Partial differential equation models in the socio-economic sciences

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Mathematical models based on partial differential equations (PDEs) have become an integral part of quantitative analysis in most branches of science and engineering, recently expanding also towards biomedicine and socio-economic sciences. The application of PDEs in the latter is a promising field, but widely quite open and leading to a variety of novel mathematical challenges. In this introductory article of the Theme Issue, we will provide an overview of the field and its recent boosting topics. Moreover, we will put the contributions to the Theme Issue in an appropriate perspective.

A model is by definition that in which nothing has to be changed, that which works perfectly; whereas reality, as we see clearly, does not work and constantly falls to pieces; so we must force it, more or less roughly, to assume the form of the model.

Italo Calvino, ‘The model of models’, in Mr. Palomar (1983)
1. Introduction

Partial differential equations (PDEs) have been used since the times of Newton and Leibniz to model physical phenomena. Famous examples are Maxwell’s formulation of the electrodynamical laws, the Boltzmann equation for rarified gases, Einstein’s general relativity theory and Schrödinger’s formulation of quantum mechanics. In the middle of the last century, PDE modelling began to be applied to certain biological processes, but only very recently was it realized that many socio-economic processes can be very successfully modelled by nonlinear PDEs.

PDE-based approaches are nowadays quite standard as pricing models in finance and insurance, always strongly related to stochastic differential equations as in the famous Black–Scholes equation. Instead PDE modelling of socio-economic processes has developed significantly in the last decade, not only by delivering new insights into qualitative and quantitative analysis of socio-economics but also by opening up a whole new range of fascinating mathematical problems, which require the development of new mathematical tools. This field may very well become one of the hot-spots of PDE centred modelling in the near future, taking its place next to physical, chemical and biological applications. Note that most challenges in the future world will have socio-economic origins and we need scientific-descriptive tools to understand this complexity.

The main idea of this Theme Issue is to present a fairly new field of PDE models in the socio-economic sciences to a broad range of applied mathematicians (and physicists), social scientists and economists representing various disciplines related to study of patterns of human behaviour. This research field has certainly been of great practical importance, namely to formalize certain areas of socio-economics and devise models with significant predictive power for pressing problems of the modern world. But also—and equally important—it has already enriched mathematics as such by presenting new and challenging PDE models to the analytical and numerical mathematics communities. We are convinced that the new research area of PDE models in the socio-economic sciences is one of the best recent success stories of cross-fertilization of totally diverse scientific fields. Consequently, we will also structure this introductory article by mathematical questions rather than specific applications. The latter will obviously appear on the way since they are driving the mathematical questions.

2. Kinetic models with non-physical interactions

Recently, ideas from statistical physics have started to permeate the socio-economic sciences, leading to mean-field and kinetic PDEs as the main tools for qualitative and quantitative research. Their applicability relies on the fact that socio-economic processes are often governed—similar to statistical physics—by the interaction of large ‘particle’ systems, where the particles are human beings (inter)acting in socio-economic scenarios. The natural step is to go from those particle models to kinetic and further macroscopic PDE models. In those, several novel issues appear due to the obvious differences compared to classical models in physics. In particular, the interactions in socio-economics can have missing conservation properties or be of unusual long-range nature, which leads to novel mathematical challenges, e.g. non-standard collision invariants to be determined.

A problem that has attracted much attention is the evolution of wealth distributions and the aim to explain experimentally observed Pareto tails in large-time limit. The models are usually based on exchange of goods and money in binary interactions, with several different approaches to the details of the modelling, all leading to different macroscopic models and stationary distributions [1–4]. In this issue, Pareschi & Toscani [5] consider the evolution of knowledge in addition to the market interactions. This leads to a second state variable and finally to a multi-dimensional non-local Fokker–Planck equation. Degond et al. [6] take a different approach to this problem by considering non-conservative game-type interactions between the individual agents. An usual macroscopic model is derived in a high-frequency trading limit, leading to several challenging open questions for the mathematical analysis along the way.
Another problem of social interaction that has received considerable attention is opinion formation and in particular the way a group reaches consensus (e.g. [7]). A major idea is that in the way of random meetings (discussions) of pairs they tend to agree (assimilate their opinion) if they are already close or rather strengthen their opinion in a debate with opposite previous opinions. This has led to novel kinetic and Fokker–Planck-type equations in domains of opinions [8,9].

A recent trend is to incorporate the heterogeneity of groups into models, in particular the existence of certain opinion leaders that are trusted more than others. Besides the fact that now systems have to be considered instead of single kinetic equations, the interactions naturally become asymmetric, which complicates the mathematical analysis (e.g. [10]). In this issue, Albi et al. [11] deal with such a problem, considering Boltzmann-type interactions with leaders also at the microscopic level, and explaining how they can control opinion from the arising macroscopic models.

3. Spatial pattern formation by consensus and herding

In the last few decades, motion of socially interacting individuals has received a lot of attention. This ranges from animals (insect swarms, fish schools and bird flocks) to human crowds. Various interesting mathematical models and questions arise from the fact that those animals adapt their spatial motion due to some consensus reached with other individuals locally around them, e.g. birds adapt their velocity to follow the flock without explicitly knowing a determined group velocity. Hence, this is a natural paradigm of self-organized formation of macroscopic patterns from microscopic interactions. The mathematical analysis of flocking has become popular after Cucker & Smale [12] introduced a simple model that is accessible to rigorous arguments. In the macroscopic setting, flocking means that—at least in some spatial region—the density looks like a Dirac delta distribution in velocity space, hence analysis of flocking is naturally linked to blow-up phenomena for PDEs (e.g. [13,14]).

In the last few years, a variety of novel techniques have been developed to treat more realistic models (e.g. [15,16]) and better understand phase transitions, i.e. critical values of certain parameters needed to successfully form flocks [17–19]. In this issue, Tadmor & Tan [20] follow those lines by considering a hydrodynamic model with non-local alignment interaction. There the global existence of solutions respectively finite-time blow-up is shown under conditions comparing the local and global variations of the initial velocity.

Another interesting example in this line is non-local aggregation [21–23], e.g. appearing in animal swarms looking for a density, but also in macroscopic models of consensus formation. This yields nonlinear integro-differential equations and has created novel insights into the interplay of the local differential operators and non-local integral operators in the equations. A particularly interesting instance is the coupling of aggregation and repulsion, where the first one is usually on a large range (to stay in the overall group) and the second acts on a long range (to keep some space needed for each agent). A variety of novel developments has been motivated by such models [24–27], in particular the large-time behaviour and analysis of stationary patterns have received considerable attention [28–30]. In this issue, Carrillo et al. [31] analyse such steady states in the case when both the repulsive as well as the aggregative force are modelled by integral operators. The existence of stationary solutions is rephrased as a variational problem for a non-local energy to be minimized over a space of non-negative measures. Of particular interest is the existence of non-concentrated measures, which is shown if the repulsion exhibits certain singularity at the zero distance.

4. Games and optimal control

Equilibria of optimization problems and games have a long tradition in socio-economic modelling (e.g. [32,33] and references therein). This has also driven some developments in differential games, related to ordinary differential equations in the time-continuous setting [34]. The standard approach is based on considering a few representative rational agents and to derive models for
those. Clearly, the basic assumptions of equal and highly rational agents are highly questionable in processes involving the behaviour of a large number of humans (often with unequal access to information); moreover, models based on those often fail to explain effects such as high volatilities observed in reality. Owing to this issue, models of large numbers of interacting agents have recently been considered, leading to optimization problems or games for large numbers of interacting individuals.

The most prominent examples are the related fields of optimal transport, mean-field games and mean-field optimal control. In particular, the idea of mean-field games introduced in the seminal paper by Lasry & Lions [35] and in parallel in the engineering literature [36] has gained enormous interest in the last few years, both with respect to applications (e.g. [36–38]) as well as due to the new kinds of mathematical and computational challenges that such games introduce (e.g. [37,39–41]). A mean-field game can be understood as the generalization of a (stochastic) differential game, where the individuals interact via some mean-field quantities accessible to every agent. A key assumption is that mean-field variable is known at the level each individual optimizes its cost, which allows one to introduce the mean-field interaction in the final optimality conditions. In the simplest case of equal type agents, a mean-field game can be written as a stochastic optimal control problem for the quantity of interest, where as well as as the coefficients in the stochastic differential equation may depend on a mean-field variable (averaging all individual quantities). Since the mean-field variable can be rewritten in dependence of the overall density function it becomes possible to derive a system of PDEs for the evolution of the overall density as well as a distributed adjoint variable. As usual in optimal control, the adjoint variable is supplemented by a terminal instead of an initial condition, which implies that the mean-field game becomes a boundary value problem for a coupled system in space–time instead of a system of evolution equations forward in time. The latter observation is crucial for understanding the analytical and computational challenges. Standard techniques for evolution equations are not applicable in the analysis of mean-field games, e.g. the question of local-in-time existence cannot even be posed reasonably. On the other hand, variational approaches suffer from missing coercivity, convexity and the fact that as usual in differential games a fixed-point operator needs to be formulated on top of the optimization problem. Novel developments coupling variational principles and viscosity solution techniques for the adjoint problem can be found in recent literature [42–45] as well as solutions with advanced probabilistic arguments [46,47]. With respect to the numerical solution, there is no simple forward integration, but the huge coupled systems in space–time need to be solved. Moreover, the coupling of the equation for the density (usually a Fokker–Planck or transport equation) with the adjoint equation (a Hamilton–Jacobi equation) needs particular care with respect to discretization. The adjoint structure of the equations has also led to novel insights beyond mean-field games, embraced under the term adjoint methods for Hamilton–Jacobi equations [48–50].

In this Theme Issue, two papers deal explicitly with mean-field games. Achdou et al. [51] provide an overview of various novel PDE models in macroeconomics, which are mainly derived in a mean-field game framework. The diversity of applications, including market shocks, wealth distributions and knowledge growth, demonstrates the relevance of the mean-field game approach as a key future tool in economics. On the other hand, most of the presented models introduce novel structures far beyond the cases currently accessible to mathematical analysis. The existence, uniqueness and structural properties of solutions naturally come up as a future challenge. Gomes et al. [52] discuss finite-state mean-field games and their applications to economics. The finite-state games are characterized by a switching of agents between a finite number of possible states, which in the mean-field setting leads to systems of coupled hyperbolic equations rather than parabolic ones. This leads to challenges in the analysis as well as in the numerical solution, for which the paper also presents novel approaches.

A slightly different approach is mean-field optimal control or control of McKean–Vlasov interacting systems, where one explicitly models a large number of agents and their interaction. Then (stochastic) differential games or optimal control problems are formulated for the agent
models, and from the optimality system a mean-field model is derived in the limit of $N \to \infty$ similar to classical interacting particle models [53]. The differences from mean-field games arising in such are further discussed in [54]. In Fornasier et al. [55], a particular instance of mean-field optimal control by sparse controls is discussed. The major idea is to find optimal controls for such systems with as few points to act as possible, marrying ideas from mean-field control with recent advances on sparsity in imaging and compressed sensing. The paper by Blanchet & Carlier [56] focuses on games with a large number of agents and discusses the connection of Nash and Cournot–Nash in the limit as the number of agents tends to infinity. In particular, they show the use of optimal transport theory (of Monge–Kantorovich type) in the analysis of this limit process.

Finally, we mention that also the above-mentioned paper [6] heavily uses mean-field game and Nash equilibria in the derivation of a kinetic wealth distribution model, thus linking these two topics in a quite novel way.

5. Non-standard free boundary problems

In classical literature, socio-economic, free boundary problems mainly appear in the context of optimal stopping of diffusions, i.e. as obstacle problems for parabolic PDEs. In their seminal paper, Lasry & Lions [35] on mean-field games also presented a parabolic free boundary problem modelling the evolution of the price in an economic market with a large number of agents, where a single good is traded. This one-dimensional free boundary problem is—not a mean field game, but it exhibits very remarkable mathematical features which make it difficult to treat by standard free boundary techniques. In fact, the Lasry–Lions price formation model has attracted a lot of attention so far: preliminary analytical results were given in [57–59] and a rather complete existence, uniqueness, regularity and long-time asymptotic analysis based on a geometric method can be found in [60,61]. A derivation from a mesoscopic Boltzmann-type system using a scaling limit technique (high trading frequency) can be found in [62] and a hyperbolic-type scaling limit in [63]. Extensions incorporating stochastically fluctuating markets are in progress [64]; however, we remark that significant further steps have to be taken in terms of modelling (e.g. incorporating utility functions) before applications to realistic economic markets can be tackled.

In this issue, Berestycki et al. [65] consider a topic which has a huge importance in the world of finance: the appearance and time evolution of financial bubbles, which are a major scare for and even a threat to the world economy. The authors devise a highly nonlinear and non-local free boundary problem of obstacle type, whose solution is that part of an asset price which can be attributed to speculation. The size of the solution then quantifies the financial bubble.

6. Inverse problems

As in other branches of sciences, inverse problems naturally arise for PDE models also in socioeconomics, as not all parameters can be determined beforehand, but need to be determined from data. Alternatively, PDE-based models offer novel opportunities for analysing data traditionally treated with all-purpose statistical models, which could not take into account the underlying mechanisms of those processes.

In [66] an inverse problem in geographical economics is considered, using the spatial Solow model as a representative example. This leads to an inverse problem for a parabolic PDE, where the parameter to be identified is the nonlinearity in the equation itself.

Woodworth et al. [67] introduce another aspect of PDE techniques, namely in spatial density estimation applied to data of criminal activity. Here, PDE-based and variational techniques recently developed in image analysis are adapted and further developed according to the special needs of the application.
References


