On a question of Rudnick: do we have square root cancellation for error terms in moment calculations?

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We answer a question of Rudnick, largely in the negative, as to whether we have square root cancellation for error terms in moment calculations.

1. Background: Lang–Weil

Start with a finite field $k$ and $X/k$ separated of finite type, which is smooth and geometrically connected, of dimension $n \geq 1$. The Lang–Weil estimate [1] is the assertion that for variable finite extensions $K$ of $k$, we have the estimate

$$
\#X(K) = (\#K)^n + O((\#K)^{n-1/2}).
$$

Lang and Weil proved this by using its truth for curves, established by Weil, together with a fibration argument. From a modern point of view, Lang–Weil is best seen as resulting from Grothendieck’s Lefschetz trace formula [2], combined with Deligne’s estimates [3, Corollary 3.3.4]. For any prime $\ell$ not the characteristic $p$ of $k$, we have

$$
\#X(K) = \sum_{i=0}^{2n} (-1)^i \text{Trace}(\text{Frob}_K|H^i_{\ell}(X_{\bar{k}}, \mathbb{Q}_\ell)).
$$

One knows that $H^i_{\ell}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is one-dimensional, with $\text{Frob}_K$ acting as $(\#K)^n$, and, thanks to Deligne, that each $H^i_{\ell}(X_{\bar{k}}, \mathbb{Q}_\ell)$ is mixed of weight $\leq i$ (for any chosen embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$).

So the formula becomes

$$
\#X(K) = (\#K)^n + \sum_{i=0}^{2n-1} (-1)^i \text{Trace}(\text{Frob}_K|H^i_{\ell}(X_{\bar{k}}, \mathbb{Q}_\ell)),
$$

with

$$
\sum_{i=0}^{2n-1} (-1)^i \text{Trace}(\text{Frob}_K|H^i_{\ell}(X_{\bar{k}}, \mathbb{Q}_\ell)) = O((\#K)^{n-1/2}).
$$
2. Background: Deligne’s equidistribution theorem

How does Deligne’s equidistribution theorem relate to this? The situation is that we have a lisse $\mathcal{Q}_\ell$-sheaf, $\ell \neq p$, $\mathcal{F}$ on $X$ which is pure of weight zero, of rank $r \geq 1$. Attached to it are its geometric and arithmetic monodromy groups $G_{\text{geom}} \leq G_{\text{arith}} \subset GL(r)$. These are algebraic groups over $\mathcal{O}_\ell$. One knows, again by Deligne, that (the identity component of) $G_{\text{geom}}$ is semi-simple, cf. [3, Corollary 1.3.9 and its proof, and Theorem 3.4.1 (iii)].

Suppose that our $\mathcal{F}$ has $G_{\text{geom}} = G_{\text{arith}}$, Embed $\mathcal{Q}_\ell$ into $\mathbb{C}$, view $G_{\text{arith}}$ as a group over $\mathbb{C}$, and choose a maximal compact subgroup $K$ of the complex Lie group $G_{\text{arith}}(\mathbb{C})$. Then for each finite extension $K/k$, and each $x \in X(K)$, (the semi-simplification, in the sense of Jordan normal form, of) the Frobenius conjugacy class $\text{Frob}_{x,K}$ meets $\mathbb{K}$ in a unique $\mathbb{K}$-conjugacy class $\theta_{x,K}$.

Deligne’s equidistribution theorem asserts that as $\# K \to \infty$, the classes $\{\theta_{x,K}\}_{x \in X(K)}$ become equidistributed in $\mathbb{K}^\#$, the space of conjugacy classes in $\mathbb{K}$, for (the direct image from $\mathbb{K}$ of) Haar measure of total mass one, cf. [3, Theorem 3.5.3], [4, Theorem 3.6] and [5, Theorem 9.2.6].

The proof goes along the now usual lines, of estimating the appropriate Weyl sums. More precisely, for each irreducible non-trivial representation $\rho$ of $G_{\text{arith}}$, we form the corresponding ‘pushout’ sheaf $\rho(\mathcal{F})$ on $X$. By Peter–Weyl, what must be shown is that the large $\# K$ limit of

$$\left(\frac{1}{\# X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\rho(\mathcal{F}))$$

vanishes.

This sum is

$$\left(\frac{1}{\# X(K)}\right) \sum_{i=0}^{2n} (-1)^i \text{Trace}(\text{Frob}_K|H^{i}_c(X_{\overline{k}}, \rho(\mathcal{F}))),$$

in which $H^{i}_c$ is mixed of weight $\leq i$, and in which the highest term $H^{2n}_c(X_{\overline{k}}, \rho(\mathcal{F}))$ is the Tate twist by $-n$ of) the space of coinvariants of $G_{\text{geom}}$ in the representation $\rho$. So the leading term vanishes

$$H^{2n}_c(X_{\overline{k}}, \rho(\mathcal{F}))(n) = 0,$$

and we get the estimate

$$\sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\rho(\mathcal{F})) = O((\# K)^{n-1/2}).$$

In view of Lang–Weil, we get

$$\left(\frac{1}{\# X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\rho(\mathcal{F})) = O\left(\frac{1}{\sqrt{\# K}}\right).$$

An equivalent formulation is this. Take any representation $\sigma$ of $G_{\text{arith}}$, and denote by $N(\sigma)$ the multiplicity of the trivial representation in $\sigma$. Thus, $N(\sigma)$ is the dimension of $H^{2n}_c(X_{\overline{k}}, \rho(\mathcal{F}))$, upon which $\text{Frob}_K$ operates as the scalar $(\# K)^n$. Write $\sigma$ as the direct sum of $N(\sigma)$ copies of the trivial representation with a finite sum of irreducible non-trivial representation $\rho$ of $G_{\text{arith}}$, say $\sigma = N(\sigma) \mathbb{I} \oplus \tau$, with $N(\tau) = 0$. For $N(\sigma) \mathbb{I}$, i.e. for the constant sheaf $\mathcal{Q}^{N(\sigma)}_\ell$, we have the tautological equality

$$\left(\frac{1}{\# X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\mathcal{Q}^{N(\sigma)}_\ell) = N(\sigma).$$

For the sheaf $\tau(\mathcal{F})$, whose $H^{2n}_c$ vanishes, the Lefschetz trace formula gives

$$\left(\frac{1}{\# X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\tau(\mathcal{F})) = \left(\frac{1}{\# X(K)}\right) \sum_{i \leq 2n-1} (-1)^i \text{Trace}(\text{Frob}_K|H^{i}_c(X_{\overline{k}}, \tau(\mathcal{F}))).$$
By Deligne (and Lang–Weil), this last sum is $O(1/\sqrt{\#K})$, so we get

$$\left(\frac{1}{\#X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\sigma(F)) = N(\sigma) + O\left(\frac{1}{\sqrt{\#K}}\right).$$

To the extent that the sum $\sum_{i \leq 2n-1} (-1)^i \text{Trace}(\text{Frob}_K|\mathcal{H}^i_{\mathcal{X}_K}(\tau(F)))$ has a better estimate, e.g. because some of its $\mathcal{H}_c^i$ vanish for large $i$, or have lower weight than allowed by Deligne’s general theorem that $\mathcal{H}_c^i$ has weight $\leq i$, we get a better estimate of the error term.

3. Rudnick’s question

Zeev Rudnick raised what is, in hindsight, the obvious question:

If $n := \dim(X) \geq 2$, when can we do better? When will we get ‘square root cancellation’, i.e. an estimate, for every irreducible non-trivial representation $\rho$ of $G_{\text{arith}}$,

$$\left(\frac{1}{\#X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\rho(F)) = O\left(\frac{1}{\sqrt{\#K}}\right).$$

Equivalently, when will we get an estimate, for every representation $\sigma$ of $G_{\text{arith}}$,

$$\left(\frac{1}{\#X(K)}\right) \sum_{x \in X(K)} \text{Trace}(\text{Frob}_{x,K}|\sigma(F)) = N(\sigma) + O\left(\frac{1}{\sqrt{\#K}}\right).$$

4. Examples showing a largely negative response

In the following sections, we will give examples in which some $\sigma$’s have square root cancellation, and in which many others do not.

Fix integers $N \geq n \geq 2$, a prime $p > 2N + 1$, and a non-trivial additive character $\psi$ of $\mathbb{F}_p$. For $K/\mathbb{F}_p$, a finite extension, $\psi_K := \psi \circ \text{Trace}_{K/\mathbb{F}_q}$ is a non-trivial additive character of $K$. Consider the $n$ parameter family of sums, for each $K$, given by

$$S(t_1,t_2,\ldots,t_n,K) := \left(-\frac{1}{\sqrt{\#K}}\right) \sum_{x \in K} \psi_K\left(x^{N+1} + \sum_{i=1}^n t_i x^i\right).$$

There is a lisse sheaf $\mathcal{F}$ on the $\mathbb{A}^n$ of $(a_1,a_2,\ldots,a_n)$ whose trace function is given by these sums:

$$\text{Trace}(\text{Frob}_{t_1,t_2,\ldots,t_n,K}|\mathcal{F}) = S(t_1,t_2,\ldots,t_n,K).$$

This sheaf $\mathcal{F}$ is lisse of rank $N$ and pure of weight zero. One knows [6, Theorem 19] that for this sheaf $\mathcal{F}$ we have

$$\text{SL}(n) \subseteq G_{\text{geom}} \subseteq G_{\text{arith}} \subseteq \text{GL}(N).$$

Lemma 4.1. After passing to a finite extension $\mathbb{F}_\eta/\mathbb{F}_p$, the sheaf $\mathcal{F}$ on $\mathbb{A}^n/\mathbb{F}_\eta$ has

$$\text{SL}(n) \subseteq G_{\text{geom}} = G_{\text{arith}} \subseteq \text{GL}(N).$$

Proof. First extend scalars to $\mathbb{F}_p$. For any finite extension $K/\mathbb{F}_p$, each $\text{Frob}_{x,K}$ has its characteristic polynomial with coefficients in $\mathbb{Q}(\zeta_p)$, so in particular has its determinant in $\mathbb{Q}(\zeta_p)$. The key point is that this field has a unique place $\mathcal{P}$ lying over $p$. So $\det(\text{Frob}_{x,K})$ has absolute value 1 at each Archimedean place (purity), and is a unit at all finite places of residue characteristic $\ell \neq p$ (existence of $\ell$-adic cohomology). By the product formula, the determinant must be a unit also at
\( \mathcal{P} \), so is a root of unity of order dividing 2p. If we take an extension \( K / \mathbb{F}_p \) of odd degree, then the square of each \( \text{Frob}_{x, K} \) has such a determinant. Thus, we have inclusions

\[
SL(n) \subset G_{\text{geom}} \subset G_{\text{arith}} \subset \{ A \in GL(N) \mid \det(A)^{\deg} = 1 \}.
\]

From these inclusions, we certainly have

\[
G_{\text{arith}} \subset G_m G_{\text{geom}} \quad (= GL(N)),
\]

so there exist an \( \ell \)-adic unit \( \alpha \) such that after the constant field twist \( \alpha^{\deg} \) of \( F \), we have \( G_{\text{geom}} = G_{\text{arith}} \), cf. [7, Lemma 3.1]. It remains only to show that any such \( \alpha \) is a root of unity. [For if \( \alpha_N = 1 \), then after extension of scalars from \( \mathbb{F}_p \) to \( \mathbb{F}_{pN} \), we will have \( G_{\text{geom}} = G_{\text{arith}} \) for \( F \).] To see that any such \( \alpha \) is a root of unity, choose any point \( x \in \mathbb{A}^n(\mathbb{F}_p) \). Then both \( \text{Frob}_{x, \mathbb{F}_p} | F \) and \( \alpha \text{Frob}_{x, \mathbb{F}_p} | F \) lie in \( G_{\text{arith}} \), indeed the latter lies in \( G_{\text{geom}} \). Comparing determinants, both of which are roots of unity of order dividing 4p, we see that \( \alpha_N \) is a root of unity of order dividing 4p.

For the remainder of this section, and in the two sections to follow, we work with the sheaf \( F \) on \( \mathbb{A}^n / \mathbb{F}_q \), with \( \mathbb{F}_q \) large enough that

\[
SL(n) \subset G_{\text{geom}} = G_{\text{arith}} \subset GL(N).
\]

We denote by \( \text{std} \) the given (‘standard’) \( n \)-dimensional representation of \( G_{\text{arith}} \), and by \( \text{std}^\vee \) the dual representation. We will be concerned with the representations

\[
\text{std}^\otimes A \otimes (\text{std}^\vee)^\otimes B
\]

of \( G_{\text{arith}} \), for each pair of integers \( (A, B) \) with \( 0 \leq A, B \leq n \) (excluding the case \( a = b = 0 \), the trivial representation). We denote

\[
M_{A,B} := \dim(\text{std}^\otimes A \otimes (\text{std}^\vee)^\otimes B)_{G_{\text{arith}}},
\]

the dimension of the space of invariants in \( \text{std}^\otimes A \otimes (\text{std}^\vee)^\otimes B \), and by

\[
M_{A,B}(\mathbb{F}_q),
\]

the ‘empirical moment’

\[
M_{A,B}(\mathbb{F}_q) := \left( \frac{1}{q^n} \right) \sum_{(t_1, \ldots, t_n) \in \mathbb{A}^n(\mathbb{F}_q)} S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^A S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^B.
\]

We know that \( M_{A,B} \) is the large \( q \) limit of \( M_{A,B}(\mathbb{F}_q) \). Our concern is with estimating the difference

\[
M_{A,B} - M_{A,B}(\mathbb{F}_q).
\]

5. Explicit calculation of \( M_{A,B}(\mathbb{F}_q) \)

For any \( (A, B) \), the empirical moments \( M_{A,B}(\mathbb{F}_q) \) and \( M_{B,A}(\mathbb{F}_q) \) are complex conjugates of each other (after any embedding of \( \overline{\mathbb{Q}_\ell} \) into \( \mathbb{C} \)). So we will assume from now on that

\[
A \geq B.
\]

In the affine space \( \mathbb{A}^A \times \mathbb{A}^B \), with coordinates \( (t_1, \ldots, x_A, y_1, \ldots, y_B) \), denote by \( V(A, B, n) \subset \mathbb{A}^A \times \mathbb{A}^B \) the closed subscheme defined by the \( n \) equations

\[
\sum_{a \leq A} x_a^d = \sum_{b \leq B} y_b^d, \quad 1 \leq d \leq n.
\]

[In the case \( B = 0 \), \( V(A, 0, n) \subset \mathbb{A}^A \) is the closed subschema defined by the \( n \) equations

\[
\sum_{a \leq A} x_a^d = 0, \quad 1 \leq d \leq n.\]
Lemma 5.1. For \( \mathbb{F}_q \) a finite field of characteristic \( p > n \), the points of \( V(A, B, n)(\mathbb{F}_q) \) have the following explicit description.

1. If \( n \geq A = B > 0 \), then a point \((x_1, \ldots, x_A, y_1, \ldots, y_A) \in \mathbb{A}^{A+A}(\mathbb{F}_q)\) lies in \( V(A, A, n)(\mathbb{F}_q) \) if and only if the two lists \((x_1, \ldots, x_A)\) and \((y_1, \ldots, y_A)\) are rearrangements of each other, i.e. if and only if the first \( A \) elementary symmetric functions agree on them.

2. If \( n \geq A > B \geq 0 \), then a point \((x_1, \ldots, x_A, y_1, \ldots, y_B) \in \mathbb{A}^{A+B}(\mathbb{F}_q)\) lies in \( V(A, B, n)(\mathbb{F}_q) \) if and only if the two lists of length \( A \), \((x_1, \ldots, x_A)\) and \((y_1, \ldots, y_B, 0, 0, \ldots)\) (the second list obtained by padding out the list of \( y_i \)'s by appending \( A - B \) zeros) are rearrangements of each other.

3. (a special case of (2) above) If \( n \geq A \) and \( B = 0 \), the only point of \( V(A, 0, n)(\mathbb{F}_q) \) is \((0, \ldots, 0)\).

Proof. Because the characteristic \( p > n \), for \( A \leq n \) the equality of the first \( A \) Newton symmetric functions is equivalent to the equality of the first \( A \) elementary symmetric functions.

Lemma 5.2. For \( n \geq A \geq B \geq 0 \), but \((A, B) \neq (0, 0)\), and \( \mathbb{F}_q \) a finite field of characteristic \( p > n \), we have

\[
M_{A, B}(\mathbb{F}_q) = \left(-\frac{1}{\sqrt{q}}\right)^{A+B} \# V(A, B, n)(\mathbb{F}_q).
\]

Proof. Expand each term \( S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^A S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^B \) of the sum defining \( M_{A, B}(\mathbb{F}_q) \). By definition, we have

\[
S(t_1, t_2, \ldots, t_n, \mathbb{F}_q) := \left(-\frac{1}{\sqrt{q}}\right) \sum_{x \in \mathbb{F}_q} \psi_{\mathbb{F}_q}(x^{N+1} + \sum_{i=1}^n t_i x^i).
\]

Its \( A \)'th power is then

\[
S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^A = \left(-\frac{1}{\sqrt{q}}\right)^A \sum_{x, y \in \mathbb{F}_q} \psi_{\mathbb{F}_q} \left( \sum_{a \leq A} \left(x_a^{N+1} + \sum_{i=1}^n t_i x_a^i\right) \right).
\]

The \( B \)th power of its complex conjugate is 1 if \( B = 0 \), and for \( B > 0 \) it is

\[
S(t_1, t_2, \ldots, t_n, \mathbb{F}_q)^B = \left(-\frac{1}{\sqrt{q}}\right)^B \sum_{x, y \in \mathbb{F}_q} \psi_{\mathbb{F}_q} \left( -\sum_{b \leq B} \left(y_b^{N+1} + \sum_{i=1}^n t_i y_b^i\right) \right).
\]

So \( M_{A, B}(\mathbb{F}_q) \) is \((-1/\sqrt{q})^{A+B} (-1/q)^n\) times

\[
\sum_{t_1, \ldots, t_n \in \mathbb{F}_q} \sum_{x, y, \ldots, y} \psi_{\mathbb{F}_q} \left( \sum_{a \leq A} \left(x_a^{N+1} - \sum_{b \leq B} y_b^{N+1} + \sum_{i=1}^n t_i \left( x_a^i - y_b^i \right) \right) \right).
\]

Reversing the order of summation, and using orthogonality of characters, we see that \( M_{A, B}(\mathbb{F}_q) \) is \((-1/\sqrt{q})^{A+B}\) times

\[
\sum_{(x_1, \ldots, x_A, y_1, \ldots, y_B) \in V(A, B, n)(\mathbb{F}_q)} \psi_{\mathbb{F}_q} \left( \sum_{a \leq A} \left(x_a^{N+1} - \sum_{b \leq B} y_b^{N+1} \right) \right).
\]

From the previous lemma, we know that for a point \((x_1, \ldots, x_A, y_1, \ldots, y_B) \) in \( V(A, B, n)(\mathbb{F}_q) \), the lists \((x_1, \ldots, x_A)\) and \((y_1, \ldots, y_B, 0, 0, \ldots)\) are rearrangements of each other. The function \(\sum_{a \leq A} x_a^{N+1} - \sum_{b \leq B} y_b^{N+1}\) vanishes at such a point, and hence this last sum is just \( \# V(A, B, n)(\mathbb{F}_q) \).

Proposition 5.3. For \( n \geq A > 0 \), \( M_{A, 0}(\mathbb{F}_q) = M_{0, A}(\mathbb{F}_q) = (-1/\sqrt{q})^A \) and \( M_{A, 0} = M_{0,A} = 0 \).

Proof. The first assertion is immediate from the previous two lemmas, and the second follows because \( M_{A,0} \) (resp. \( M_{0,A} \)) is the large \( q \) limit of \( M_{A,0}(\mathbb{F}_q) \) (resp. of \( M_{0,A}(\mathbb{F}_q) \)).

Corollary 5.4. If \( N = n \), the group \( G_{geom} \) for our sheaf \( \mathcal{F} \) is \( \{A \in GL(n)| \det(A)^n = 1\} \).
Proof. In the previous section, we have seen that over \( \mathbb{F}_p^2 \) we have inclusions (remember \( N = n \) here)
\[
SL(n) \subset G_{\text{geom}} \subset G_{\text{arith}} \subset \{ A \in GL(n) \mid \det(A)^2p = 1 \}.
\]
Hence \( \det(F)^{\otimes p} \) is a lisse rank one sheaf on \( \mathbb{A}_p^n \), which is of order dividing 2. But the group \( H^1(\mathbb{A}_p^n, \mu_2) \) vanishes, because \( p \) is odd. So we have inclusions
\[
SL(n) \subset G_{\text{geom}} \subset \{ A \in GL(n) \mid \det(A)^p = 1 \}.
\]
We must rule out the possibility that \( G_{\text{geom}} \) is \( SL(n) \). But if it were, then \( \det(F) \), would be a geometrically trivial summand of \( F^{\otimes n} \), and \( M_{n,0} \) would be non-zero. \( \blacksquare \)

**Proposition 5.5.** Suppose \( n \geq A > B > 0 \). For \( C := A - B \), we have \( M_{A,B} = 0 \),
\[
M_{A,B}(\mathbb{F}_q) = O\left(\left(\frac{1}{\sqrt{q}}\right)^C\right),
\]
and \( (\sqrt{q})^CM_{A,B}(\mathbb{F}_q) \) has a non-zero large \( q \) limit.

Proof. In this case, \( 0 < A - B < n \leq N \), so already the scalars in \( SL(N) \), namely \( \mu_N \), act by a non-trivial character, namely the \( A - B \)th power of the ‘identical’ character \( \xi \mapsto \xi \), in the representation
\[
\text{std}^{\otimes A} \otimes (\text{std}^{\otimes B}).
\]
A point in \( V(A,B,n)(\mathbb{F}_q) \) is of the form \((x_1, \ldots, x_A, y_1, \ldots, y_B)\) such that at least \( C := A - B \) of the \( x_i \) vanish, and such that the list of (at most \( B \)) non-vanishing \( x_a \)'s is a rearrangement of the list of non-vanishing \( y_b \)'s. Now break up \( V(A,B,n)(\mathbb{F}_q) \) by the number \( d \) of distinct non-zero \( x_a \) in a point. There is exactly one point whose \( d \) is zero. For given \( d \) with \( B \geq d \geq 1 \), the number of points with \( d \) distinct non-zero \( x_a \) is the product of \( \prod_{i=1}^d (q - i) \) with a strictly positive combinatorially defined integer, call it \( D(A,B,n,d) \). Thus, we have
\[
\#V(A,B,n)(\mathbb{F}_q) = D(A,B,n,B)q^B + O(q^{B-1}).
\]
Dividing by \( \sqrt{q}^{A+B} \), we see that
\[
M_{A,B}(\mathbb{F}_q) = \left(\frac{1}{\sqrt{q}^{A+B}}\right) \#V(A,B,n)(\mathbb{F}_q)
\]
is
\[
= \frac{(-1)^{A+B}D(A,B,n,B)}{\sqrt{q}^C} + O\left(\frac{1}{q^{C+2}}\right).
\]
\( \blacksquare \)

**Proposition 5.6.** For \( n \geq A \geq 1 \), we have the following results.

1. For \( A = 1 \), \( M_{1,1}(\mathbb{F}_q) = 1 \) and \( M_{1,1} = 1 \).
2. For \( A = 2 \), we have
\[
M_{2,2}(\mathbb{F}_q) = 2 - \frac{1}{q}.
\]
3. For \( n \geq A \geq 3 \), we have
\[
M_{A,A}(\mathbb{F}_q) = A! - \frac{A(A - 1)!}{4q} + O\left(\frac{1}{q^2}\right).
\]

Proof. Assertion (1) is immediate from the fact that \( \#V(1,1,1)(\mathbb{F}_q) = q \). Assertion (2) is immediate from the fact that \( \#V(2,2,1)(\mathbb{F}_q) = 2q(q - 1) + q = 2q^2 - q \).

For \( n \geq A \geq 3 \), we break up \( V(A,A,n)(\mathbb{F}_q) \) by the number \( d \) of distinct coordinates \( x_a \) in a point. The number of points with precisely \( d \) distinct \( x_a \)'s is the product of \( \prod_{i=1}^{d-1} (q - i) \) with a strictly positive combinatorially defined integer, call it \( D(A,A,n,d) \).
We have $D(A, A, n, A) = A!$ and $D(A, A, n, A - 1) = \left(\frac{A}{2}\right)^2 \times (A - 2)!$ [The term $\left(\frac{A}{2}\right)^2$ is to specify on each side the placement of the double root, and the term $(A - 2)!$ is to specify the reordering of the $A - 2$ simple roots.]

So looking at the two highest order terms, we have

\[
\#V(A, A, n)(\mathbb{F}_q) = A! \prod_{i=0}^{A-1} (q - i) + \left(\frac{A}{2}\right)^2 (A - 2)! \prod_{i=0}^{A-2} (q - i) + O(q^{A-2}).
\]

Expanding out $\prod_{i=0}^{A-1} (q - i)$, we get

\[
\prod_{i=0}^{A-1} (q - i) = q^A - \left(\frac{A(A - 1)}{2}\right)q^{A-1} + \text{lower terms}.
\]

Thus $\#V(A, A, n)(\mathbb{F}_q)$ is

\[
A!q^A - \left(\frac{A(A - 1)}{2}\right)q^{A-1} + \left(\frac{A}{2}\right)^2 (A - 2)!q^{A-2} + O(q^{A-2}) = A!q^A - \left(\frac{A(A - 1)a!}{4}\right)q^{A-1} + O(q^{A-2}).
\]

Dividing through by $q^A$ gives the assertion. ■

6. Cohomological consequences

We have seen in Lemma 5.2 that, up to a factor $(-1/\sqrt{q})^{A+B}$, $M_{A,B}$ is a polynomial in $q$, in principle quite explicit. A natural question is the extent to which we can infer from such information the vanishing, or non-vanishing, of various cohomology groups. Here are some results along this line.

Let us begin with the fact that $M_{1,1}(\mathbb{F}_q) = 1$. By the Lefschetz Trace Formula, this is equivalent to

\[
\sum_{i=0}^{2n} (-1)^i \text{Trace}(\text{Frob}_{\mathbb{F}_q}|H^i_c(\mathbb{A}^n_{\overline{\mathbb{F}}_q}, \mathcal{F} \otimes \mathcal{F}^\vee)) = q^n.
\]

Already the trace on the $H^{2n}_c$ is $q^n$. This suggests that $H^i_c(\mathbb{A}^n_{\overline{\mathbb{F}}_q}, \mathcal{F} \otimes \mathcal{F}^\vee)$ vanishes for $i \neq 2n$. We will now show that this is in fact the case. Here is an equivalent formulation.

The sheaf $\mathcal{F} \otimes \mathcal{F}^\vee = \text{End}(\mathcal{F})$ has a direct sum decomposition

\[
\text{End}(\mathcal{F}) = \mathcal{Q}_\ell \oplus \text{End}^0(\mathcal{F}),
\]

in which $\text{End}^0(\mathcal{F})$ is the subsheaf of endomorphisms of trace zero. The fact that $M_{1,1}(\mathbb{F}_q) = 1$ is thus equivalent to

\[
\sum_{i=0}^{2n} (-1)^i \text{Trace}(\text{Frob}_{\mathbb{F}_q}|H^i_c(\mathbb{A}^n_{\overline{\mathbb{F}}_q}, \text{End}^0(\mathcal{F}))) = 0.
\]

**Lemma 6.1.** The cohomology groups $H^i_c(\mathbb{A}^n_{\overline{\mathbb{F}}_q}, \text{End}^0(\mathcal{F}))$ all vanish.

**Proof.** Compute the cohomology via the Leray spectral sequence for the projection

\[
pr : \mathbb{A}^n \to \mathbb{A}^{n-1}, \quad (a_1, \ldots, a_n) \mapsto (a_2, \ldots, a_n).
\]
It suffices to show that all the $R^i \text{pr}_! \text{End}^0(\mathcal{F})$ vanish. By proper base change, it suffices to do this fibre by fibre. On the fibre over the point $\bar{a} := (a_2, \ldots, a_n)$, say with values in some finite extension $k/\mathbb{F}_q$, we have the polynomial
\[
f_{\bar{a}}(x) := x^n + \sum_{i=2}^{n} a_i x^i \in k[x],
\]
and our sheaf $\mathcal{F}$ on this fibre is the (naive) Fourier Transform of $\mathcal{L}_{\psi}(f_{\bar{a}})$. So the restriction of $\mathcal{F}$ to this fibre is geometrically irreducible, and its $M_{1,1}(k)$ is 1, by the same calculation as above. Therefore, the restriction to this fibre of $\text{End}^0(\mathcal{F})$ has no $H^2$ (because $\mathcal{F}$ on this fibre is geometrically irreducible), and the alternating sum of traces of $\text{Frob}_k$ on its $H^2$ is zero. On the other hand, its $H^0$ vanishes (because $\text{End}^0(\mathcal{F})$ is lisse on an open curve), and hence its $H^1_\iota$ must vanish, as all powers of $\text{Frob}_k$ have trace zero on this $H^1_\iota$. □

At the opposite extreme, we have the following result.

**Lemma 6.2.** For $n \geq A \geq 1$, the cohomology group
\[
H^2_{c,n-1}(\mathbb{P}^n, \mathcal{F} \otimes A \otimes (\mathcal{F}^\vee) \otimes A^{-1})
\]
is non-zero and its subspace of highest weight $2n - 1$ is non-zero.

**Proof.** This is immediate from proposition 5.5. First it gives the vanishing of the $H^2_{c,n}$. Then it tells us that
\[
\sum_{i=0}^{2n-1} (-1)^i \text{Trace}(\text{Frob}_k | H^i_c(\mathbb{P}^n, \mathcal{F} \otimes A \otimes (\mathcal{F}^\vee) \otimes A^{-1}))
\]
is $O(\sqrt{q^{2n-1}})$, and that after division by $\sqrt{q^{2n-1}}$, its large $q$ limit is non-zero. By Deligne, the $H^i_c$ for $i < 2n - 1$ have lower weight, so we get the asserted non-vanishing of the weight $2n - 1$ part of the $H^2_{c,n-1}$.

**Lemma 6.3.** For $n \geq A \geq 2$, the weight $2n - 2$ part of
\[
H^2_{c,n-1}(\mathbb{P}^n, \mathcal{F} \otimes A \otimes (\mathcal{F}^\vee) \otimes A^{-1})
\]
is non-zero, and has dimension at least $A(A-1)A!/4$, but its weight $2n - 1$ part vanishes.

**Proof.** By proposition 5.6, we have
\[
\sum_{i=0}^{2n-1} (-1)^i \text{Trace}(\text{Frob}_k | H^i_c(\mathbb{P}^n, \mathcal{F} \otimes A \otimes (\mathcal{F}^\vee) \otimes A))
\]
\[
= -\left(\frac{A(A-1)A!}{4}\right) q^{n-1} + \text{a polynomial in } q \text{ of lower degree}.
\]
This already shows that the weight $2n - 1$ part of $H^2_{c,n-1}$ vanishes. If we look at the parts of weight $2n - 2$, only $H^2_{c,n-1}$ with wt.$=2n-2$ and $H^2_{c,n-2}$ with wt.$=2n-2$ are possibly non-zero, and we get
\[
-\text{Trace}(\text{Frob}_k | (H^2_{c,n-1})_{\text{wt}.=2n-2}) + \text{Trace}(\text{Frob}_k | (H^2_{c,n-2})_{\text{wt}.=2n-2}) = -\left(\frac{A(A-1)A!}{4}\right) q^{n-1}.
\]
We rewrite this as
\[
\text{Trace}(\text{Frob}_k | (H^2_{c,n-1})_{\text{wt}.=2n-2}) = \left(\frac{A(A-1)A!}{4}\right) q^{n-1} + \text{Trace}(\text{Frob}_k | (H^2_{c,n-2})_{\text{wt}.=2n-2}),
\]
which gives the asserted result. □
7. Another example

We fix an odd integer \( n \geq 3 \), and a prime \( p \) not dividing \( n(n - 1) \). We consider, in characteristic \( p \), the two parameter family of hyperelliptic curves

\[ y^2 = x^n + ax + b, \]

over the open set of \( \mathbb{A}^2 \), parameters \((a, b)\), where the discriminant of \( x^n + ax + b \), namely

\[ \Delta = \Delta(a, b) := (n - 1)^{n-1}a^n + n^n b^{n-1}, \]

is invertible. For this family of curves, its \( H^1 \) along the fibres, Tate twisted by \( \frac{1}{n} \), is a lisse sheaf \( \mathcal{F} \) on \( \mathbb{A}^2[1/\Delta] \) of rank \( 2g = n - 1 \) which is pure of weight zero. Its trace function at a point \((a, b)\) with values in a finite extension \( \mathbb{F}_q \) is given by

\[ \text{Trace}(\text{Frob}_{(a,b),\mathbb{F}_q}|\mathcal{F}) = \left( -\frac{1}{\sqrt{q}} \right) \sum_{x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b). \]

Here \( \chi_{2,\mathbb{F}_q} \) denotes the quadratic character of \( \mathbb{F}_q^\times \), extended by zero to all of \( \mathbb{F}_q \). To define \( \sqrt{q} \), we fix a choice of \( \sqrt{p} \) in \( \mathbb{Q}_\ell \) and then define \( \sqrt{q} \) to be the appropriate power of \( \sqrt{p} \).

One knows that for this \( \mathcal{F} \), we have \( G_{\text{geom}} = G_{\text{arith}} = \text{Sp}(n - 1) \), cf. [8, Theorem 5.4 (1)]. In particular, the standard representation is irreducible, and hence \( M_{1,0} = 0 \), i.e. \( H^1_{\text{ét}}(\mathbb{A}^2_{\mathbb{F}_p}[1/\Delta], \mathcal{F}) \) vanishes. Moreover, we have

**Lemma 7.1.** For any finite extension \( \mathbb{F}_q/\mathbb{F}_p \), \( M_{1,0}(\mathbb{F}_q) = 0 \).

**Proof.** By definition, \( M_{1,0}(\mathbb{F}_q) \) is \((1/\#\mathbb{A}^2[1/\Delta](\mathbb{F}_q))(-1/\sqrt{q}) \) times the sum

\[ \sum_{(a,b) \in \mathbb{A}^2[1/\Delta](\mathbb{F}_q), x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b). \]

If this sum extended over all \((a, b) \in \mathbb{A}^2(\mathbb{F}_q)\), it would vanish; simply reverse the order of summation, i.e. write it as

\[ \sum_{(a,x) \in \mathbb{A}^2(\mathbb{F}_q)} \sum_{b \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b), \]

and note that the innermost sum \( \sum_{b \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b) \) vanishes.

So it remains to show that

\[ \sum_{(a,b) \in \mathbb{A}^2(\mathbb{F}_q) \setminus \Delta(a,b)=0, x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b) = 0. \]

The condition \( \Delta(a, b) = 0 \) is the condition

\[ (n - 1)^{n-1}a^n + n^n b^{n-1} = 0, \]

which we rewrite as

\[ (-\frac{a}{n})^n = \left( \frac{b}{(n - 1)} \right)^{n-1}. \]

This means precisely that \((-a/n, b/(n - 1))\) is of the form \((t^{n-1}, t)\) for a unique \( t \in \mathbb{F}_q \). So our sum is

\[ \sum_{t \in \mathbb{F}_q, x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n - nt^{n-1}x + (n - 1)t^l). \]

For \( t = 0 \), the inner sum becomes \( \sum_{x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n) \), which vanishes because \( n \) is odd. For \( t \neq 0 \), we use the fact that \( x^n - nt^{n-1}x + (n - 1)t^l \) is homogeneous in \( x, t \) of degree \( n \), so we write it as
\( t^n(X^n - nX + n - 1) \) with \( X := x/t \). The sum over \( t \neq 0 \) becomes

\[
\sum_{t \in \mathbb{F}^*} \chi_{2, \mathbb{F}_q}(t^n(X^n - nX + n - 1)),
\]

which is the product

\[
\left( \sum_{t \in \mathbb{F}^*_q} \chi_{2, \mathbb{F}_q}(t^n) \right) \left( \sum_{X \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(X^n - nX + n - 1) \right),
\]

in which the first factor vanishes (again because \( n \) is odd).

In fact, we have the following explanation of this vanishing.

**Lemma 7.2.** The cohomology groups \( H^i_c(\hat{\Lambda}_p^2 \mathbb{F}_p / \Delta_1, \mathcal{F}) \) all vanish.

**Proof.** The idea is simply to imitate, cohomologically, the argument given above.

We first define a sheaf \( \mathcal{F} \) on all of \( \Lambda^2 \) which agrees with our previously defined \( \mathcal{F} \) on \( \Lambda^2[1/\Delta] \) and whose trace function at any point \((a, b) \in \Lambda^2[\mathbb{F}_p] \) is

\[
\left( -\frac{1}{\sqrt{p}} \right) \sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(x^n + ax + b).
\]

For this, we consider the sheaf \( \mathcal{L}_{X \langle x^n + ax + b \rangle} \) on the \( \Lambda^3 \) of \((x, a, b)\), with the understanding that this sheaf has been extended by zero across the points where \( x^n + ax + b = 0 \). For the projection of \( \Lambda^3 \) onto \( \Lambda^2 \) given by \( pr(x, a, b) := (a, b) \), \( R^i pr_!(\mathcal{L}_{X \langle x^n + ax + b \rangle}) \) vanishes for \( i \neq 1 \) (check fibre by fibre).

The Tate-twisted sheaf \( R^i pr_!(\mathcal{L}_{X \langle x^n + ax + b \rangle})(1/2) \) is the desired \( \mathcal{F} \).

We wish to show that all the groups \( H^i_c(\hat{\Lambda}_p^2 \mathbb{F}_p / \Delta_1, \mathcal{F}) \) vanish. Using the excision long exact sequence

\[
\cdots \to H^i_c(\hat{\Lambda}_p^2 \mathbb{F}_p / \Delta_1, \mathcal{F}) \to H^i_c(\Lambda^2 \mathbb{F}_p, \mathcal{F}) \to H^i_c((\Delta = 0) \mathbb{F}_p, \mathcal{F}) \to \cdots
\]

we are reduced to showing the vanishing of all the groups \( H^i_c(\Lambda^2 \mathbb{F}_p, \mathcal{F}) \) and of all the groups \( H^i_c((\Delta = 0) \mathbb{F}_p, \mathcal{F}) \).

To show the vanishing of the groups \( H^i_c(\Lambda^2 \mathbb{F}_p, \mathcal{F}) \), we note first that, from the construction of \( \mathcal{F} \) as (a Tate twist of) the only non-vanishing \( R^i pr_!(\mathcal{L}_{X \langle x^n + ax + b \rangle}) \), namely the \( R^1 \), we have

\[
H^i_c(\Lambda^2 \mathbb{F}_p, \mathcal{F}) = H^{i+1}_c(\Lambda^3 \mathbb{F}_p, \mathcal{L}_{X \langle x^n + ax + b \rangle},(1/2))
\]

To show that these groups vanish, we use the projection \( pr_{1,2} \) of \( \Lambda^3 \) onto \( \Lambda^2 \) given by \((x, a, b) \mapsto (x, a)\).

For this projection, all the \( R^i(pr_{1,2})!\mathcal{L}_{X \langle x^n + ax + b \rangle} \) vanish, as one sees looking fibre by fibre (the cohomological version of summing over \( b \)).

To show that the groups \( H^i_c((\Delta = 0) \mathbb{F}_p, \mathcal{F}) \) all vanish, we use the construction of \( \mathcal{F} \) once again, this time to write

\[
H^i_c((\Delta = 0) \mathbb{F}_p, \mathcal{F}) = H^{i+1}_c(\Lambda^2 \mathbb{F}_p, \mathcal{L}_{X \langle x^n - nX + n - 1 \rangle}),
\]

where the \( \Lambda^2 \) in question is that of \((x, t)\). By excision on this \( \Lambda^2 \), it suffices to treat separately the open set \( \Lambda^1 \times \mathbb{G}_m \), coordinates \( x \) and \( t \), and the line \( t = 0 \). On this line, with coordinate \( x \), we are looking at the groups

\[
H^{i+1}_c(\Lambda^1 \mathbb{F}_p, \mathcal{L}_{X \langle x^n \rangle}),
\]

which all vanish. On the product \( \Lambda^1 \times \mathbb{G}_m \), we make the \((t, x/t)\) substitution to write our sheaf as the external tensor product of \( \mathcal{L}_{X \langle X^n - nX + n - 1 \rangle} \) on the first \( \Lambda^1 \) factor with \( \mathcal{L}_{X \langle t^n \rangle} \) on the \( \mathbb{G}_m \) factor. The vanishing then results from Kunneth, because on the second factor all the groups \( H^i_c((\mathbb{G}_m) \mathbb{F}_p, \mathcal{L}_{X \langle t^n \rangle}) \) vanish (again because \( n \) is odd).
Thanks to a marvelous formula of Davenport–Lewis, we do have square root cancellation for $M_{1,1}(\mathbb{F}_q)$.

**Lemma 7.3.** We have $M_{1,1}(\mathbb{F}_q) = 1 + O(1/q)$.

**Proof.** Davenport and Lewis prove (cf. [9, eqn (19), p. 55] or [10, Lemma 8]) that for any $n \geq 0$, we have
\[
\sum_{(a,b) \in \mathbb{A}^2(\mathbb{F}_q)} \left( \sum_{x \in \mathbb{F}_q} \chi_2(\mathbb{F}_q)(x^n + ax + b) \right)^2 = q^2(q - 1).
\]
The sum over $(a,b) \in \mathbb{A}^2(\mathbb{F}_q)$ with $\Delta = 0$ is, as we have seen above, the sum
\[
\sum_{t \in \mathbb{F}_q^*} \left( \sum_{x \in \mathbb{F}_q} \chi_2(\mathbb{F}_q)(tx^n - nx + n - 1) \right)^2 = (q - 1) \left( \sum_{x \in \mathbb{F}_q} \chi_2(\mathbb{F}_q)(x^n - nx + n - 1) \right)^2 = O(q^2).
\]
Thus, the sum over $(a,b) \in \mathbb{A}^2(\mathbb{F}_q)$ is $q^2(q - 1) + O(q^2)$. Dividing by $\# A^2[1/\Delta](\mathbb{F}_q) = q^2(q - 1)$, we find the asserted result. 

\[\blacksquare\]

8. A third example

We fix an even integer $n \geq 4$, and a prime $p$ not dividing $n(n - 1)$. We consider, in characteristic $p$, the two parameter family of hyperelliptic curves
\[y^2 = x^n + ax + b,
\]
over the open set of $\mathbb{A}^2$, parameters $(a,b)$, where the discriminant of $x^n + ax + b$, namely
\[\Delta = \Delta(a,b) := (n - 1)^{n-1} a^{n-1} + n^n b^{n-1},
\]
is invertible. For this family of curves, its $H^1$ along the fibres, Tate twisted by $1/2$, is a lisse sheaf $\mathcal{F}$ on $\mathbb{A}^2[1/\Delta]$ of rank $2g = n - 2$ which is pure of weight zero. Its trace function at a point $(a,b)$ with values in a finite extension $\mathbb{F}_q$ is given by
\[\text{Trace(Frob}_{(a,b),\mathbb{F}_q}, \mathcal{F}) = \left( -\frac{1}{\sqrt{q}} \right) \left( 1 + \sum_{x \in \mathbb{F}_q} \chi_2(\mathbb{F}_q)(x^n + ax + b) \right).
\]
One knows [8, Theorem 5.17 (1)] that for this $\mathcal{F}$, we have $G_{\text{geom}} = G_{\text{arith}} = Sp(n - 2)$. In particular, the standard representation is irreducible, and hence $M_{1,0} = 0$, i.e., $H^4_{\text{sp}}(\mathbb{A}^2[1/\Delta], \mathcal{F})$ vanishes. However, in contradistinction to the case when $n$ is odd, we have the following lemma.

**Lemma 8.1.** We have
\[M_{1,0}(\mathbb{F}_q) = -\frac{1}{\sqrt{q}} + O\left( \frac{1}{q} \right).
\]

**Proof.** Here the discriminant $\Delta(a,b)$ vanishes precisely when
\[(n - 1)^{n-1} a^{n-1} = n^n b^{n-1},
\]
in other words when \((a, b)\) is of the form \((a, b) = (nt^{n-1}, (n-1)t^n)\) for a unique \(t \in \mathbb{F}_q\). Thus, there are \(q\) points in \(\mathbb{A}^2(\mathbb{F}_q)\) at which \(\Delta\) vanishes. By definition, \(M_{1,0}(\mathbb{F}_q)\) is \((-1/\sqrt{q})(1/(q(q-1)))\) times the sum

\[
\sum_{(a, b) \in \mathbb{A}^2(\mathbb{F}_q)} \left( 1 + \sum_{x \in \mathbb{F}_q^*} \chi_{2, \mathbb{F}_q}(x^n + ax + b) \right).
\]

If this sum extended over all points \((a, b)\) in \(\mathbb{A}^2(\mathbb{F}_q)\), it would be \(q^2\) (from summing the term 1); the sum over all \((a, b, x)\) of \(\chi_{2, \mathbb{F}_q}(x^n + ax + b)\) vanishes (for each \((a, x)\), sum over \(b\)).

The sum over the \(\mathbb{F}_q\) points where \(\Delta\) vanishes is the sum

\[
\sum_{(t, x) \in \mathbb{A}^2(\mathbb{F}_q)} (1 + \chi_{2, \mathbb{F}_q}(x^n + nt^{n-1}x + (n-1)t^n))
\]

\[
= q + \sum_{(t, x) \in \mathbb{A}^2(\mathbb{F}_q)} \chi_{2, \mathbb{F}_q}(x^n + nt^{n-1}x + (n-1)t^n).
\]

In this second sum, the sum over the points \((0, x)\) is \(q - 1\) (because \(n\) is even). For each \(t \neq 0\), we write

\[
x^n + nt^{n-1}x + (n-1)t^n = t^n(X^n + nX + n - 1),
\]

with \(X := x/t\). Because \(n\) is even, for each \(t \neq 0\) the sum over \(x\) of \(\chi_{2, \mathbb{F}_q}(x^n + nt^{n-1}x + (n-1)t^n)\) is independent of \(t\), equal to the quantity

\[
\sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(x^n + nx + n - 1).
\]

So all in all, the sum over the \(\mathbb{F}_q\) points where \(\Delta\) vanishes is

\[
2q - 1 + (q - 1) \sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(x^n + nx + n - 1).
\]

So \(M_{1,0}(\mathbb{F}_q)\) is \((-1/\sqrt{q})(1/(q(q-1)))\) times the quantity

\[
q^2 - 2q + 1 - (q - 1) \sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(x^n + nx + n - 1).
\]

One checks easily that the polynomial \(x^n + nx + n - 1\) has no triple roots, and that its unique double root is \(x = -1\). We readily compute that

\[
x^n + nx + n - 1 = (x + 1)^2 P_{n-2}(x), \quad P_{n-2}(x) = x^{n-2} - 2x^{n-3} + 3x^{n-4} + \cdots + (n-1).
\]

Thus \(P_{n-2}(x)\) is square free. As \(x^n + nx + n - 1\) vanishes at \(x = -1\), we have

\[
\sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(x^n + nx + n - 1) = \sum_{x \in \mathbb{F}_q, x \neq -1} \chi_{2, \mathbb{F}_q}(P_{n-2}(x)).
\]

The value of \(P_{n-2}(x)\) at \(x = -1\) is \(n(n-1)/2\) (L’Hôpital’s rule), so we get

\[
\sum_{x \in \mathbb{F}_q, x \neq -1} \chi_{2, \mathbb{F}_q}(P_{n-2}(x)) = -1 - \chi_{2, \mathbb{F}_q} \left( \frac{n(n-1)}{2} \right) - S_{n-2}(\mathbb{F}_q)
\]

with

\[
S_{n-2}(\mathbb{F}_q) = - \left( 1 + \sum_{x \in \mathbb{F}_q} \chi_{2, \mathbb{F}_q}(P_{n-2}(x)) \right).
\]

Here \(S_{n-2}(\mathbb{F}_q)\) is the trace of \(\text{Frob}_{\mathbb{F}_q}\) on \(H^1\) of the complete non-singular model of the hyperelliptic curve \(y^2 = P_{n-2}(x)\) of genus \((n - 4)/2\). In particular, \(S_{n-2}(\mathbb{F}_q) = O(\sqrt{q})\).
Thus, \( M_{1,0}(\mathbb{F}_q) \) is \((-1/\sqrt{q})(1/(q(q - 1)))\) times the quantity
\[
(q - 1)^2 - (q - 1)(-1 - \chi_{2, \mathbb{F}_q}(n(n - 1)/2) - S_{n-2}(\mathbb{F}_q)) = q(q - 1) + O(q^{3/2}).
\]
Thus,
\[
M_{1,0}(\mathbb{F}_q) = -\frac{1}{\sqrt{q}} + O\left(\frac{1}{q}\right).
\]

**Lemma 8.2.** The cohomology group \( H^2_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{F}) \) vanishes, but the weight 3 part of \( H^3_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{F}) \) is one-dimensional, and \( \text{Frob}_{\mathbb{F}_q} \) acts on it as \( q^{3/2} \).

**Proof.** The vanishing of the \( H^2_c \) is the fact that \( M_{1,0} = 0 \). By the Lefschetz trace formula, \( M_{1,0}(\mathbb{F}_q) \) is \((1/(q(q - 1)))\) times the two term sum
\[
-\text{Trace}(\text{Frob}_{\mathbb{F}_q}|H^2_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{F})) + \text{Trace}(\text{Frob}_{\mathbb{F}_q}|H^2_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{F})).
\]
From our estimate that \( M_{1,0}(\mathbb{F}_q) = -1/\sqrt{q} + O(1/q) \), we see that this sum is \(-q^{3/2} + O(q)\). As \( H^3_c \) is mixed of weight \( \leq i \), we get the asserted result.

**Remark 8.3.** The reader may be concerned by the apparent sign ambiguity in the statement above, that the eigenvalue of \( \text{Frob}_{\mathbb{F}_q} \) on the weight three part of \( H^3_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{F}) \) is \( q^{3/2} \). Here is a more intrinsic way to say this. Instead of \( \mathcal{F} \), consider the sheaf \( \mathcal{H} \) which is the \( H^1 \) along the fibres of our family of curves \( y^2 = x^n + ax + b \). In terms of \( \mathcal{H} \), we defined \( \mathcal{F} \) to be the one-half Tate twist \( \mathcal{H}(1/2) \), which involved a choice of \( \sqrt{\mathbb{F}} \) and a consequent determination of \( \sqrt{\mathbb{F}} \). The sheaf \( \mathcal{H} \) is pure of weight one, the cohomology group \( H^2_c(\mathbb{A}^2_{\mathbb{F}_q}[1/\Delta], \mathcal{H}) \) is mixed of weight less than or equal to 4, and what is being asserted is that its weight 4 part is one-dimensional, with \( \text{Frob}_{\mathbb{F}_q} \) acting as \( q^2 \).

Exactly as in lemma 7.3, the Davenport–Lewis formula gives square root cancellation for \( M_{1,1}(\mathbb{F}_q) \).

**Lemma 8.4.** We have \( M_{1,1}(\mathbb{F}_q) = 1 + O(1/q) \).

**Proof.** By definition, \( M_{1,1}(\mathbb{F}_q) \) is \((1/q)(1/(q(q - 1)))\) times the sum
\[
\sum_{(a,b)\in \mathbb{A}^2[1/\Delta]\langle\mathbb{F}_q\rangle} \left(1 + \sum_{x\in \mathbb{F}_q} \chi_2, \mathbb{F}_q(x^n + ax + b)\right)^2.
\]
Expanding the square, this is
\[
q(q - 1) + 2 \sum_{(a,b)\in \mathbb{A}^2[1/\Delta]\langle\mathbb{F}_q\rangle} \sum_{x\in \mathbb{F}_q} \chi_2, \mathbb{F}_q(x^n + ax + b) + \sum_{(a,b)\in \mathbb{A}^2[1/\Delta]\langle\mathbb{F}_q\rangle} \left(\sum_{x\in \mathbb{F}_q} \chi_2, \mathbb{F}_q(x^n + ax + b)\right)^2.
\]
If these last two summations extended over all \((a,b)\in \mathbb{A}^2(\mathbb{F}_q)\), the first would vanish, and the second would be \( q^2(q - 1) \) by the Davenport–Lewis formula. So our sum is
\[
q(q - 1) - 2 \sum_{(a,b)\in \mathbb{A}^2(\mathbb{F}_q), \Delta(a,b)=0} \sum_{x\in \mathbb{F}_q} \chi_2, \mathbb{F}_q(x^n + ax + b)
\]
\[
+ q^2(q - 1) - \sum_{(a,b)\in \mathbb{A}^2(\mathbb{F}_q), \Delta(a,b)=0} \left(\sum_{x\in \mathbb{F}_q} \chi_2, \mathbb{F}_q(x^n + ax + b)\right)^2.
\]

The summands for \((a,b) = (0,0)\) are \( q - 1 \) and \( (q - 1)^2 \), respectively, so both are \( O(q^2) \). For each of the \( q - 1 \) summands with \((a,b) \neq (0,0)\) but \( \Delta(a,b) = 0 \), the polynomial \( x^n + ax + b \) has precisely
$n - 1$ roots, of which $n - 2 > 0$ are simple roots. In particular, this polynomial is not geometrically a square, so the Weil bound gives
\[
\left| \sum_{x \in \mathbb{F}_q} \chi_{2,\mathbb{F}_q}(x^n + ax + b) \right| \leq (n - 1)\sqrt{q}.
\]
So all in all, the total contribution of the $\Delta = 0$ terms is $O(q^2)$, and our sum over $\mathbb{A}^2[1/\Delta](\mathbb{F}_q)$ is $q^2(q - 1) + O(q^2)$. Dividing through by $q^2(q - 1)$ gives the asserted result.

References