Locally covariant quantum field theory and the problem of formulating the same physics in all space–times

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The framework of locally covariant quantum field theory is discussed, motivated in part using ‘ignorance principles’. It is shown how theories can be represented by suitable functors, so that physical equivalence of theories may be expressed via natural isomorphisms between the corresponding functors. The inhomogeneous scalar field is used to illustrate the ideas. It is argued that there are two reasonable definitions of the local physical content associated with a locally covariant theory; when these coincide, the theory is said to be dynamically local. The status of the dynamical locality condition is reviewed, as are its applications in relation to (i) the foundational question of what it means for a theory to represent the same physics in different space–times and (ii) a no-go result on the existence of natural states.

1. Introduction

Quantum field theory (QFT) was originally developed as a theory of particle physics in Minkowski space, in which the Poincaré symmetry group plays a key role. It appears in the practical computations of Lagrangian QFT, with pervasive use of momentum space techniques, in the classification of particle species via representation theory, and also in axiomatic approaches to the subject in which a unitary Hilbert space representation of the Poincaré group and an invariant vacuum vector take centre stage [1,2].

Our universe, however, is not Minkowski space, but instead is well described by a curved space–time; accordingly, much work has been devoted to the extension of QFT to such backgrounds. Even where the starting point is a classical Lagrangian, for which minimal coupling can suggest a natural extension to
curved space–time, the formulation of the QFT raises numerous conceptual issues which have gradually been solved over the past 40 years (e.g. [3]). In the process, reliance on space–time symmetries, a preferred vacuum state and even the notion of particles have all had to be jettisoned. The axiomatic setting faces an additional problem. When one strips out the axioms in the Wightman–Gårding or Haag–Kastler–Araki frameworks [1,2] that relate to the Poincaré group, one is left with a rather meagre residue. Moreover, the aim of QFT in curved space–time is to permit the formulation of, in some sense, the ‘same’ physical theory in arbitrary (sufficiently well-behaved) space–time backgrounds. What axiom can be given to capture this idea of formulating the same physics in all space–times (a phrase we will abbreviate as SPASs)?

Perhaps for this reason, the axiomatic development of QFT in curved space–times has been comparatively underdeveloped, in contrast both to axiomatic approaches in Minkowski space and to the investigation of concrete QFT models in curved space–time. The main purpose of this contribution is to present the axiomatic framework of local covariance for physical theories on general space–time backgrounds, introduced by Brunetti et al. [4], and to describe the extent to which it addresses the problem of SPASs [5]. The ideas will be illustrated using the inhomogeneous scalar field model, following the recent treatment [6]. There are two side themes: first, the absence of any viable notion of natural state compatible with local covariance to replace the Minkowski vacuum state—a general model-independent result proved in [5]—will be described, but in addition a new and self-contained argument will be given for the inhomogeneous scalar field; second, I will attempt to motivate some of the ideas presented as a constructive use of ‘ignorance principles’.

2. Ignorance principles

The task of science is to reduce the complexity of the real world to basic ideas and principles from which progressively more detailed models of reality can be built. The success of this endeavour is all the more remarkable, given that there are many things about the world that we do not know, and moreover many things that we cannot know. Of course, science aims to remedy contingent ignorance, but even here progress has been greatly assisted by a fortuitous separation of scales in the structure of matter, permitting the development of fluid dynamics, for example, without the need to understand atoms, and of chemistry without the need to understand quarks.

On the other hand, it is much less obvious that science can proceed at all in a world where there are things that cannot be known. Imagine a world in which influences of which we had no control or knowledge were at work, permeating physical phenomena on all scales and without restriction. That world would appear capricious, perhaps not even displaying statistical regularity despite the best efforts of experimentalists. A prerequisite for the success of science, therefore, would appear to be the principle: Anything that we cannot know may be neglected, which we dignify with the title of the ignorance meta-principle.

The unavoidable ignorance to be discussed here is imposed by the bounded speed of propagation of all influences and signals. To the best of our knowledge, this is a law of nature, which absolutely denies us direct access to regions of space–time that are space-like-separated from our own. Application of the ignorance meta-principle leads to familiar principles of locality: that observations in a space–time region $O$ should be independent of any made in the causal complement of $O$, and that the equations of motion should be consistent with the finite speed of light. Each can be regarded as a consistency mechanism in the theory that maintains the ignorance of experimenters in $O$ of the unknowable world beyond. To these familiar principles, one can add two more ideas. Namely, the description of local physics in $O$ should be independent of space–time or other background structures outside $O$ to the extent that (a) the description would not change if there was no space–time beyond $O$ and (b) it would also be unchanged if the background structures were changed outside $O$. Here, we have in mind that $O$ is causally convex, i.e. every causal curve with endpoints in $O$ lies entirely within $O$.

In stating (a) and (b), we have switched from statements about the causal complement of $O$ to statements about its complement. Certainly no controlled experiment in $O$ can have any
direct access to the complement, as a result of causal convexity. The causal complement of \( O \) is completely unknowable (from \( O \)), while the causal future and past of \( O \) are each only partially knowable, because information is lost from the causal past to the causal complement, which also supplies information to the causal future. It seems reasonable to apply the ignorance principle also in this case, because the description of local physics in \( O \) should not require a precise knowledge of the prior history of the world. Certainly, this prior history may contribute to the determination of the state of physical systems in \( O \) and can be passively observed from within \( O \) (e.g. astronomical observations), but the local physical processes in \( O \) should be independent of it, and should be susceptible to experiments in which this background is controlled for or screened out, and which can be repeated at a later stage in the history of the world provided the local conditions are recreated. In any case, the main aim of this discussion is to motivate ideas that can be turned into precise technical statements in the framework of local covariance to yield definite consequences. In this way, we are making constructive use of our ignorance to guide the formulation of physical theories. We now turn to the formal development of the framework.

3. Local covariance

(a) General setting

The discussion above provides a motivation for the formulation of locally covariant physical theories introduced by Brunetti et al. [4] and further developed by Fewster & Verch [5]. The underlying ideas first appeared in [7,8].

The fundamental idea is that a physical theory \( \mathcal{A} \) should be formulated on general space–time backgrounds, so that to each background \( M \) there is a mathematical object \( \mathcal{A}(M) \) describing the theory \( \mathcal{A} \) in \( M \). From the perspective of QFT in curved space–times, or general relativity, it is natural to aim for a description on as wide a class of space–times as possible. However, it may also be motivated from ignorance of the precise geometry or topology outside the region of immediate interest: the description ought to be valid regardless of how the space–time is continued (or whether it continues at all).

The space–times of interest should form the objects of a category \( \mathcal{B}_k\text{Grnd} \), in which the morphisms indicate allowed space–time embeddings: \( \psi : M \to N \) in \( \mathcal{B}_k\text{Grnd} \) indicates that \( \psi \) is a way of embedding space–time \( M \) as a sub-space–time of \( N \); equivalently, we may think of \( \psi \) as specifying a particular continuation of \( M \) as a space–time, with the idea that physics in \( M \) should be indistinguishable from physics within its image in \( N \).

Successive space–time embeddings \( \varphi : L \to M \) and \( \psi : M \to N \) naturally provide a means of embedding \( L \) in \( N \), which is given by the composite morphism \( \psi \circ \varphi \). Of course, every space–time \( M \) can be regarded as a sub-space–time of itself in a trivial way; this corresponds to the identity morphism \( \text{id}_M \) of \( M \). As we intend that a morphism from \( M \) to \( N \) embeds \( M \) as a sub-space–time of \( N \) it is reasonable to demand that all morphisms in \( \mathcal{B}_k\text{Grnd} \) are monic; that is, \( \psi \circ \varphi_1 = \psi \circ \varphi_2 \) implies that \( \varphi_1 = \varphi_2 \).

The precise specification of the category \( \mathcal{B}_k\text{Grnd} \) can vary. The main example studied to date is the category \( \text{loc} \), whose objects are oriented and time-oriented globally hyperbolic space–times of fixed (but arbitrary) dimension \( n \geq 2 \). Each object \( M \) is thus a quadruple \( M = (\mathcal{M}, g, o, t) \) in which \( \mathcal{M} \) is a smooth \( n \)-dimensional orientable manifold, equipped with smooth Lorentz metric \( g \) of signature \( + - \cdots - \), and orientation \( o \) (a component of the set of smooth nowhere vanishing \( n \)-form fields) and a time orientation \( t \) (a component of the cone of smooth nowhere vanishing time-like 1-form fields), so that the condition of global hyperbolicity holds: there are no closed causal curves, and all sets of the form \( \mathcal{I}^+_M(p) \cap \mathcal{I}^-_M(q) \) are compact for \( p, q \in \mathcal{M} \).

A morphism \( \psi : (\mathcal{M}, g, o, t) \to (\mathcal{M}', g', o', t') \) in \( \text{loc} \) is given by a smooth embedding \( \psi : \mathcal{M} \to \mathcal{M}' \) of the underlying manifolds that is isometric and preserves the orientation and time-orientation

\[
\psi^*g' = g, \quad \psi^*o' = o, \quad \psi^*t' = t,
\]
and such that the image $\psi(M)$ is a causally convex subset of the codomain space–time. In particular, $\psi$ is injective as a function and monic as a morphism in $\text{BkGrnd}$. Depending on the precise application, one could allow for additional background structure, such as background source fields (e.g. [6,9]) or more general bundle structures (e.g. [10]). Alternatively, one could equally allow for different models of space–time structure, for example, taking $\text{BkGrnd}$ to be a category of discrete causal sets with morphism injective maps respecting causal order and with a causally convex image.

The mathematical objects describing the given theory in specific space–times are also required to be objects within a category $\text{Phys}$. Here, the interpretation of a morphism $f : P \rightarrow Q$ in $\text{Phys}$ is that $f$ embeds the physical system $P$ as a subsystem of $Q$; accordingly, we assume that all morphisms in $\text{Phys}$ be monic (in some applications this has been relaxed, e.g. [9,11–13]). The specification of $\text{Phys}$ reflects the type of physical theory under consideration. Commonly employed examples include $\text{Sympl}$, the category of symplectic real vector spaces with symplectic maps as morphisms (e.g. to model linear dynamical systems), or the category $\mathcal{A}lg$ of unital $*$-algebras with unit-preserving injective $*$-homomorphisms as morphisms, which would be a natural setting for a description of quantum theory in terms of local algebras of fields and/or observables. This setting is deliberately general and allows for many variations, e.g. restricting to the subcategory $\mathcal{C}^*\mathcal{A}lg$ of $\mathcal{A}lg$, whose objects are required to be $\mathcal{C}^*$-algebras. It should be clear that we are describing a framework for physical theories, rather than any particular theory.

Returning to the description of a theory $\mathcal{A}$, let us consider a morphism $\psi : M \rightarrow N$ in $\text{BkGrnd}$. For each of $M$ and $N$, there should be a mathematical object $\mathcal{A}(M)$ and $\mathcal{A}(N)$ of $\text{Phys}$. Now our basic idea is that $\psi$ embeds $M$ in $N$ in such a way that physics in $M$ ought to be indistinguishable from physics in the image $\psi(M)$ in $N$. Thus, there should be a way of embedding $\mathcal{A}(M)$ as a physical subsystem of $\mathcal{A}(N)$, represented by a morphism $\mathcal{A}(\psi) : \mathcal{A}(M) \rightarrow \mathcal{A}(N)$, which protects the ignorance of an experimenter in $M$ as to whether the experimental region is really just a portion of the larger space–time $N$. There may, of course, be many ways of embedding $\mathcal{A}(M)$ in $\mathcal{A}(N)$, but our assumption is that the theory should specify one of these. We require

1. $\mathcal{A}(\text{id}_M) = \text{id}_{\mathcal{A}(M)}$ for every $M$; i.e. trivial embeddings of backgrounds correspond to trivial subsystem embeddings and
2. for successive embeddings $\varphi : L \rightarrow M$ and $\psi : M \rightarrow N$, the subsystem embeddings obey

$$\mathcal{A}(\psi) \circ \mathcal{A}(\varphi) = \mathcal{A}(\psi \circ \varphi),$$

i.e. the composite of the two subsystem embeddings should be the subsystem embedding of the composite space–time embedding.

The second part again reflects the ignorance principle: if one cannot detect whether space–time has been extended at all, one should certainly not be able to distinguish whether it was extended in one step or in two successive stages. These demands together amount to the following definition.

**Definition 3.1.** A locally covariant physical theory is a covariant functor $\mathcal{A} : \text{BkGrnd} \rightarrow \text{Phys}$. 

(b) **Example: scalar field with sources**

As an example, we describe a simple theory: the Klein–Gordon equation with an external source. This model was studied in some detail recently in [6]—the presentation here is a streamlined account and most details are suppressed. The discussion at the end of §3d and theorem 3.6 are new.

As the category of backgrounds we take a category $\mathcal{Loc}$ whose objects are pairs $(M, J)$ where $M$ is an object of $\mathcal{M}$ and $J \in \mathcal{C}^\infty(M)$ is a smooth background field. A morphism from $(M, J)$ to $(M', J')$ is defined by a morphism $\psi : M \rightarrow M'$ in $\mathcal{Loc}$ that also obeys $\psi^* J' = J$. 
The classical theory we wish to describe has Lagrange density

\[ \mathcal{L}(M,J) = \rho_g \left( \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{2} m^2 \phi^2 - f \phi \right), \tag{3.1} \]

where \( \rho_g \) is the canonical volume density induced by \( g \). The corresponding equation of motion is

\[ (\Box_M + m^2) \phi + J = 0 \]

on background \((M, J)\). As \( M \) is globally hyperbolic, there are unique advanced (−) and retarded (+) Green functions \( E_M^\pm : C_0^\infty(M) \rightarrow C^\infty(M) \) such that, for each \( f \in C_0^\infty(M) \), \( \phi = E_M^\pm f \) is the unique smooth solution to \(((\Box_M + m^2) \phi) = f \) with support in \( f(M) \).

For the QFT, we define on each \((M, J)\) a unital \(*\)-algebra \( \mathcal{A}(M, J) \), with unit \( 1_{\mathcal{A}(M, J)} \), generated by elements \( \Phi(M, J)(f) \ (f \in C_0^\infty(M)) \) and subject to the relations

- complex linearity of \( f \mapsto \Phi(M, J)(f) \);
- hermiticity: \( \Phi(M, J)(f)^* = \Phi(M, J)(f) \) for all \( f \in C_0^\infty(M) \);
- field equation:

\[ \Phi(M, J)((\Box_M + m^2)f) + \left( \int_M f \, d\text{vol}_M \right) 1_{\mathcal{A}(M, J)} = 0 \]

for all \( f \in C_0^\infty(M) \); and
- commutation relation

\[ [\Phi(M, J)(f), \Phi(M, J)(f')] = i E_M(f, f') 1_{\mathcal{A}(M, J)} \]

for all \( f, f' \in C_0^\infty(M) \), where

\[ E_M(f, f') = \int_M f(E_M^- - E_M^+) f' \, d\text{vol}_M. \]

The interpretation of this algebra is that the generator \( \Phi(M, J)(f) \) is to be thought of as the quantum field smeared against test function \( f \). The form of the commutation relations may be motivated as the Dirac quantization of the classical Peierls bracket of the corresponding smeared classical fields.

This completes the construction of the algebra in each space–time. Given a morphism \( \psi : (M, J) \rightarrow (M', J') \) in \( \text{LocStr} \), we may define a map \( \mathcal{A}(\psi) : \mathcal{A}(M, J) \rightarrow \mathcal{A}(M', J') \) by

\[ \mathcal{A}(\psi) \Phi(M, J)(f) = \Phi(M', J')(\psi f) \]

for \( f \in C_0^\infty(M) \), where \( \psi_* \) denotes the push-forward of compactly supported functions

\[ (\psi_* f)(p) = \begin{cases} f(\psi^{-1}(p)) & p \in \psi(M) \\ 0 & \text{otherwise.} \end{cases} \]

The above expression defines \( \mathcal{A}(\psi) \) on the generators of \( \mathcal{A}(M, J) \) and, because it is compatible with the relations imposed in the two algebras, it extends to a unit-preserving \(*\)-homomorphism, which is injective (\( \mathcal{A}(M, J) \) can be shown to be a simple algebra). Thus, \( \mathcal{A}(\psi) \) is a morphism in \( \text{Alg} \).

It is clear from the definition and properties of the push-forward that the functorial conditions \( \mathcal{A}(\text{id}(M, J)) = \text{id}_{\mathcal{A}(M, J)} \) and \( \mathcal{A}(\psi \circ \varphi) = \mathcal{A}(\psi) \circ \mathcal{A}(\varphi) \) are met. Accordingly, we have defined the theory as a functor \( \mathcal{A} : \text{LocStr} \rightarrow \text{Alg} \).

The definition of the morphisms \( \mathcal{A}(\psi) \) seems almost an afterthought, but it is actually crucial to the definition of the theory. Indeed, the algebras by themselves do not really specify the theory at all. To see this, fix a background \((M, J)\) and also choose a particular real-valued solution
φ ∈ C∞(M) to the classical equation of motion. Now define a map κφ : ℰ(M, J) → ℰ(M, 0) by

\[ κφΦ_{(M,J)}(f) = Φ_{(M,0)}(f) + \left( \int_M f\phi \, d\text{vol}_M \right) 1_{{ℰ}(M,0)} \]

(3.2)

and κφ1_{{ℰ}(M,J)} = 1_{{ℰ}(M,0)}. One may check that this map on generators is compatible with the relations of the two algebras and therefore extends to a morphism in Alg; in fact, it is an isomorphism, with inverse defined on generators by

\[ κφ^{-1}Φ_{(M,0)}(f) = Φ_{(M,J)}(f) - \left( \int_M f\phi \, d\text{vol}_M \right) 1_{{ℰ}(M,J)}. \]

As the algebras ℰ(M, J) and ℰ(M, 0) are isomorphic, they carry no specific information about the background source f. Note, however, that κφ depends on the special choice of a particular solution φ, and there is no canonical way of choosing such a solution in a general background.

Developing this point a bit further, let λ ∈ ℝ and define a functor \( Z_λ : \text{LocSt} \to \text{LocSt} \) so that \( Z_λ(M, J) = (M, λJ) \) and so that, if \( ψ : (M, J) \to (M', J') \), then \( Z_λ(ψ)(M, λJ) \to (M', λJ') \) has the same underlying map as ψ. Then the functor \( Z_λ \) := ℰ(M, J) → Alg is a new theory, which assigns algebra ℰλ(M, J) = ℰ(M, λJ) to the background (M, J); namely, ℰλ is the theory of the inhomogeneous field with a coupling strength λ, corresponding to classical field equation \((□_M + m^2)φ + λJ = 0 \) on background (M, J). For λ ≠ μ, we would expect that the theories ℰλ and ℰμ should represent different physics; however, the arguments above show that ℰλ(M, J) and ℰμ(M, J) are isomorphic for each background. Therefore, it is the specification of the morphisms of the theories, rather than the objects (algebras, in this case) that distinguishes them. We return to this point in §3d.

The theory ℰ determines, on each background, the algebra of smeared fields of the inhomogeneous scalar field, suitable elements of which are observables. To complete the physical description, we need to specify allowed states—our discussion is based on [4,14]. A state space for \( A ∈ \text{Alg} \) is a subset \( S \) of linear functionals \( ω \) on \( A \) that are normalized \((ω(1_A) = 1)\), positive \((ω(σA) > 0 \text{ for all } A ∈ A)\) and so that \( S \) is closed under convex linear combinations. As usual, \( ω(A) \) is interpreted as the expectation value of observable \( A \) in state \( ω \)—using the GNS theorem, each state induces a Hilbert space representation \( π_ω \) of \( A \) on a Hilbert space \( ℌ_ω \), with a distinguished vector \( Ω_ω \in ℌ_ω \) so that \( ω(A) = \langle Ω_ω \mid π(A)Ω_ω \rangle \) for all \( A ∈ A \), recovering the familiar Born probability rule. In particular, the set of all states \( A^+_{1,2} \) of \( A \) is a state space.

We may now introduce a category \( \text{AlgSts} \), whose objects are pairs \((A, S)\), where \( A ∈ \text{Alg} \) and \( S \) is a state space for \( A \). A morphism in \( \text{AlgSts} \) between objects \((A, S) \) and \((B, T)\) is an \( \text{Alg} \)-morphism \( α : A \to B \) with the additional property \( α^∗T \subseteq S \), where \( α^* \) is the dual map to \( α \). Composition of morphisms in \( \text{AlgSts} \) is inherited from \( \text{Alg} \).

Our inhomogeneous scalar field theory ℰ may be augmented to a theory with values in \( \text{AlgSts} \) in various ways. The simplest is to define \( \tilde{ℰ} : \text{LocSt} \to \text{AlgSts} \) so that \( \tilde{ℰ}(M, J) = (ℰ(M, J), ℰ(M, J)^+_{1,2}) \) for each object \((M, J)\) of \( \text{LocSt} \), and taking \( \tilde{ℰ}(ψ) \) to be the morphism induced by \( ℰ(ψ) \) for each morphism \( ψ : (M, J) \to (M', J') \) in \( \text{LocSt} \) (note that \( ℰ(ψ)^* \) maps any state of \( ℰ(M', J') \) to a state of \( ℰ(M, J) \)). A more interesting possibility is to equip each \( ℰ(M, J) \) with the corresponding set of Hadamard states, which are the standard choice of physically acceptable states of the scalar field (e.g. [3]).

1For example, let \( A = Φ_{(M,J)}((□_M + m^2)φ) + \left( \int_M f\phi \, d\text{vol}_M \right) 1_{{ℰ}(M,J)} \). Then our definitions give

\[ κφA = Φ_{(M,0)}((□_M + m^2)φ) + \left( \int_M f\phi \, d\text{vol}_M \right) 1_{{ℰ}(M,0)} + \left( \int_M f\phi \, d\text{vol}_M \right) 1_{{ℰ}(M,0)} = 0 \]

using the relations in ℰ(M, 0) and the field equation obeyed by φ, which is consistent with the fact that \( A = 0 \) by the relations in ℰ(M, J).

2By the time-slice property (discussed later) there are isomorphisms ℰ(M, J) = ℰ(M', J) whenever the Cauchy surfaces of \( M \) and \( M' \) can be related by an orientation-preserving diffeomorphism.

3For simplicity of presentation, we suppress a further condition often imposed on state spaces: namely that \( S \) should be closed under operations induced by \( A \), i.e. to each \( ω ∈ S \) and \( B ∈ A \) with \( ω(B^*B) > 0 \), the state \( \omega_B(\hat{A}) := ω(B^*AB)/ω(B^*B) \) is also an element of \( S \).
Definition 3.2. A state \( \omega \) on \( \mathcal{A}(M, J) \) is said to be Hadamard if the corresponding two-point function \( W^{(2)}_\omega : C_0^\infty(M) \times C_0^\infty(M) \to \mathbb{C} \) defined by \( W^{(2)}_\omega(f, f') = \omega(\Phi(M, J)(f)\Phi(M, J)(f')) \) is a distribution in \( \mathcal{D}'(M \times M) \) with wavefront set

\[
WF(W^{(2)}_\omega) \subset N^- \times N^+,
\]

where \( N^+/N^- \) is the bundle of future/past-directed null covectors on \( M \). The set of all Hadamard states on \( \mathcal{A}(M, J) \) will be denoted \( S(M, J) \).

It would take us too far from our main purpose to give the definition of the wavefront set here (see [15] for details) but the main points are that:

- the wavefront set \( WF(u) \) of a distribution \( u \in \mathcal{D}'(X) \) on manifold \( X \) is a subset of the cotangent bundle \( T^*X \) encoding the singular structure of \( u \)—in particular smooth distributions have empty wavefront sets;
- under pull-backs by smooth functions the wavefront set obeys \( WF(\kappa^*u) \subset \kappa^*WF(u) \);
- in the QFT context, the wavefront set condition (3.3) on the two-point function is sufficient to fix the wavefront sets of all \( n \)-point functions exactly (combining [16, Prop. 6.1] and [17]) and also ensures that the two-point function differs from the ‘Hadamard parametrix’ by a smooth function [18]; and
- the form of the wavefront set condition recalls the fact that Minkowski two-point functions are positive frequency in the first variable and negative frequency in the second.\(^4\)

It is easily seen that \( S(M, J) \) is a state space for \( \mathcal{A}(M, J) \). Now consider a morphism \( \psi : (M, J) \to (M', J') \) and a Hadamard state \( \omega \in S(M', J') \). Then the \( n \)-point functions of \( \omega \) and \( \mathcal{A}(\psi)^*\omega \) are related by

\[
W^{(n)}_{\mathcal{A}(\psi)^*\omega}(f_1, \ldots, f_n) = (\mathcal{A}(\psi)^*\omega)(\Phi(M, J)(f_1) \cdots \Phi(M, J)(f_n))
\]

\[
= \omega((\mathcal{A}(\psi^*\Phi(M, J)(f_1)) \cdots (\mathcal{A}(\psi^*\Phi(M, J)(f_n)))
\]

\[
= \omega(\Phi(M', J')(\psi f_1) \cdots \Phi(M', J')(\psi f_n)) = W^{(n)}_{\omega}(\psi f_1, \ldots, \psi f_n),
\]

i.e. \( W^{(n)}_{\mathcal{A}(\psi)^*\omega} = (\psi \times \cdots \times \psi)^*W^{(n)}_{\omega} \), from which it follows that \( \mathcal{A}(\psi)^*\omega \) is Hadamard. Hence, setting \( \mathcal{A}(M, J) = (\mathcal{A}(M, J), S(M, J)) \), \( \mathcal{A}(\psi) \) induces a morphism \( \mathcal{A}(\psi) : \mathcal{A}(M, J) \to \mathcal{A}(M', J') \). This defines a new theory \( \mathcal{A} : \text{LocSts} \to \mathcal{A}(M, J) \).

One might reasonably wonder how small a state space can be: can we choose a state space consisting of a single state \( \omega_{(M, J)} \) for each background, so that \( \mathcal{A}(\psi)^*\omega_{(M, J)} = \omega_{(M, J)} \) for every morphism \( \psi : (M, J) \to (M', J') \)? QFT in Minkowski space is so tightly framed around the vacuum state that it is only natural to seek a replacement in general space–time backgrounds. However, as will be discussed later, such natural states can be ruled out under general circumstances for QFTs. This has two consequences for physics: First, the particle interpretation of the theory is based on excitations of ‘the vacuum’, and so the loss of a preferred state also means the loss of a preferred notion of particles. Second, any procedure that does define a state in all space–time, such as a path integral prescription (setting aside the difficulties of making this precise) or the recently proposed construction [19], must depend in a non-local way on the space–time including those portions outside the experimental region controlled by an observer. It would seem to run counter to the ignorance meta-principle to ascribe operational significance to such a state. Further discussion on these lines appears in [20, §5].

\(^4\)The correspondence between ‘positive frequency’ and \( N^- \) is an unfortunate by-product of the standard conventions for Fourier transform, used in [15]. Radzikowski [18] and some of the other literature use non-standard conventions to remove this issue.
(c) Relations between theories

The example of the theories $\mathcal{A}_\lambda$ shows that physical equivalence of two theories $\mathcal{A}, \mathcal{B} : \text{BkGrnd} \to \text{Phys}$ is not simply a matter of the existence of isomorphisms between $\mathcal{A}(M)$ and $\mathcal{B}(M)$ for each background $M$.

**Definition 3.3.** A theory $\mathcal{A} : \text{BkGrnd} \to \text{Phys}$ is a subtheory of $\mathcal{B} : \text{BkGrnd} \to \text{Phys}$ if there is a natural transformation $\zeta : \mathcal{A} \to \mathcal{B}$. The theories are equivalent if there is a natural isomorphism between them.

Recall that a natural transformation between functors $\mathcal{A}$ and $\mathcal{B}$ is a collection of natural isomorphisms $\zeta_M : \mathcal{A}(M) \to \mathcal{B}(M)$ for each $M \in \text{BkGrnd}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{A}(M) & \xrightarrow{\zeta_M} & \mathcal{B}(M) \\
\mathcal{A}(N) & \xrightarrow{\zeta_N} & \mathcal{B}(N)
\end{array}
$$

commutes whenever $\psi : M \to N$ is a morphism in $\text{BkGrnd}$. In other words, the operations of passing between space–times and passing between theories must commute. For $\zeta$ to be a natural isomorphism, each $\zeta_M$ must be a Phys-isomorphism. In the case where $\mathcal{A}$ and $\mathcal{B}$ coincide, the natural automorphisms of $\mathcal{A}$ turn out to have a physically natural interpretation: they are the global gauge transformations of the theory [14].

In theorem 3.6, we will show that the theories $\mathcal{A}_\lambda$ are inequivalent for distinct $\lambda \in \mathbb{R}$. As a mild digression, we show that this rules out the existence of natural states in the theory. For suppose that the theory $\mathcal{A}_\lambda$ admits a natural state $(\omega_{(M,J)}, \omega_{(M,J)})_{\text{elad}\mathcal{A}_\lambda}$ for some $\lambda \neq 0$. In each background $(M, J) \in \text{elad}\mathcal{A}_\lambda$, the one-point function of the natural state $W_{(M,J)}^{(1)}(f) = \omega_{(M,J)}(\Phi_{(M,J)}(f))$ solves the classical field equation in the sense that

$$
W_{(M,J)}^{(1)}\left(\Box_M + m^2\right)f + \int_M J f \, d\text{vol}_M = 0.
$$

By analogy with (3.2), we may define an Alg-isomorphism $\kappa_{(M,J)} : \mathcal{A}_\lambda(M, J) \to \mathcal{A}_0(M, J)$ acting on generators by

$$
\kappa_{(M,J)}(\Phi_{(M,J)}(f)) = \Phi_{(M,0)}(f) + W_{(M,J)}^{(1)}(f)1_{\mathcal{A}(M,0)},
$$

(recall that $\mathcal{A}_\lambda(M, J) = \mathcal{A}(M, J)$ has generators $\Phi_{(M,J)}(f), f \in C_0^\infty(M)$). The morphisms $\kappa_{(M,J)}$ yield a natural isomorphism between $\mathcal{A}_\lambda$ and $\mathcal{A}_0$. To see this, consider a morphism $\psi : (M, J) \to (M', J')$. For any $f \in C_0^\infty(M)$,

$$
\mathcal{A}_0(\psi) \circ \kappa_{(M,J)}(\Phi_{(M,J)}(f)) = \Phi_{(M',0)}(\psi f) + W_{(M,J)}^{(1)}(f)1_{\mathcal{A}(M',0)},
$$

while

$$
\kappa_{(M',J')} \circ \mathcal{A}_\lambda(\psi)(\Phi_{(M,J)}(f)) = \Phi_{(M',0)}(\psi f) + W_{(M,J)}^{(1)}(\psi f)1_{\mathcal{A}(M',0)}.
$$

However, the naturality of the state entails precisely that

$$
W_{(M,J)}^{(1)}(\psi f) = \omega_{(M',J')}\left(\Phi_{(M',J')}\left(\psi f\right)\right) = \omega_{(M',J')}\left(\mathcal{A}_\lambda(\psi)\Phi_{(M,J)}(f)\right)
$$

$$
= \omega_{(M,J)}\left(\Phi_{(M,J)}(f)\right) = W_{(M,J)}^{(1)}(f).
$$

As $f$ was arbitrary, it follows that

$$
\mathcal{A}_0(\psi) \circ \kappa_{(M,J)} = \kappa_{(M',J')} \circ \mathcal{A}_\lambda(\psi),
$$

which shows that the $\kappa_{(M,J)}$ cohere to form a natural isomorphism $\kappa : \mathcal{A}_\lambda \to \mathcal{A}_0$. If natural states existed for all the theories $\mathcal{A}_\lambda (\lambda \in \mathbb{R})$, then one would be able to establish equivalence between all of them. Thus, the inequivalence of these theories precludes the existence of natural states.
(d) Time-slice axiom and relative Cauchy evolution

So far, we have only imposed the condition of local covariance on physical theories. A much stronger condition is the time-slice axiom, which can be regarded as encoding the existence of a dynamical law in the theory. For our presentation, we restrict to categories such as \( \mathcal{Loc} \) and \( \mathcal{LocSt} \) that are based on globally hyperbolic space-times, but the ideas can be extended to more general settings. A morphism \( \psi : M \to M' \) in \( \mathcal{Loc} \) whose image \( \psi(M) \) contains a Cauchy surface of \( M' \) will be called a Cauchy morphism; similarly, a morphism \( \psi : (M, J) \to (M', J') \) in \( \mathcal{LocSt} \) is called Cauchy under the same condition.

**Definition 3.4.** A locally covariant theory \( \mathcal{A} : \mathcal{BkGrnd} \to \mathcal{Phys} \) (where \( \mathcal{BkGrnd} \) is \( \mathcal{Loc} \) or \( \mathcal{LocSt} \)) obeys the time-slice axiom if \( \mathcal{A} \) maps every Cauchy morphism of \( \mathcal{BkGrnd} \) to an isomorphism in \( \mathcal{Phys} \).

If \( \psi : M \to M' \) is a Cauchy morphism, every aspect of the physics on \( M' \) can be predicted from the physics on \( M \), provided the time-slice axiom holds.

It was realized by Brunetti et al. that the time-slice axiom allows the comparison of dynamics on different backgrounds, in terms of relative Cauchy evolution \([4]\). Here we describe the adaptation to \( \mathcal{LocSt} \) given in \([6]\), with some slight modifications. Let \( (M, J) \) be an object of \( \mathcal{LocSt} \), with \( M = (M, g, o, t) \). Let \( h \) be a compactly supported rank-2 covariant symmetric tensor field such that \( (M, g + h) \) is a globally hyperbolic space–time, with respect to the (unique) time-orientation \( t[h] \) that agrees with \( t \) outside \( \text{supp } h \). Then \( M[h] = (M, g + h, o, t[h]) \) is an object of \( \mathcal{Loc} \), and \( (M[h], J[j]) := (M[h], J + j) \) is an object of \( \mathcal{LocSt} \) for any \( j \in C^\infty_0(M) \). Under these circumstances, we write \( (h, j) \in H(M, J) \). Choose open \( g \)-causally convex sets \( M^+/\sim \) of \( M \) lying to the future/past of the \( \text{supp } (h) \cup \text{supp } (j) \),\(^5\) and containing Cauchy surfaces of \( M \), as in figure 1. These sets are therefore also causally convex with respect to \( g + h \) and, defining \( M^\pm \) to be the sets \( M^\pm \) equipped with causal structure and orientation inherited from \( M \), and \( J^\pm = J|_{M^\pm} \), there are \( \mathcal{LocSt} \) Cauchy morphisms

\[
i^\pm : (M^\pm, J^\pm) \to (M, J) \tag{3.4a}
\]

and

\[
j^\pm : (M^\pm, J^\pm) \to (M[h], J[j]) \tag{3.4b}
\]

induced by the set inclusions of \( M^\pm \) in \( M \). A theory \( \mathcal{A} : \mathcal{LocSt} \to \mathcal{Phys} \) that obeys the time-slice axiom converts each of these morphisms to an isomorphism. The relative Cauchy evolution of \( \mathcal{A} \) induced by \( (h, j) \in H(M, J) \) is defined as the automorphism

\[
rce_{(M, J)}[h, j] := \mathcal{A}(i^+) \circ \mathcal{A}(j^-)^{-1} \circ \mathcal{A}(j^+) \circ \mathcal{A}(i^+)^{-1} \tag{3.5}
\]

of \( \mathcal{A}(M, J) \). One may show that \( \text{rce}_{(M, J)}[h, j] \) is independent of the choices of \( M^\pm \) made in the construction (cf. \([5, \S3]\)).

\(^5\)That is, there should be Cauchy surfaces \( S^\pm \) such that \( M^\pm \subset J^\pm(S^\pm) \), \( \text{supp } (h) \cup \text{supp } (j) \subset J^\pm(S^\pm) \).
The significance of relative Cauchy evolution can be explained as follows. Owing to the dynamical law of the theory on \((M, J)\), any observable \(A\) can be measured in the region \(M^+\). We fix that description, and proceed to modify the background to the past of \(M^+\), and future of \(M^-\), obtaining an observable in the \(M^+\) region of the perturbed background \((M[h], J[j])\). In turn, this observable can also be measured in the \(M^-\) region of \((M[h], J[j])\), owing to the dynamical law of the theory on the perturbed background. Transferring that description to the \(M^-\) region of the unperturbed space–time, we obtain a new observable on \((M, J)\) which will not in general coincide with our original observable \(A\). The discrepancy is precisely measured by the relative Cauchy evolution, which applies in this way to all aspects of the theory, not just observables.

The relative Cauchy evolution is particularly interesting for infinitesimal perturbations: its functional derivative with respect to the background metric yields a derivation related to the stress–energy tensor \([4]\) and similar results are obtained for other background sources. For example, in the case of the inhomogeneous scalar field theory, one finds that \([6, \S 7.3]\)

\[
\frac{d}{ds} \text{rce}_{(M,J)}[sh, sj]A \bigg|_{s=0} = i \frac{1}{2} T_{(M,J)}(h) + \Phi_{(M,J)}(j), \quad (3.6)
\]

for all \(A \in \mathcal{A}(M, J)\), where \(T_{(M,J)}(h) = \int_M h_{ab} T_{ab}^{\text{th}}(M,J) \, d\text{vol}_M\) is the smearing with \(h_{ab}\) of the quantization of the stress–energy tensor\(^6\)

\[
T_{ab}^{\text{th}}(\phi) := -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{ab}(x)} = \nabla^a \phi \nabla^b \phi - \frac{1}{2} g^{ab} \nabla_c \phi \nabla^c \phi + \frac{1}{2} m^2 g^{ab} \phi^2 + g^{ab} \frac{1}{2} \phi, \quad (3.7)
\]

and \(S\) is the classical action obtained from the Lagrangian \((3.1)\). In \((3.6)\), the derivative is understood with respect to be taken as a weak derivative in suitable Hilbert space representations \([4]\). Thus, the relative Cauchy evolution can be taken as a replacement for the classical action, because its functional derivative corresponds to quantities normally obtained from the functional derivative of the action with respect to the background.

A key observation \([5, \text{Prop. 3.8}]\) is the following.

**Theorem 3.5.** If \(\mathcal{A} : \text{LocSK} \to \text{Phys}\) and \(B\) are two locally covariant theories, obeying the time-slice axiom, and \(\zeta : \mathcal{A} \to B\) embeds \(\mathcal{A}\) as a subtheory of \(B\), then

\[
\zeta_{(M,J)} \circ \text{rce}_{(M,J)}^{(\mathcal{A})}(h, j) = \text{rce}_{(M,J)}^{(\mathcal{B})}(h, j) \circ \zeta_{(M,J)}
\]

for all background perturbations \((h, j) \in H(M, J)\).

This provides a strong and practical constraint that can be used to rule out or classify natural transformations between theories and particularly their automorphisms \([14]\). Here, we indicate how it distinguishes the inhomogeneous theories \(\mathcal{A}_\lambda\) for different values of the coupling constant \(\lambda\). In addition to the differential formula \((3.6)\) it will be useful to use the formula

\[
\text{rce}_{(M,J)}^{(\mathcal{A}_\lambda)}(0, j) \Phi_{(M,J)}(f) = \Phi_{(M,J)}(f) + E_M(f, j) 1_{\mathcal{A}(M,J)} \quad (3.8)
\]

also obtained in \([6]\). Now the theory \(\mathcal{A}_\lambda\) is defined so that \(\mathcal{A}_\lambda(M, J) = \mathcal{A}(M, \lambda J)\). Hence

\[
\text{rce}_{(M,J)}^{(\mathcal{A}_{\lambda})}(h, j) = \text{rce}_{(M,J)}^{(\mathcal{A}_{\lambda})}(h, j),
\]

the analogue of \((3.8)\) is

\[
\text{rce}_{(M,J)}^{(\mathcal{A}_{\lambda})}(h, j) \Phi_{(M,J)\lambda}(f) = \Phi_{(M,J)\lambda}(f) + \lambda E_M(f, j) 1_{\mathcal{A}(M,J)} \quad (3.9)
\]

and \((3.6)\) entails the formula

\[
\frac{d}{ds} \text{rce}_{(M,J)}^{(\mathcal{A}_{\lambda})}(0, sj)A \bigg|_{s=0} = i\lambda [\Phi_{(M,J)\lambda}(f), A]. \quad (3.10)
\]

\(^6\)The renormalized stress–energy tensor is not an element of \(\mathcal{A}(M, J)\), but we write \(A \mapsto [T_{M,J}(h), A]\) as convenient notation for the outer derivation of \(\mathcal{A}(M, J)\) obtained by regularizing the stress–energy tensor by point-splitting, computing the commutator within \(\mathcal{A}(M, J)\) and then removing the regulation. In \([6, \S 7.3]\), the analogue of \((3.6)\) was stated only for the case \(A = \Phi_{(M,J)}(f)\), but it extends immediately to the form given here.
Theorem 3.6. If $\lambda, \mu \in \mathbb{R}$ are distinct, then $\mathcal{A}_\lambda$ and $\mathcal{A}_\mu$ are inequivalent.

Proof. Without loss, assume $\mu \neq 0$ and that there is a natural transformation $\xi : \mathcal{A}_\lambda \to \mathcal{A}_\mu$. For each background $(M, J)$ and all $j, f \in C^\infty_0(M)$, theorem 3.5 gives

$$
\xi_{(M, J)} \circ \text{rec}_{(M, J)}[0, j] \Phi_{(M, J)}(f) = \text{rec}_{(M, J)}[0, j] \circ \xi_{(M, J)} \Phi_{(M, J)}(f)
$$

which, together with (3.9), yields

$$
\xi_{(M, J)} \Phi_{(M, J)}(f) + \lambda E_M(f, j) 1_{\mathcal{A}_{(M, J)}} = \text{rec}_{(M, J)}[0, j] \circ \xi_{(M, J)} \Phi_{(M, J)}(f).
$$

However, $\mu E_M(f, j) 1_{\mathcal{A}_{(M, J)}} = \text{rec}_{(M, J)}[0, j] \Phi_{(M, J)}(f) - \Phi_{(M, J)}(f)$, so after rearrangement, we may deduce that $X(f) = \xi_{(M, J)} \Phi_{(M, J)}(f) - (\lambda/\mu) \Phi_{(M, J)}(f)$ obeys

$$
\text{rec}_{(M, J)}[0, j] X(f) = X(f)
$$

for all $j \in C^\infty_0(M)$. By (3.10), it follows that $[\Phi_{(M, J)}(j), X(f)] = 0$ for all $j \in C^\infty_0(M)$ and that $X(f)$ is central. Hence

$$
[\xi_{(M, J)} \Phi_{(M, J)}(f), \xi_{(M, J)} \Phi_{(M, J)}(f')] = \left(\frac{\lambda}{\mu}\right)^2 [\Phi_{(M, J)}(f), \Phi_{(M, J)}(f')]
$$

$$
= \left(\frac{\lambda}{\mu}\right)^2 \xi_{(M, J)}[\Phi_{(M, J)}(f), \Phi_{(M, J)}(f')],
$$

contradicting the assumption that $\xi_{(M, J)}$ is a homomorphism. Hence $\mathcal{A}_\lambda$ cannot be embedded as a subtheory of $\mathcal{A}_\mu$, and in particular is not equivalent to it.

This result demonstrates the power of the functorial viewpoint; recall that the individual algebras $\mathcal{A}_\lambda(M, J)$ and $\mathcal{A}_\mu(M, J)$ are isomorphic for all $\lambda, \mu \in \mathbb{R}$. Thus, the distinction between the theories lies in the fact that there is no way of choosing such isomorphisms in a natural way. In this sense, the physics represented by a theory is encoded in its functorial structure.

4. Local physical content and dynamical locality

We turn to the description of the local physical content of a theory $\mathcal{A} : \mathcal{C} \to \text{Phys}$, in a causally convex region $O$ of space–time $M$ (there seems to be no obstruction to generalizing the background category if desired). Two ignorance principles apply: ignorance of whether the space–time extends at all beyond $O$, and ignorance of what the metric might be outside $O$. According to the ignorance meta-principle, the local content associated with $O$ ought to be independent of each factor, and suggests two characterizations.

First, we may consider the local physical content to be the full content we would have if the space–time were coterminous with $O$. To quantify this idea, let $i_O$ be the embedding $O \to M$, which induces a $\mathcal{C}$-morphism $i_O : M|_O \to M$, where $M|_O$ is the set $O$ with the metric and causal structure induced from $M$. Then the functor describing the theory assigns a morphism $\mathcal{A}_i(M|_O) : \mathcal{A}(M|_O) \to \mathcal{A}(M)$ whose image $\mathcal{A}_{\text{kin}}(M; O)$ is called the kinematic subalgebra of $\mathcal{A}(M)$ corresponding to $O$. The correspondence $O \mapsto \mathcal{A}_{\text{kin}}(M; O)$ defines a net of local algebras with properties generalizing those of algebraic QFT [2] to curved space–time [4].

Alternatively, instead of ignoring the background outside $O$ altogether, one could take the view that it might be altered, and that this should have no impact on the physics within $O$. We employ the relative Cauchy evolution as a quantitative measure of the response of such observables to changes in the geometry. For any compact subset $K$ of $M$, set

$$
\mathcal{A}^*(M; K) := \{ A \in \mathcal{A}(M) : \text{rec}_M[h] A = A \forall h \text{ supported in } K^\perp \},
$$

where $K^\perp = M \setminus J_M(K)$ is the causal complement of $K$. The dynamical algebra $\mathcal{A}_{\text{dyn}}(M; O)$ associated with an open causally convex region $O$ is then the subalgebra of $\mathcal{A}(M)$ generated by the $\mathcal{A}^*(M; K)$ as $K$ ranges over a suitable set of compact subsets of $O$ (see [5, §5] for details).
The reader might wonder why only metric perturbations in the causal complement are considered instead of arbitrary perturbations outside $O$. The reason for the restriction lies in our use of the relative Cauchy evolution. As mentioned in §3d, the relative Cauchy evolution uses the time-slice property to fix an equivalent description of an observable in the causal future of the perturbation region. It is this initially equivalent description that is held fixed during the perturbation, rather than the original description of the observable in $O$, so such observables are not in general invariant under the relative Cauchy evolution metric perturbations supported in $f_M(O)$. However, the relative Cauchy evolution induced by perturbations in the causal complement does provide a usable test of stability under background perturbations.

If the kinematic and dynamical descriptions agree, i.e. $\mathcal{A}_{\text{kin}}(M;O) = \mathcal{A}_{\text{dyn}}(M;O)$ for all open causally convex subsets $O$ of $M$ with finitely many connected components, the theory is said to be dynamically local. This property is known to hold for the following quantized theories:

- the free Klein–Gordon field $(\Box + m^2 + \xi R)\phi = 0$ in dimensions $n \geq 2$ provided either the mass $m$ or curvature coupling $\xi$ is non-zero [21,22], and the corresponding extended algebra of Wick polynomials for $m > 0$ at least for minimal or conformal coupling [22] (there is no reason to expect failure for other values of $\xi$); 
- the free massless current in dimensions $n \geq 2$ (restricting to connected space–times) or $n \geq 3$ (allowing disconnected space–times) [21];
- the inhomogeneous minimally coupled Klein–Gordon field, for $m \geq 0$, $n \geq 2$—here one adapts the setting to LocSrc by defining the dynamical algebras to be those invariant under all background perturbations (metric and source) in the causal complement [6];
- the free Dirac field with mass $m \geq 0$ [23]; and
- the free Maxwell field in dimension $n = 4$, in a ‘reduced formulation’ [12].

The known cases in which dynamical locality fails are: the free Klein–Gordon field with $m = 0$, $\xi = 0$ in dimensions $n \geq 2$, which may be traced to the rigid $\phi \mapsto \phi + \text{const.}$ gauge symmetry [21]; the free massless current in two dimensions allowing disconnected space–times [21]; and the free Maxwell field in dimension $n = 4$, in a ‘universal formulation’ [12]. The main difference between the two Maxwell formulations is that the universal formulation allows for topological electric and magnetic charges in space–times with non-trivial second de Rham cohomology, whereas the reduced formulation does not. In fact, the topological charges also fail to satisfy the injectivity property normally required of a locally covariant theory—this reflects their non-local nature, of course (see [9,13] for more discussion of the injectivity issue in related models). The emerging pattern is that dynamical locality can be expected to fail where a theory admits a broken rigid gauge symmetry or has charges stabilized by topological or other constraints; otherwise, it appears to be reasonable to expect dynamical locality to hold—it also seems that even trivial ‘interactions’ such as a mass or curvature coupling are sufficient to restore dynamical locality. With the exception of the example of the massless current in disconnected space–times, all known failures of dynamical locality are related to the existence of elements that are invariant under arbitrary relative Cauchy evolution; conceivably the treatment of the massless current might be modified to restore dynamical locality.

5. The same physics in all space–times

We have seen that the locally covariant approach provides a criterion for whether two theories represent the same physics as each other, namely, the existence or otherwise of a natural isomorphism between the corresponding functors. In this section, we turn to the question of what can be said about whether an individual theory is one that represents the same physics in different space–times (see [5], and [24] for a summary). This is a foundational question for theories of physics in curved space–times, but one that does not seem to have been addressed in any axiomatic way before. For theories defined by a Lagrangian, one may of course write down ‘the same’ Lagrangian in different space–times (although there can be subtleties with this [24])—what
we want to understand is on what grounds one might declare that this is a sound procedure. There are a number of difficulties: we lack a definition of ‘the same physics’, and it is unclear whether there might be many possible definitions, or indeed whether any suitable definition exists. We also face the question of how one should formulate a notion of SPASs mathematically.

The last question is the easiest to answer: we may represent any possible definition of SPASs extensionally by the class of theories that obey it. We then reason as follows: suppose \( \mathcal{T} \) is a class of locally covariant theories representing a definition of SPASs, that \( \mathcal{A} \) and \( \mathcal{B} \) are theories in \( \mathcal{T} \) and there is a space–time \( M \) in which \( \mathcal{A} \) and \( \mathcal{B} \) represent identical physics. Then, because the physical content of \( \mathcal{A} \) is assumed to be the same across all space–times, and the same is assumed of \( \mathcal{B} \) (according to a common notion, expressed by \( \mathcal{T} \)) the two theories ought to coincide in every space–time. The spirit of this argument may be captured mathematically as follows.

**Definition 5.1.** A class of theories \( \mathcal{T} \) has the **SPASs property** if, whenever \( \mathcal{A} \) and \( \mathcal{B} \) are theories in \( \mathcal{T} \), such that a natural \( \zeta : \mathcal{A} \rightarrow \mathcal{B} \) embeds \( \mathcal{A} \) as a subtheory of \( \mathcal{B} \), and there is a space–time \( M \) in which \( \zeta_{M} \) is an isomorphism, then \( \zeta \) is a natural isomorphism making \( \mathcal{A} \) and \( \mathcal{B} \) equivalent.

Note that the SPASs property is intended as a necessary criterion for \( \mathcal{T} \) to be a satisfactory notion of SPASs. Perhaps surprisingly, the collection of all locally covariant theories, \( \mathcal{LCT} \), does not have the SPASs property. Here, we assume \( \text{Phys} \) has a monoidal structure understood as composition of independent systems, so that any \( \mathcal{A} \in \mathcal{LCT} \) can be ‘doubled’ to give \( \mathcal{A} \otimes 2 \in \mathcal{LCT} \) by

\[
\mathcal{A} \otimes 2(M) = \mathcal{A}(M) \otimes \mathcal{A}(M) \quad \text{and} \quad \mathcal{A} \otimes 2(\psi) = \mathcal{A}(\psi) \otimes \mathcal{A}(\psi)
\]

for all objects \( M \) and morphisms \( \psi \) of \( \mathcal{LCT} \). Let us assume that \( \mathcal{A} \) is inequivalent to \( \mathcal{A} \otimes 2 \).\(^7\) Then we may form a new theory \( \mathcal{B} \in \mathcal{LCT} \) by

\[
\mathcal{B}(M) = \begin{cases} 
\mathcal{A}(M) & \text{if } \Sigma_{M} \text{ non-compact} \\
\mathcal{A}(M) \otimes 2 & \text{if } \Sigma_{M} \text{ compact}
\end{cases}
\]

\[
\psi_{A} = \begin{cases} 
\mathcal{A}(\psi)A & \text{if } \Sigma_{N} \text{ non-compact} \\
\mathcal{A}(\psi) \otimes 2A & \text{if } \Sigma_{M} \text{ compact} \\
\mathcal{A}(\psi)A \otimes 1 & \text{if } \Sigma_{N} \text{ compact}, \text{ but not } \Sigma_{M}
\end{cases}
\]

for \( \psi : M \rightarrow N \), where \( \Sigma_{M} \) denotes a Cauchy surface of \( M \). Owing to a theorem of Lorentzian geometry [25, theorem 1], there are no morphisms in which \( \Sigma_{M} \) is compact, but \( \Sigma_{N} \) is non-compact—indeed, if \( \Sigma_{M} \) is compact then it is diffeomorphic to \( \Sigma_{N} \) [5, proposition A.1].

The theory \( \mathcal{B} \) represents one copy of \( \mathcal{A} \) in space–times with non-compact Cauchy surfaces, but two copies in space–times with compact Cauchy surfaces. It is not hard to show that there are subtheory embeddings \( \zeta : \mathcal{A} \rightarrow \mathcal{B} \) and \( \eta : \mathcal{B} \rightarrow \mathcal{A} \otimes 2 \)

\[
\zeta_{M}A = \begin{cases} 
A & \text{if } \Sigma_{M} \text{ non-compact} \\
A \otimes 1 & \text{if } \Sigma_{M} \text{ compact} 
\end{cases}
\]

\[
\eta_{M}A = \begin{cases} 
A & \text{if } \Sigma_{M} \text{ compact} \\
A \otimes 1 & \text{if } \Sigma_{M} \text{ non-compact} 
\end{cases}
\]

and that \( \zeta_{M} \) is an isomorphism if \( \Sigma_{M} \) is non-compact, while \( \eta_{M} \) is an isomorphism if \( \Sigma_{M} \) is compact. If \( \mathcal{LCT} \) had the SPASs property, then there would be natural isomorphisms \( \mathcal{A} \cong \mathcal{B} \cong \mathcal{A} \otimes 2 \), contradicting the assumed inequivalence of \( \mathcal{A} \) and \( \mathcal{A} \otimes 2 \). This proves the following.

**Theorem 5.2.** \( \mathcal{LCT} \) does not have the SPASs property, nor does any class of theories containing \( \mathcal{A} \), \( \mathcal{B} \), \( \mathcal{A} \otimes 2 \).

We note that the theory \( \mathcal{B} \) is just one among many potential pathological theories that exist in \( \mathcal{LCT} \), for which there are general constructions [5, §4].

Evidently, in order to find classes of theories that do have the SPASs condition, we need a way of excluding theories like \( \mathcal{B} \). A clue is that dynamical locality fails in \( \mathcal{B} \); if \( M \) has compact Cauchy

---

\(^7\)This holds, for example, if \( \mathcal{A} \) is the theory of a free massive scalar field, because \( \mathcal{A} \) has automorphism group \( \mathbb{Z}_{2} \), while \( \mathcal{A} \otimes 2 \) has automorphism group \( O(2) \) [14].
surfaces and $O \subset M$ has non-trivial causal complement, then

$$\mathcal{B}^{\text{kin}}(M; O) = \mathcal{A}^{\text{kin}}(M; O) \otimes \mathbb{1}$$

and

$$\mathcal{B}^{\text{dyn}}(M; O) = \mathcal{A}^{\text{dyn}}(M; O) \otimes \mathbb{2}.$$  

We see that $\mathcal{B}^{\text{dyn}}(M; O)$ captures the degrees of freedom available in the ambient space–time, owing to the use of the relative Cauchy evolution in its definition. This clue turns out to be fruitful.

**Theorem 5.3.** The class of dynamically local theories has the SPASs property.

Aside from its intrinsic interest, this result has an application to the question of the existence of natural states [5, theorem 6.13] (we state a simplified and slightly weaker version).

**Theorem 5.4.** Suppose $\mathcal{A} : \text{Loc} \to \text{Alg}$ is a dynamically local theory that also obeys extended locality and admits a natural state $(\omega_M)_{M \in \text{Loc}}$. If, in Minkowski space $M_0$, the state $\omega_{M_0}$ induces a faithful GNS representation of $\mathcal{A}(M_0)$ with the Reeh–Schlieder property, then $\mathcal{A}$ is equivalent to the trivial theory that assigns the trivial unital $*$-algebra $\mathbb{C}$ to every space–time.

Sketch arguments for non-existence of natural states of the real scalar field appear in [4,8]; however, theorem 5.4 was the first complete argument and, moreover, applies to a general class of theories, including those listed in §4. Its proof makes use of theorem 5.3—the trivial theory is certainly a subtheory of $\mathcal{A}$, and the hypotheses entail that these theories coincide in Minkowski space; hence, as both $\mathcal{A}$ and the trivial theory are dynamically local, the result is proved.

### 6. Concluding remarks

This paper has described the framework of locally covariant physical theories, providing a motivation in terms of ‘ignorance principles’. This framework has opened up axiomatic QFT in curved space–time: in addition to the results discussed above, there is a general spin-statistics theorem [7] and versions of the Reeh–Schlieder theorem [28], Haag duality [29] and the split property [30], the global gauge group is understood [14] and superselection sectors have been investigated [31]. Moreover, the underlying ideas play an important role in the perturbative construction of interacting models in curved space–time [8,32] among other applications. Although the formalism allows us to begin to address the issue of SPASs, more can be done in this direction: while the class of dynamically local theories has the SPASs property, it is unknown whether it may contain pathological theories that should be ruled out by further axioms.

As far as natural states are concerned, the general result of theorem 5.4 shows that they cannot be expected in reasonable models of QFT; we have also given a relatively straightforward proof of this for the inhomogeneous scalar field. As with any no-go theorem, one can always seek to by-pass the hypotheses. The obvious condition to drop is the requirement that the preferred state depend locally on the background, and indeed a proposal of this type has been discussed recently for the scalar field [19]. However, the states constructed turn out not to be Hadamard and to have a number of other defects [20,33] (see also [34] for related discussion for Dirac fields). While there is an ingenious modification of [19] that does yield Hadamard states—see [35] (and [34] in the Dirac case)—this is achieved at the expense of introducing a whole family of states, none of which is canonically preferred. The no-go theorem is not so easily evaded.

**Competing interests.** I declare I have no competing interests.

**Funding.** I thank the Royal Society for funding the workshop.
Acknowledgements. I thank the organizers and participants of the workshop New Geometric Concepts in the Foundations of Quantum Physics for stimulating discussion and comments. I also thank Francis Wingham and the referees for their careful reading of the manuscript.

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