x \)-axis on the \((x, y)\)-plane. Here \( E_0(x, y) = X_0(x, y) + iY_0(x, y) \) and \( N_0(x, y) \) describes the stationary axially symmetric LS profile, which corresponds to the time-independent solution of equations (3.1) and (3.2) with \( \tau = 0 \). Although formally this solution depends on the feedback parameters \( \eta \) and \( \phi \) we neglect this dependence assuming that the feedback rate is sufficiently small, \( \eta \ll 1 \). Substituting this expansion into equations (3.1) and (3.2) and collecting the first-order terms in small parameter \( v \), we obtain

\[
L \begin{pmatrix}
\text{Re}E_1 \\
\text{Im}E_1 \\
N_1
\end{pmatrix} = \begin{pmatrix}
\text{Re}[\partial_x E_0(1 - \eta \tau e^{i\phi})] \\
\text{Im}[\partial_x E_0(1 - \eta \tau e^{i\phi})] \\
\gamma^{-1}\partial_x N_0
\end{pmatrix},
\]

where the linear operator \( L \) is given by

\[
L = \begin{pmatrix}
\mu - 2C(N_0 - 1) & \nabla^2 - \theta - 2C\alpha(N_0 - 1) & -2C(X_0 + \alpha Y_0) \\
-\nabla^2 + \theta + 2C\alpha(N_0 - 1) & \mu - 2C(N_0 - 1) & -2C(Y_0 - \alpha X_0) \\
2(N_0 - 1)X_0 & 2(N_0 - 1)Y_0 & -d\nabla^2 + 1 + |E_0|^2
\end{pmatrix}.
\]

By applying the solvability condition to the right-hand side of equation (4.3), we obtain the drift instability threshold

\[
\eta \tau = \frac{1 + \gamma^{-1}(b/c)}{\sqrt{1 + (a/c)^2} \cos[\varphi + \arctan (a/c)]}
\]
The dependence of the critical feedback rate \( \eta \) corresponding to the drift instability threshold defined by equation (4.4) on the feedback phase \( \varphi \) and carrier relaxation rate \( \gamma \) is illustrated by figure 4. In this figure, the curves labelled by different numbers correspond to different values of \( \gamma \). Considering the fact that the feedback in equation (3.1) is introduced with the minus sign, we see that the drift instability takes place only for those feedback phases when the interference between the cavity field and the feedback field is destructive, i.e. when the \( \cos \) function in the denominator of the right-hand side of equation (4.4) is positive. On the contrary, when this interference is constructive the feedback has a stabilizing effect on the LS. Furthermore, the slower the carrier relaxation rate, the higher the drift instability threshold. Since the stationary LS solution does not depend on the carrier relaxation rate \( \gamma \), the coefficients \( a \) and \( b \) in the threshold condition (4.4) are also independent of \( \gamma \). Therefore, (4.4) gives an explicit dependence of the threshold feedback rate on the carrier relaxation rate. In particular, in the limit of very fast carrier response, \( \gamma \gg 1 \), and zero feedback phase, \( \varphi = 0 \), we recover from (4.4) the threshold condition \( \eta \tau = 1 \) which was obtained earlier for the LS drift instability induced by a delayed feedback in the real Swift–Hohenberg equation [33]. Note that at \( \gamma \to \infty, a \neq 0 \), and \( \varphi = -\arctan a \) the critical feedback rate appears to be smaller than that obtained for the real Swift–Hohenberg equation, \( \eta \tau = (1 + a^2)^{-1/2} < 1 \).
As demonstrated above, the bifurcation threshold responsible for self-induced drift of LS in the VCSEL transverse section is obtained by expanding the slowly moving localized solution in the small velocity \( v \), substituting this expansion into the model equations (3.1), (3.2) and matching the first-order terms in \( v \). In order to describe the slow evolution of the LS velocity slightly above the bifurcation threshold, one needs to perform a similar procedure with \( E = E_0(x - x_0(t), y) + \sum_{k=1}^{3} \epsilon^k E_k(x - x_0(t), y, t) + \cdots \) and \( N = N_0(x - x_0(t), y) + \sum_{k=1}^{3} \epsilon^k N_k(x - x_0(t), y, t) + \cdots \), where \( \frac{dx}{dt} = v(t) = \mathcal{O}(\epsilon) \), \( \frac{dv}{dt} = \mathcal{O}(\epsilon^3) \) and \( \epsilon \) is a small parameter characterizing the distance from the bifurcation point. Then, omitting detailed calculations, in the third order in \( \epsilon \), we obtain the normal form equation for the LS velocity

\[ \frac{p}{2} \frac{dv}{dt} = v(\delta q - \eta \tau^2 rv^2), \]  

(4.7)
where $\delta \eta$ is the deviation of the feedback rate from the bifurcation point. The coefficients $q$, $p$ and $r$ are given by $q = a \sin \phi + c \cos \phi$, $p = q + b$ and $r = f \sin \phi + g \cos \phi + O(\tau^{-1})$, respectively. Here $a$, $b$ and $c$ are defined by equation (4.5) and $f = \langle \psi_{\mathbf{1}}^{\dagger} \partial_{xx} Y_0 \rangle - \langle \psi_{\mathbf{2}}^{\dagger} \partial_{xx} X_0 \rangle$, $h = \langle \psi_{\mathbf{3}}^{\dagger} \partial_{xx} N_0 \rangle$, $g = \langle \psi_{\mathbf{1}}^{\dagger} \partial_{xx} X_0 \rangle + \langle \psi_{\mathbf{2}}^{\dagger} \partial_{xx} Y_0 \rangle$. The stationary LS velocity above the drift instability threshold is obtained by calculating the non-trivial steady state of equation (4.7), $v = \sqrt{\delta \eta} Q$, where the coefficient $Q = (1/\tau) \sqrt{q/(\eta)}$ determines how fast the LS velocity increases with the square root of the deviation from the critical feedback rate. The dependence of this coefficient on the feedback phase is illustrated in figure 5.

5. Conclusion

In the first part of the paper, we investigated experimentally the formation of transverse LSs of light in a medium-sized bottom-emitting InGaAs multiple quantum well VCSEL operated in an injection-locked regime. Creation and annihilation of a single LS have been demonstrated by changing the injection beam power. In the experimental part, the localized peaks were stationary since delayed feedback was not applied to the laser.

In the second part, we have analysed theoretically the effect of time-delayed feedback from an external mirror on the stability of transverse LSs in a broad area VCSEL. We have shown that depending on the phase of the feedback it can have either a destabilizing or stabilizing effect on the LSs. In particular, when the interference between the LS field and the feedback field is destructive, the LS can be destabilized via a pitchfork bifurcation, where a branch of uniformly moving LS bifurcates from the stationary one. We have calculated analytically the threshold value of the feedback rate corresponding to this bifurcation and demonstrated that the faster the carrier relaxation rate in the semiconductor medium, the lower the threshold of the spontaneous drift instability induced by the feedback. Finally, we have derived the normal form equation (4.7) governing the slow dynamics of the LS velocity. This generic destabilization mechanism is robust in one and two spatial dimensions and could be applied to a large class of far from equilibrium systems under time-delay control.

Our future work will be focused on the experimental investigation of the effect of delayed feedback on the spontaneous motion of LSs. This study would allow us to check the theoretical predictions of §4. We are also planning to investigate the role of local polarization dynamics in the formation of LSs in the transverse plane of the VCSEL. This would allow us to study the spontaneous motion of vector LSs with different polarizations, which is induced by delayed feedback.

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