By way of introduction, the general invariant integral (GI) based on the energy conservation law is presented, with mention of cosmic, gravitational, mass, elastic, thermal and electromagnetic energy of matter application to demonstrate the approach, including Coulomb’s Law generalized for moving electric charges, Newton’s Law generalized for coupled gravitational/cosmic field, the new Archimedes’ Law accounting for gravitational and surface energy, and others. Then using this approach the temperature track behind a moving crack is found, and the coupling of elastic and thermal energies is set up in fracturing. For porous materials saturated with a fluid or gas, the notion of binary continuum is used to introduce the corresponding GIs. As applied to the horizontal drilling and fracturing of boreholes, the field of pressure and flow rate as well as the fluid output from both a horizontal borehole and a fracture are derived in the fluid extraction regime. The theory of fracking in shale gas reservoirs is suggested for three basic regimes of the drill mud permeation, with calculating the shape and volume of the local region of the multiply fractured rock in terms of the pressures of rock, drill mud and shale gas.

1. Introduction

The following discourse is necessarily limited to a disposition of the author’s own work. Let us consider physical fields that are stationary in the Cartesian frame of coordinates $Ox_1x_2x_3$. It is assumed that the mathematical image of the matter under study is represented by the field and mass energy and by basic parameters of state. We confine ourselves by the combined field of cosmic, gravitational, thermal,
elastic and electromagnetic fields in dielectrics. The field parameters under consideration are as follows: $\varphi(x_1, x_2, x_3)$, the potential of coupled gravitational/cosmic field (per unit mass); $U(\varepsilon_{ij}, D_i, b_i, T)$, the field energy per unit volume; $T(x_1, x_2, x_3)$, temperature; $v_i(x_1, x_2, x_3)$, the velocity components ($i, j = 1, 2, 3$); $u_{ij}(x_1, x_2, x_3)$, the distortions; $\varepsilon_{ij}(x_1, x_2, x_3)$, the strains; $\sigma_{ij}(x_1, x_2, x_3)$, the stresses; $E_i, D_i, H_i$ and $B_i$, the components of the vectors of electromagnetic field (some functions of $x_1, x_2$ and $x_3$); and $K(v)$, the energy density of a moving mass (the kinetic energy plus the Einstein energy when $v \ll c$)

$$K = \frac{\rho_m c^2}{\sqrt{1 - v^2/c^2}} \left( K = \rho_m c^2 + \frac{1}{2} \rho_m v^2 \quad \text{when } v \ll c \right); \quad (1.1)$$

here $\rho_m$ is the mass density at rest; $c$, the speed of light in vacuum; and $v$, the value of the local velocity of matter.

The state equations of elastic dielectrics are

$$E_i = \frac{\partial U}{\partial D_i}, \quad H_i = \frac{\partial U}{\partial B_i} \quad \text{and} \quad \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}. \quad (1.2)$$

In linear approximation, function $U(\varepsilon_{ij}, D_i, b_i, T)$ is a sum of some quadratic functions of $\varepsilon_{ij}, D_i$ and $b_i$ plus term $K_0 \alpha T \varepsilon_{ij}(T - T_0)$, where $K_0$ is the bulk modulus, $\alpha_T$ is the coefficient of thermal expansion and $T_0$ is the reference temperature.

### 2. General invariant integral

The law of energy conservation for the combined field under consideration can be written in the shape of the general invariant integral (GI) as follows [1,2]

$$\Gamma_k = \oint_{\Omega} \left[ (8\pi G)^{-1} \left( \varphi, \rho \varphi, \varphi_{ij} - 2 \varphi_i \varepsilon_{ij} \varphi_{k} \right) + \kappa \varphi \varphi + K \rho_m c H \varepsilon_{nk} + U \varepsilon_{nk} \right. \left. + \frac{K}{v^2} T_i n_i \varphi_k - \varepsilon_{ij} n_j \varepsilon_{ik} + E_i D_i \varepsilon_{nk} + H_i B_i \varepsilon_{nk} - D_i n_i E_k - B_i n_i H_k \right] d\Omega \quad (i, j, k = 1, 2, 3); \quad (2.1)$$

here $\Gamma_k$ is the external energy spent to move the matter inside surface $\Omega$ on unit length along axis $x_k$; $\Omega$, an arbitrary closed surface of integration; $G$, the gravitational constant; $\Lambda$, the cosmological constant; $\varepsilon_{nk}$ and $K$, the specific heat and thermal conductivity of the matter. In physical terms, quantity $\Gamma(\Gamma_1, \Gamma_2, \Gamma_3)$ called the driving force can be a real equivalent force upon the matter inside $\Omega$, or a configuration force, or an energy loss vector [1]. It measures the exchange of energy between the field at hand and other coupled fields, which takes place at field singularities.

If there are no field singularities inside $\Omega$, then $\Gamma_k = 0$, and equation (1.2) as well as Maxwell’s and thermoelasticity equations, which can be derived from equation (2.1) by using the divergence theorem, are valid at every point inside $\Omega$ [1]. If there is a field singularity inside $\Omega$, then $\Gamma_k \neq 0$, and $\Gamma$ is called the singularity driving force. By field singularities, the coupling binds different forms of energy.

The common field singularities are point charges, linear currents, point holes, the front of cracks and dislocations, point inclusions, vacancies, concentrated forces, moments and torques, etc. When using the procedure of the $\Gamma$-integration for divergent invariant integrals, the GI is very effective to calculate the force upon a field singularity, and so to get any physical law of the interaction forces [1]. In a sense, based on the mass–energy ($E_a = M_a c^2$) and space–time ($L_a = c T_a$) dualisms, mass and momentum are some forms of energy which, under common conditions, are independent of other forms of energy [1]. Here, $M_a, E_a, L_a$ and $T_a$ are characteristic mass, energy, length and time, respectively.

Let us provide some examples of the application of the GI in equation (2.1).
(a) Newton’s Law of inertia

Suppose a rigid body moves along axis $x_1$ at some variable velocity $v = v(x_1)$. In this case, from equations (2.1) and (1.1) we find

$$\Gamma_1 = \oint_{\Omega} K_{n1} \, d\Omega = \int_V K_{11} \, dV = \int_V \frac{1}{v} \frac{dK}{dv} \, dv = \frac{d}{dt} \left( \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \right); \quad (2.2)$$

Here $t$ is time; $V$ and $\Omega$, the volume and surface of the body; $m_0$, its mass at rest; and $\Gamma_1$, the force upon the body. In equation (2.2), we used the following equality

$$\frac{1}{v} \frac{dK}{dv} \, dv = \frac{1}{c^2} \frac{d}{dt} (v K).$$

The original Newton’s Law follows from equation (2.2) when $v \ll c$. Some basic invariant integrals of relativistic physics were presented in [3].

(b) Generalized Coulomb’s Law for moving electric charges

Suppose two point electric charges $q_1$ and $q_2$ move along axis $x_1$ at constant speed $v$ in their own electromagnetic field. In this case, from equation (2.1), it follows that the driving force upon the back charge is equal to [1,4]

$$\Gamma_1 = \frac{q_1 q_2}{\varepsilon_0 R^2} \left( \frac{v^2}{a^2} \right) - \frac{1}{1 - v^2/c^2}; \quad (2.3)$$

Here $R$ is the distance between the charges in the proper coordinate frame; $\varepsilon_0$, the dielectric constant and $a$, the speed of light in the matter ($a < c$).

At $v < a$, the opposite force of same value acts upon the front charge. At $a < v < c$, the force upon the front charge equals zero, and the force upon the back charge changes its sign so that the driving force attracts the back charge to the front charge of same sign. These laws allowed us to explain some unusual features of the fracturing of various materials subject to the radiation of powerful relativistic electron beams [1,4]. When $v = 0$, equation (2.3) is Coulomb’s Law.

(c) Generalized Newton’s Law of gravitation

Suppose two point masses $m_1$ and $m_2$ are on axis $x_1$ at some distance $R$, one from the other, in their own coupled gravitational/cosmic field described by potential $\varphi$. In this case, from equation (2.1), it follows that the force upon mass $m_1$ is equal to [1,5]

$$\Gamma_1 = \frac{G m_1 m_2}{R^2} - \frac{4 \pi}{3} G \Lambda m_1 R; \quad (2.4)$$

Here, the first term describes the mass gravitation, and the second term the cosmic repulsion as a natural property of space. When $\Lambda = 0$, we arrive at Newton’s Law of gravitation.

In the scale of solar system, the second term is $10^{23}$ times less than the first term; in the galactic scale, the second term is $10^3$ times less than the first term and in the scale of super-cluster of galaxies, the repulsion becomes essential. The cosmic component of the field is a carrier of negative mass–energy uniformly distributed in space (there are about 0.01 g of this substance in the Earth). The law in equation (2.4) allowed us to explain the accelerated expansion of our Universe and the singular density of matter at the centre of galaxies [1,5]. From equation (2.4), it follows that the orbital speed of stars in spiral galaxies is constant and equal to $(G k_G)^{1/2} \approx 250 \text{ km s}^{-1}$, where $k_G$ is the galactic constant that equals $10^{21} \text{ kg m}^{-1}$ [1,5]. The extrapolation of equation (2.4) leads to the conclusion that our Universe has a finite size and volume of about $10^{78} \text{ m}^3$, and its total mass–energy equals zero so that it represents a gigantic fluctuation that came from nothing [1,5].
(d) Generalized Archimedes’s Law

Using the GI of hydrostatics the vertical force upon a body lying on the horizontal surface of a heavy fluid was found to be equal to [2]

\[ \Gamma = \Gamma_A \pm \gamma \int \frac{\sin(\theta - \alpha)}{\sin \theta} \, ds; \]

(2.5)

here \( \Gamma_A \) is Archimedes’ force; \( \gamma \), the free surface energy of fluid per unit surface; \( ds \), the element of the wetting contour of integration; \( \alpha \), the edge angle of non-wetting; and \( \theta \), the angle between the body surface and horizontal plane in the normal cross-section of the wetting contour. Sign plus is valid for hydrophobic fluids and sign minus for hydrophilic ones.

(e) Dynamic cracks

Suppose an open mode crack front \( x_1 = x_2 = 0 \) moves along axis \( x_1 \) at a constant speed \( v < c_T \) in an elastic isotropic homogeneous material. In this case, from equation (2.1), it follows that the crack-driving force is equal to [1,6,7]

\[ \Gamma_1 = \int \left[ (K(v) + U)n_1 - \sigma_{ij n_j u_{i,1}} \right] \, d\Omega = \frac{(1 + \delta) \nu T^2 K_1^2 \sqrt{1 - \nu T^2}}{8 E R_A} \]

\[ \times \left( R_A = \sqrt{(1 - \nu T^2)(1 - \nu T^2)} - \left( 1 - \frac{1}{2} \nu T^2 \right)^2, \quad \nu_T = \frac{v}{c_T}, \quad \nu_T = \frac{v}{c_L}, \quad c_T^2 = 1 - 2\delta \right); \]

(2.6)

here \( K_1 \) is the stress intensity factor; \( E \) and \( \delta \), Young’s modulus and Poisson’s ratio; \( c_L \) and \( c_T \), the longitudinal and transverse velocities of elastic waves.

From equation (2.6), it follows that \( \Gamma_1 = K_1^2 E^{-1}(1 - \delta^2) \) when \( v = 0 \), which is Irwin’s equation. The crack-driving force was also found for inhomogeneous anisotropic elastic solids and for various structures of solids, shells, plates and membranes [1,2]. Many other singularities of elastic field were studied in [1,2].

It is appropriate to mention here that the early forerunner of the GI in equation (2.1) was Eshelby’s path-independent integral [8] used in [6].

3. Temperature track behind a moving crack or dislocation

Let us study the local stationary temperature field near the front of arbitrary cracks or dislocations. In this case, the field does not depend on \( x_3 \), and a crack or dislocation in an elastic solid is at \( x_2 = 0, x_1 < 0 \) so that its front in the coordinate frame \( Ox_1 x_2 \) moves along \( x_1 \) with respect to the solid. The value of force \( \Gamma_1 \) driving a crack or dislocation turns into heat because the losses of energy for acoustic and electromagnetic radiation, for latent residual stresses and for surface energy, are negligibly small. The work spent on local plastic deformations turns into heat. And so, the moving front of a crack or dislocation is a heat source.

Driving force \( \Gamma_1 \) in equation (2.6) can be calculated from the local elastic field of stresses \( \sigma_{ij}(x_1, x_2) \) and displacements \( u_i(x_1, x_2) \) near the crack front using the following formula [1]:

\[ \Gamma_1 = \frac{\pi}{2} \lim_{\delta \to 0, j = 1, 2, 3} \left[ \sigma_{2j}^{0} B_{j} \right]; \]

(3.1)

This equation is valid for dynamic and static cracks in arbitrary anisotropic linearly elastic materials and for interface cracks. Besides, it is valid for nonlinear power-law hardening incompressible materials, with replacing coefficient \( \pi/2 \) by another coefficient dependent of power [1]. For a dislocation, equation \( \Gamma_1 = \sigma_2^0 B_j \) is valid, where \( B_j \) is the displacement discontinuity and \( \sigma_2^0 \) are the stresses at the front location without the dislocation.

And so, we come to the problem of a linear source of heat \( \Gamma_1 \) moving at constant speed \( v_1 = v \) along axis \( x_1 \). From equation (2.1), it follows that the corresponding GI has the following form in
this case
\[ \Gamma_1 = \oint \left( \rho_m c_H T n_1 + \frac{k_T}{v} T_j n_j \right) \, d\Omega; \quad (3.2) \]
here, \( T(x_1, x_2) \) is the temperature increase owing to the heat source at point \( O \), and \( \Gamma_1 \) is the value of the force driving a crack or dislocation.

If there are no heat sources inside the integration contour, then \( \Gamma_1 = 0 \), so that applying the divergence theorem provides the following equation which is valid at all regular points of the stationary temperature field:
\[ k_T (T_{,11} + T_{,22}) = -\rho_m c_H v T_{,1}; \quad (3.3) \]
A similar equation emerged in a more complicated problem of heat/mass transfer in a fluid flow past a cylinder of an arbitrary cross-section [9]. The solution of equation (3.3), which is singular at point \( O \), vanishes at infinity, and is an even function of \( x_2 \), has the following form [9]:
\[ T = A \lambda^{-2} e^{\lambda x_1} K_0(\lambda r) \left( r = \sqrt{x_1^2 + x_2^2}, \lambda = \frac{\nu \rho m c_H}{2 k_T} \right); \quad (3.4) \]
here, \( A \) is a constant to be found, and \( K_0(\lambda r) \) is the modified Bessel function which has the following asymptotes
\[ K_0(\lambda r) \to -\ln \frac{\lambda r}{2}, \quad \text{when} \, \lambda r \to 0 \quad \text{and} \quad K_0(\lambda r) \to \sqrt{\frac{\pi}{2\lambda r}} e^{-\lambda r}, \quad \text{when} \, \lambda r \to \infty. \quad (3.5) \]
Let us substitute function \( T(x_1, x_2) \) in equation (3.2) by its asymptotic value for small \( \lambda r \), see equations (3.5) and (3.4). By taking a circle of infinitely small radius as the integration contour in equation (3.2) and by calculating the integral, we get
\[ A = \frac{\nu \lambda^2}{2 \pi k_T} \Gamma_1. \quad (3.6) \]
And so, the local temperature field produced by a moving crack or dislocation is
\[ T = \frac{\nu \Gamma_1}{2 \pi k_T} e^{\lambda x_1} K_0(\lambda r) \left( r = \sqrt{x_1^2 + x_2^2}, \lambda = \frac{\rho_m \nu c_H}{2 k_T} \right). \quad (3.7) \]
The temperature has a logarithmic singularity at the front of a moving crack or dislocation, with intensity, in the case of cracks, being directly proportional to the square of fracture toughness, see equation (2.6).

This implies that the crack can grow owing to the fluidization or vapourization of an infinitely small amount of the material at the crack tip. It provides an alternative to Griffith’s view on the fracturing as a reversible exchange of elastic and surface energies. The irreversible exchange of elastic and thermal energies is a better choice. The growth of a through crack in a thin shell or membrane at low tensile loads owing to a heat source at the crack tip produced by a laser beam is an example of cooperating effects of thermal and elastic energy in fracturing.

4. Fluid/gas flow in porous materials: binary continuum

Let us model a stationary process of the flow of viscous fluid or gas in a porous material by a binary continuum so that two different continua are assumed to be at each point of space. One of them is an elastic solid characterized by the following GIs
\[ \Gamma_k = \oint (Un_k - \sigma_{ij} n_i \mu_{j,k}) \, d\Omega, \quad \oint (\sigma_{ij} n_j + \epsilon_p p n_i) \, d\Omega = 0, \quad (4.1) \]
and the other continuum is a fluid or gas flow characterized by the following GIs
\[ \Gamma_k = \oint (f_{ki} n_i - \rho_i v_i n_i v_k) \, d\Omega, \quad \oint \rho_i v_i n_i \, d\Omega = 0 \quad \text{and} \quad \oint (f_{ki} n_i - \epsilon_p p n_k) \, d\Omega = 0 \quad (4.2) \]
and by the following state equations for fluid and gas
\[ f_{ij} = -p \delta_{ij} + \eta_i (v_{ij} + v_{ji}), \quad p \rho_i^T = C_p. \quad (4.3) \]
here \( e_p \) is the effective porosity; \( \eta_f \), the dynamic fluid/gas viscosity; \( v_i \), the fluid/gas flow rate (real velocity equals \( v_i/e_p \); \( \rho_f, p \) and \( f_{ij} \), the density, pressure and stresses in fluid/gas continuum; \( \chi \) and \( C_p \), constants characterizing the polytropic process for gas.

In equations (4.1) and (4.2), the value of \( \Gamma_k \) is the same owing to the interconnection of both continua. The term \( e_p \rho_l \) of volume force follows from the tubular model of porous materials. The effective porosity accounts only for the volume of interconnected pores where the fluid flows. At regular points, the equations of the theory of elasticity and fluid or gas dynamics can be obtained from equations (4.1) to (4.3) by means of the divergence theorem \[1\]. We study some problems arising from the horizontal drilling of porous rocks in the following section.

(a) Horizontal borehole in oil deposit: stationary extraction regime

Let axis \( x_3 \) coincide with the axis of a vertical borehole so that plane \( x_1 x_2 \) is parallel to the day surface. Suppose there is also a horizontal cylindrical borehole along axis \( x_1 = x \) issuing from the vertical borehole. Let us use the cylindrical coordinate frame \( O x r \) where \( r^2 = x_2^2 + x_3^2 \) and point \( O \) is the issue of the horizontal borehole (HB). We assume that a HB of radius \( r_0 \) is embedded inside a fluid deposit whose size is much greater than length \( l_H \) of the HB.

The porous rock is subject to stress \( \sigma_{33} = -w_r \) which is equal to the weight of higher rocks per unit square and to stresses \( \sigma_{11} = \sigma_{22} = -\delta_T w_r \) where lateral thrust coefficient \( \delta_T \) equals \( \delta/(1 - \delta) \) in terms of Poisson’s ratio \( \delta \), in the plane-strain model of rock structure. Since \( 1 \geq \delta_T \geq 0 \), fracking wins the best advantage from horizontal drilling because fractures in rocks tend to grow along planes which are perpendicular to the day surface.

Let us find the fluid flow field ignoring elasticity of the porous medium. Luckily, in this three-dimensional problem there are two small dimensionless parameters \( \lambda_1 \) and \( \lambda_2 \)

\[
\lambda_1 = \frac{r_0}{l_H} \quad \text{and} \quad \lambda_2 = \frac{k_p}{r_0}, \quad (\lambda_1 \ll 1, \ \lambda_2 \ll 1); \quad (4.4)
\]

here, \( k_p \) is the permeability of the porous medium. Parameter \( \lambda_2 \) is small, and the fluid transport through a HB is much faster than through the porous rock.

These small parameters signal that there is a boundary layer in the domain \( 0 < x < l_H, \ r_0 < r < r_s \), where \( r_s \) is the thickness of the boundary layer \[1,10\]. Calculating the GI in equation (4.2) over the surface of this boundary layer, the fluid pressure can be expressed as follows \[1,10\]:

\[
p = P_B(x) + \frac{p_\infty - P_B(x)}{\ln(r_s/r_0)} \ln \frac{r}{r_0} \left( v_r = -\frac{k_p}{\eta_f} \frac{\partial p}{\partial r}, \quad v_x = -\frac{k_p}{\eta_f} \frac{\partial p}{\partial x} \right); \quad (4.5)
\]

here \( v_r \) and \( v_x \) are the fluid flow rates; \( p_\infty \), the initial pressure in the deposit and \( P_B(x) \), the pressure in the HB.

From here, we arrive at the following ordinary differential equations

\[
\frac{dV_B}{dx} = 2\pi r_0 q_B, \quad V_B(x) = \frac{\pi r_0^4}{8\eta_f} \frac{dP_B}{dx} \quad \text{and} \quad q_B(x) = \frac{k_p}{\eta_f} \cdot \frac{p_\infty - P_B(x)}{r_0 \ln(r_s/r_0)}; \quad (4.6)
\]

here \( V_B(x) \) is the fluid flow rate through the HB cross-section and \( q_B(x) \), the inflow rate of fluid into the HB.

The solution of equation (4.6) can be written in the following form

\[
P_B = p_\infty - \frac{p_\infty - p_b}{\ln(l_H/l_1)} sh(\frac{l_H - x}{l}), \quad q_B = \frac{k_p}{r_0 \eta_f} \cdot \frac{p_\infty - p_b}{\ln(r_s/r_0)} \cdot \frac{1}{sh(l_H/l_1)} \frac{l_H - x}{l} \quad \text{and} \quad (4.7)
\]

\[
V_B = \frac{\pi r_0^4}{8\eta_f} \cdot \frac{p_\infty - p_b}{sh(l_H/l_1)} ch(\frac{l_H - x}{l}), \quad \left( l = \frac{1}{4}\left[ l_H + \frac{1}{k_p} \ln(\frac{r_s}{r_0}) \right] \right)
\]

here \( p_b \) is the pressure at the issue of the HB where \( x = 0 \).
And so, the fluid output of the HB without fractures per unit time is

$$Q_b = \frac{\pi r_0^2}{2 \eta \tau} \sqrt{k_p \left( \ln \frac{r_s}{r_0} \right)^{-1/2} (p_\infty - p_b) c_h \frac{l_H}{1 - \frac{r}{R}}}.$$ (4.8)

To find $r_s/r_0$, it is convenient to use equality $r_s/r_0 = (l_H/r_0)\alpha$, where $\alpha$ is a fitting constant to be found from one numerical solution of the problem [1]. It is usually equal to approximately 0.7. This approach allowed us to get some high accuracy analytical solutions, e.g. [1,10,11].

(b) Penny-shaped fracture in oil deposit: stationary extraction regime

Let a penny-shaped fracture of radius $R_0$ be issuing from the horizontal borehole at $x = x_0$ so that its centre is on the $x$-axis and its plane is perpendicular to this axis. We move the frame $O\xi r$ along the $x$-axis to the centre of the fracture and designate it as $O\xi r$ where $\xi = x - x_0$.

The distance between opposite banks of the fracture at $r_0 < r < R < R_0$ can be taken constant equal to $d_p$, where $d_p$ is the diameter of solid particles (proppants) in the drill mud used to make the fracture by fracking. The particles remain inside the open fracture after the mud is removed and the rock pressure is closing the opening. These particles keep the fracture open like a wedge does. The value of $R$ is determined by the fracking process while the difference $R_0 - R$ can be found from the corresponding plane-strain problem of fracture mechanics when $R_0 - R \ll R$. We provide the result of its solution [1]

$$\frac{Ed_p}{2(1 - \delta^2)^2} \sqrt{\frac{1}{\pi (R_0 - R)}} - \delta_T w r \frac{1}{2} \pi (R_0 - R) = K_{IC};$$ (4.9)

here $K_{IC}$ is the rock fracture toughness. And so, from equation (4.9) it follows that $R_0 - R$ is less than $Ed_p[2\pi \delta_T w r (1 - \delta^2)]^{-1}$, i.e. less than about 0.1 m for sandstones at depth 1 km and $d_p \approx 0.5$ cm. Thus, $R_0 - R \ll R$ indeed. In what follows we assume that $R \gg r_s$; otherwise, the fracking has no advantage.

The fluid flow near the fracture has the structure of a boundary layer $|\xi| < x_s, r_0 < r < R$, where $x_0 = x_s/R \ll 1$ and $d_p \gg k_s$ so that $V_F \gg d_p \eta r$ ($V_F$ is the flow rate through the fracture cross-section per unit length). In the boundary layer, the previous approach provides the following basic equations:

$$\frac{d}{dr} (r V_F) = r q_F, \quad V_F = -\frac{d^3_p}{12 \eta} \cdot \frac{dP_F}{dr} \quad \text{and} \quad q_F = 2 \frac{k_p}{\eta} \cdot \frac{p_\infty - P_F(r)}{x_s};$$ (4.10)

here $P_F(r)$ and $q_F(r)$ are the pressure in and the inflow rate into the fracture.

From equation (4.10), it follows that

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dP_F}{dr} \right) = -\frac{1}{b^2} \left[ p_\infty - P_F(r) \right] \quad \left( b = \frac{d_p}{2 \sqrt{6 k_p}} \right)$$ (4.11)

$$\left\{ \begin{array}{l} \text{when} \quad r = R, \quad \text{and} \quad P_F = p_b, \quad \text{when} \quad r = r_0; \end{array} \right.$$ 

here $p_b$ is the pressure at the issue of the fracture on the HB.

The solution of the boundary problem in equation (4.11) can be written as follows:

$$P_F = p_\infty - D (p_\infty - p_b) \left[ I_0 \left( \frac{R}{b} \right) K_0 \left( \frac{R}{b} \right) - K_0 \left( \frac{R}{b} \right) I_0 \left( \frac{R}{b} \right) \right],$$ (4.12)

where $D = [K_0(r_0/b)I_0(R/b) - K_0(R/b)I_0(r_0/b)]^{-1}$; here $I_0(r/b)$ is the modified Bessel function so that

$$I_0 \left( \frac{r}{b} \right) = \exp \left( \frac{r}{b} \right) \left( 2 \pi \frac{r}{b} \right)^{-1/2}, \quad \text{when} \quad \frac{r}{b} \to \infty \quad \text{and} \quad I_0 \left( \frac{r}{b} \right) \to 1, \quad \text{when} \quad \frac{r}{b} \to 0.$$ (4.13)
According to equations (4.10) and (4.12), the fluid output from the fracture into the HB per unit time is equal to

\[ Q_F = \frac{\pi r_0^2 d}{6b\eta_f} D(p_{\infty} - p_b) \left[ I_0 \left( \frac{R}{b} \right) K_1 \left( \frac{r_0}{b} \right) + K_0 \left( \frac{R}{b} \right) I_1 \left( \frac{r_0}{b} \right) \right]; \quad (4.14) \]

here \( K_1(\frac{r_0}{b}) \) and \( I_1(\frac{r_0}{b}) \) are the corresponding modified Bessel functions.

For very large fractures, when \( R \gg b \gg r_0 \), it is reduced to the simple equation

\[ Q_F = \frac{\pi d^3 (p_{\infty} - p_b)}{6\eta_f \ln(b/r_0)}, \quad (4.15) \]

and so the output of a very large fracture significantly depends on \( d_p, p_{\infty} - p_b \) and \( \eta_f \), and much less on \( b/r_0 \). The value of the latter fitting parameter is to be found from one numerical solution to the problem under some typical conditions.

Evidently, this extraction process can be productive only for large fractures in rocks of high permeability. In the case of several fractures, when the distance between any two neighbouring fractures is greater than \( x_* \), the total output is given by summation of equations (4.8) and (4.14). The other case requires more study.

5. The theory of fracking

Let us study the hydrofracturing in shale gas reservoirs [12]. Shales are characterized by high porosity, low permeability and low fracture toughness. They are fractured by minor tensile stresses so that the shale destruction opens the way to extract gas stored in closed pores. Because of low permeability the fluid flow in rock beyond the fractured zone can be ignored. Horizontal boreholes in shales can be as long as 2 km. High pressure of the drill mud upon the HB surface and chemicals dissolving links between the rock fragments at the front of fractures produce a well-fractured volume in the local vicinity of the HB. Practically all gas can be extracted from this volume. And so the capacity of the HB depends on the volume of fractured rock. The fractures keep open using proppants embedded by the drill mud inside fractures during fracking.

When the drill mud pressure is low, the horizontal cylindrical channel in an elastic rock is subject to the following stresses in the surface layer of the channel

\[ \sigma_r = -p_m \quad \text{and} \quad \sigma_\theta = p_m - w_r \left[ 1 + \delta_T + 2(1 - \delta_T) \cos 2\theta \right]; \quad (5.1) \]

here \( p_m \) is the drill mud pressure and \( \theta \) is the angle between the horizontal plane and radius in the polar system of coordinates \( Or\theta \) in the cylinder cross-section with the centre at the axis. The stresses far from this channel are

\[ \sigma_{11} = \sigma_{22} = -\delta_T w_r \quad \text{and} \quad \sigma_{33} = -w_r. \quad (5.2) \]

The fracturing starts at the top point \( \theta = \frac{\pi}{2} \) when \( p_m > (3\delta_T - 1)w_r \).

We study three basic regimes of fracking in the following section. In the permeation regime, the drill mud penetrates everywhere inside fractures, whereas in the non-permeation regime, it penetrates nowhere in the rock. The most practical regime is that of partial permeation. In all cases, the zone of fractured rock is assumed to enclose the HB. Also, we assume that many fractures issue from the HB, all being radial, i.e. propagating along planes \( \theta = \text{const} \). Evidently, in the cylinder cross-section the contour of the zone of fractured rock always represents an oval extended in the vertical direction.

(a) The permeation regime of fracking

The friable shale is fractured by minor tensile stresses caused by the pressure of the drill mud that permeates multiple fractures. As a result the hydrostatic pressure \( p_m \) is setting in everywhere in
the well-fractured rock so that
\[ \sigma_{11} = \sigma_{22} = \sigma_{33} = -p_m \quad \text{(inside } Z_F\text{)}; \] (5.3)

here \( Z_F \) is the closed contour of fractured rock in the normal cross-section \( Ox_1x_2 \) of the HB. This stress state is similar to a specific fluidized state \[13\]. The rock outside \( Z_F \) is intact and elastic and is in a pre-fractured state on \( Z_F \) so that
\[ \sigma_n = -p_m \quad \text{and} \quad \sigma_i = -\sigma_s \quad \text{if} \quad \sigma_t \leq 0 \quad \text{and} \quad \sigma_{nt} = 0 \quad \text{(on } Z_F\text{)}; \] (5.4)

here \( \sigma_n, \sigma_{nt} \) and \( \sigma_t \) are the normal, shear and tangential stresses, respectively, on \( Z_F \) satisfying a failure criterion, e.g. von Mises criterion (\( k_s \) is in a pre-fractured state on \( Z_F \)).

The rock outside \( Z_F \) here the Kolosov–Muskhelishvili potentials \( \Phi(z) \) and \( \Psi(z) \) are analytic functions outside contour \( Z_F \) that is unknown beforehand and has to be found.

From equations (5.2), (5.4) and (5.7), it follows that
\[ 4\Phi(z) = -(1 + \delta_T)w_t = -p_m - \sigma_s. \] (5.8)

Hence, the pressure of the drill mud necessary for this regime of fracking is
\[ p_m = (1 + \delta_T)w_t - \sigma_s. \] (5.9)

Using equations (5.4), (5.7), (5.8) and the equation
\[ \sigma_t - \sigma_n + 2i\sigma_{nt} = \alpha e^{2iz}(\sigma_{33} - \sigma_{22} + 2i\sigma_{23}), \]
we have the following boundary value problem
\[ 2e^{2iz}\Psi(z) = p_m - \sigma_s \quad (z \in Z_F); \] (5.10)

here \( \alpha \) is the angle between the external normal to \( Z_F \) and axis \( x_2 \) being counted from the axis to the normal.

Let the conformal mapping of domain \( |\zeta| \geq 1 \) onto the domain outside \( Z_F \) be provided by function \( z = \omega(\zeta) \), where \( \zeta \) is a new parametric complex variable. Since \( \overline{\omega'(\zeta)}e^{2iz} = -\zeta^2\omega'(\zeta) \) on \( |\zeta| = 1 \), the boundary condition (5.10) can be written as
\[ (p_m - \sigma_s)\overline{\omega'(\zeta)} + 2\zeta^2\omega'(\zeta)\Psi(\omega(\zeta)) = 0 \quad (\zeta = 1). \] (5.11)

Using the method of functional equations introduced in \[14\]—also see \[15\] for more detail and for many other nonlinear problems solved by this method—we get the solution to this boundary
value problem in the following shape

\[ \omega(\zeta) = A \left( \zeta - \frac{\kappa}{\zeta} \right) \quad \text{and} \quad \Psi(\omega(\zeta)) = -\frac{1}{2}(p_m - \sigma) \frac{1 + \kappa \xi^2}{\kappa + \xi^2}; \]

(5.12)

here \( A \) is an arbitrary constant, and \( \kappa \) is equal to

\[ \kappa = \frac{1 - \delta_T}{1 + \delta_T}, \quad (0 < \kappa < 1, \ 0 < \delta_T < 1). \]

(5.13)

And so the boundary of the fractured rock has a shape of ellipse which diameters in the vertical and horizontal directions are

\[ D_V = 2A(1 + \kappa) \quad \text{and} \quad D_H = 2A(1 - \kappa). \]

(5.14)

The output of shale gas in this regime is directly proportional to \( \pi (1 - \kappa^2) A^2 l_B \).

For \( A \gg r_0 \), the value of \( A \) is directly proportional to the square root of the volume of the drill mud pumped into the HB. In this regime of fracking, the shale gas output is directly proportional to the drill mud volume pumped into the HB.

(b) The non-permeation regime of fracking

Let us also study an extreme case when the permeation of the drill mud into the fractured rock can be ignored. In the continuum approximation, for many radial fractures inside contour \( Z_F \), we get

\[ \sigma_r = -p_m \frac{r_0}{r}, \quad \sigma_\theta = 0 \quad \text{and} \quad \sigma_{r\theta} = 0, \quad (r \geq r_0 \quad \text{and inside} \ Z_F). \]

(5.15)

Similar to the previous problem, we use the conformal mapping of domain \(|\zeta| \geq 1\) onto the domain outside \( Z_F \) by function \( z = \omega(\zeta) \) and arrive at the following boundary value problem when \(|\zeta| = 1\)

\[ 4 \text{Re} \Phi(\omega(\zeta)) = -\frac{r_0 p_m}{|\omega(\zeta)|}, \quad \frac{\omega(\zeta)}{\omega'(\zeta)} \frac{d}{d\zeta} \Phi(\omega(\zeta)) + \Psi(\omega(\zeta)) = \frac{r_0 p_m \omega(\zeta)}{2 \omega(\zeta)|\omega(\zeta)|}. \]

(5.16)

The method of functional equations provides the following solution to this boundary value problem [14,15]:

\[ \omega(\zeta) = B(2 \zeta^2 - \lambda)^2 \zeta^{-3}, \quad \Phi(\omega(\zeta)) = w_r(1 + \delta_T) \left[ \frac{1}{4} - \frac{\zeta^2}{2 \zeta^2 - \lambda} \right] \]

\[ \Psi(\omega(\zeta)) = -\frac{1}{2} \lambda w_r(1 + \delta_T) \left[ \frac{\zeta^4(1 + \zeta^2)(2(4 + \lambda^2)\zeta^2 + \lambda(4 - 3\lambda^2))}{(2\zeta^2 - \lambda)^3(2\zeta^2 + 3\lambda)} \right] \]

\[ \left( B = \frac{r_0 p_m}{(1 + \delta_T)(4 - \lambda^2)w_r}, \quad \lambda^3 + 4\lambda = 8\kappa \right). \]

(5.17)

(5.18)

(Notice of erratum: the denominator of the second equation (5.2.23) in book [15] should be equal to \( \sigma_\lambda^\infty + \sigma_\beta^\infty - 2\sigma_\gamma \) instead of \( \sigma_\gamma - p \).)

Contour \( Z_F \) is described by the following equations

\[ x_2 = B[4(1 - \lambda) \cos \beta + \lambda^2 \cos 3\beta], \quad (2\pi \geq \beta \geq 0) \]

(5.19)

and

\[ x_3 = B[4(1 + \lambda) \sin \beta - \lambda^2 \sin 3\beta]. \]

The vertical and horizontal diameters of the fractured zone are as follows:

\[ D_V = 2B(2 + \lambda)^2 \quad \text{and} \quad D_H = 2B(2 - \lambda)^2. \]

(5.20)

Contour \( Z_F \) encloses the HB when

\[ \frac{p_m}{w_r} \geq (1 + \delta_T) \frac{2 - \lambda}{2 + \lambda}. \]

(5.21)
It can be shown that the solution (5.17)–(5.20) is valid when
\[
\frac{2}{3} \geq \lambda \geq 0, \quad \text{when} \quad \frac{17}{37} \leq \delta_T \leq 1.
\]

When \( \lambda = 2/3 \), cusps appear at points \( x_3 = \pm(1/2)D_V \) of contour \( Z_F \) (at this state \( D_V = 4D_H \)). This feature signals that for \( \lambda > 2/3 \) fractures grow in the intact rock from the cusps along the vertical plane \( x_1x_3 \) because of the rise of the square-root singularity of tensile elastic stresses at the cusps.

According to equation (5.19) the area of the cross-section of fractured zone is equal to
\[
\pi B^2(16 - 16\lambda^2 - 3\lambda^4) - \pi r_G^2.
\]

And so, in the non-permeation regime, parameters \( \delta_T \) and \( p_m/w_T \) control the fracking process. When \( p_m \gg w_T \), the volume of fractured rock in this regime is equal approximately to \( l_H r_G^2(p_m/w_T)^2 \).

(c) The general regime of fracking

The general regime of partial permeation occurs when the drill mud penetrates into some part of fractures at \( r_0 \leq r \leq r_s \) while the gas liberated from fractured pores permeates all the remaining part of fractures. This regime is of most practical importance. The stresses in the rock between any two neighbouring radial fractures meet the following equation:
\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \text{and} \quad \sigma_\theta = 0, \quad (\theta_0 \leq \theta \leq \theta_0 + \Delta \theta).
\]

Since \( \Delta \theta \ll 1 \), we can put \( \sigma_\theta = -p_m \) in the area where the drill mud wets the fracture surface, i.e. when \( r_0 \leq r \leq r_s \). This is an axisymmetric area because it is determined by the axisymmetric conditions in the vicinity of the HB. In the remaining part inside \( Z_F \) when \( r \geq r_s \), we can put \( \sigma_\theta = -p_G \), where \( p_G \) is the pressure of shale gas liberated from fractured pores.

And so from here and equation (5.24), in the continuum approximation for all fractured area inside \( Z_F \), we get
\[
\sigma_r = \sigma_\theta = -p_m \quad \text{and} \quad \sigma_\theta = 0, \quad (r_0 \leq r \leq r_s)
\]
and
\[
\sigma_r = -p_G + (p_G - p_m)\frac{r_s}{r}, \quad \sigma_\theta = -p_G \quad \text{and} \quad \sigma_\theta = 0, \quad (r \geq r_s \text{ in } Z_F).
\]

Evidently, \( r_s \) increases if \( p_m > p_G \), and \( r_s \) decreases if \( p_m < p_G \), but the boundary velocity is much less than \( \sigma_T \).

In this case, the method of functional equations provides the following solution
\[
z = \omega(\xi) = D(2\xi^2 - \mu)^2 \xi^{-3}, \quad (|\xi| \geq 1)
\]
\[
\Phi(\omega(\xi)) = -p_G + \frac{1}{4} (1 + \delta_T) \omega_T - \frac{(\omega_T + \delta_T w_T - 2p_G)\xi^2}{2\xi^2 - \mu},
\]
and
\[
D = \frac{(p_m - p_G)r_s}{(4 - \mu^2)(1 + \delta_T - 2p_G)\omega_T}, \quad \mu^2 + 4\mu = \frac{8(1 - \delta_T)}{1 + \delta_T - 2p_G}, \quad \text{and} \quad p_G = \frac{p_G}{\omega_T}.
\]

Function \( \Psi(\omega(\xi)) \) coincides with that in equation (5.17) if in equation (5.17): factor \( \omega_T(1 + \delta_T) \) is replaced by \( (1 + \delta_T)\omega_T - 2p_G \), and \( \lambda \) is replaced by \( \mu \).

Contour \( Z_F \) of the fractured zone and its vertical and horizontal diameters are provided by equations (5.19) and (5.20) where \( \lambda \) has to be replaced by \( \mu \), and \( B \) by \( D \). It can be shown that the solution (5.25)–(5.29) exists for \( 2/3 \geq \mu \geq 0 \). At \( \mu = 2/3 \), cusps appear at points \( x_3 = \pm D(2 + \mu)^2 \) so that for \( \mu > 2/3 \) two fractures grow in the intact rock along plane \( x_1x_3 \) issuing from those points.
In this general case, the cross-sectional area of fractured rock is equal to
\[ \pi D^2(16 - 16\mu^2 - 3\mu^4) - \pi r_0^2. \]  
(5.30)

And so this regime of fracking is determined by four dimensionless parameters \( \delta_T, p_m/w_r, p_G/w_r \) and \( r_s/r_0 \) which have to meet the following conditions
\[ 37\delta_T \geq 17 + 20p_G, \quad \frac{r_s}{r_0} \geq 1 \quad \text{and} \quad D(2 - \mu)^2 \geq r_0. \]  
(5.31)

The fractured area grows when \( p_m > p_G \). When \( p_G > p_m \) the gas pushes out the drill mud and fractures close up to the level supported by proppants.

6. Conclusion

The basic invariant integral first introduced into fracture mechanics in [16]—see also [6–8,17–23]—is presented in a more general manner, beginning with initial mention of some broader applications in physics. Specific applications of the invariant integral based method are described for the thermal field induced by a moving crack or dislocation and, especially, for deriving the theory of fracking. In the latter case, attention is given to the complex variables based calculation of the shape and volume of the multiply fractured region near a horizontal borehole in a shale gas reservoir under three basic regimes of the mud permeation into this region.

Acknowledgements. The author thanks Ron Armstrong for helpful remarks and suggestions.

References


