The analysis of shape optimization problems involving the spectrum of the Laplace operator, such as isoperimetric inequalities, has known in recent years a series of interesting developments essentially as a consequence of the infusion of free boundary techniques. The main focus of this paper is to show how the analysis of a general shape optimization problem of spectral type can be reduced to the analysis of particular free boundary problems. In this survey article, we give an overview of some very recent technical tools, the so-called shape sub- and supersolutions, and show how to use them for the minimization of spectral functionals involving the eigenvalues of the Dirichlet Laplacian, under a volume constraint.

1. Introduction

The notion of the ‘shape optimization’ problem emerged formally in the early seventies as a mathematical branch devoted to the analysis of problems issued from engineering and mixing questions from mechanics and optimization to partial differential equations on unknown geometric domains. Roughly speaking, a shape optimization problem can be formally written as

$$\min\{J(\Omega) : \Omega \subseteq D\},$$

where $D$ is a bounded open set in $\mathbb{R}^d$, $\Omega$ stands for an unknown open subset of $D$ and $J$ is a cost functional depending on the solution of a partial differential equation set on $\Omega$, or the spectrum of some operator defined on a functional space related to $\Omega$. In this general formulation, the first attempts were to produce numerical methods in order to give good approximations of
optimal sets, while questions about existence of solutions (optimal shapes) and their regularity started to be studied much later.

In recent years, shape optimization has become more and more involved in questions issued from spectral geometry, being an effective tool to understand different qualitative and numerical properties of a range of isoperimetric inequalities of spectral type, a classical example being the Faber–Krahn inequality (we refer to [1] for a detailed account on this topic and other inequalities involving the spectrum of the Laplace operator). The currently available existence results for this kind of shape optimization problem, generally provide optimal shapes with no regularity (the set is just measurable/open/quasi-open). Meanwhile, those shapes are expected to be smooth. In the literature, regularity is proved very often starting from some assumption (see for instance [2,3]), e.g. if the optimal set is known to be Lipschitz, then it is analytic. We are particularly interested in this ‘primary’ regularity for general shape optimization problems as it is the key ingredient for higher regularity.

The main way to deal with the regularity of optimal shapes is to see them as free boundary (or free discontinuity) problems. The fundamental difficulty is that a general shape optimization problem is not necessarily of energy minimization type and has very often a strong non-local character so that one cannot replace the unknown shape by an unknown function, whose level set is the shape, and then rephrase the problem as a function optimization one!

We shall focus in this paper on a class of geometric spectral optimization problems, involving the spectrum of the Dirichlet–Laplacian. For every (quasi)-open set of finite measure \( \Omega \subseteq \mathbb{R}^d \), the Dirichlet–Laplacian admits a compact resolvent and so a discrete spectrum consisting of eigenvalues, given by

\[
\lambda_k(\Omega) := \min_{S_k} \max_{\gamma \in S_k} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} u^2 \, dx},
\]

where the minimum ranges over all \( k \)-dimensional subspaces \( S_k \) of \( H^1_0(\Omega) \).

The question is to solve

\[
\min_{\Omega \subseteq \mathbb{R}^d} \{ F(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)) + |\Omega| \}
\]

for a suitable function \( F : \mathbb{R}^k \to \mathbb{R} \). For instance, taking \( F(\lambda) = \lambda_k \), we deal with the optimization problem for the \( k \)th eigenvalue of \(-\Delta\):

\[
\min_{\Omega \subseteq \mathbb{R}^d} \{ |\lambda_k(\Omega)| + |\Omega| \}.
\]

For \( k = 1 \) and \( k = 2 \), the solution is known to be a ball, respectively, two equal disjoint balls, but the exact shape of the optimal set is not known for higher values of \( k \). A natural question in spectral geometry is to prove the existence of a minimizing set \( \Omega \) and to study its regularity. Numerical computations in the last 10 years, by various methods (shape derivative, level set, relaxation on measures and genetic shape algorithms) gave robust answers suggesting uniqueness of the optimal shape and interesting geometric properties.

In order to prove (local) existence, the reference result is due to Buttazzo & Dal Maso [4], while for regularity, one has to go back to the seminal work of Alt & Caffarelli [5]. Alt and Caffarelli dealt with the minimization of the capacity with a measure penalization: given a compact set \( K \subseteq \mathbb{R}^d \), solve

\[
\min \left\{ \int_{\Omega} |\nabla u|^2 \, dx + ||u > 0|| \mid u \in H^1(\mathbb{R}^d), \ u = 1 \text{ on } K \right\}.
\]

The free boundary here is the boundary of the set where the solution \( u \) is strictly positive. Denoting this set \( \Omega := \{ u > 0 \} \), one can rewrite this problem as a shape optimization problem

\[
\min \{ \text{cap}(K, \Omega) + |\Omega| \mid \Omega \subseteq \mathbb{R}^d \},
\]

where \( \text{cap}(K, \Omega) \) stands for the capacity of the set \( K \) in \( \Omega \)

\[
\text{cap}(K, \Omega) = \min \left\{ \int_{\Omega} |\nabla u|^2 \, dx \mid u \in H^1_0(\Omega), \ u \equiv 1 \text{ on } K \right\}.
\]
The Alt–Caffarelli method was extended by Briançon et al. [6] to the Dirichlet energy $E_f(\Omega)$ (see also [7] for the first eigenvalue): given $f \in L^\infty(\mathbb{R}^d)$, let us denote

$$E_f(\Omega) = \inf \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega fu \, dx : u \in H^1_0(\Omega) \right\}.$$ 

Given a bounded open set $D \subseteq \mathbb{R}^d$, the shape optimization problem reads

$$\min\{E_f(\Omega) + |\Omega| : \Omega \subset D\}. \quad (1.4)$$

In this case, stating the problem as a shape optimization problem or as a function minimization problem makes no difference, problem (1.4) being equivalent to

$$\min \left\{ \int_{\mathbb{R}^d} \left( \frac{|\nabla u|^2}{2} - fu + \chi_{\{|u|>0\}} \right) \, dx : u \in H^1_0(D) \right\}.$$ 

The starting point for proving the regularity of the free boundary of the optimal set relies on the analysis of the state function, solution of the PDE

$$-\Delta u = f \quad \text{in } \Omega, \quad u \in H^1_0(\Omega).$$

The case of a general spectral functional (1.2) involving higher eigenvalues leads to a new kind of difficulties related to the fact that each eigenvalue of order higher than 1 is itself a saddle point and not a minimizer of an unconstrained energy! This fact makes very difficult any attempt to use the Alt–Caffarelli techniques in a direct manner. For example, the natural choice of a state function in the $\lambda_k$ optimization problem (1.3) is a corresponding eigenfunction, solution of

$$-\Delta u_k = \lambda_k(\Omega)u_k \quad \text{in } \Omega, \quad u_k \in H^1_0(\Omega), \quad \int_\Omega u_k^2 \, dx = 1.$$ 

Even in this simple case, the relationship between the set $\Omega$ and the eigenfunction $u_k$ is not trivial, $\Omega$ being, in general, only a set containing $\{u_k \neq 0\}$ (in fact, if $\Omega$ is just a quasi-open set, it is not even known if the nodal line $\{u_k = 0\} \cap \Omega$ has a zero Lebesgue measure). The knowledge of $u_k$ alone gives the value of the eigenvalue but not its actual position in the spectrum and does not characterize the set $\Omega$. To obtain this information, one also needs to understand the behaviour of the first $k-1$ eigenfunctions, even tough they do not appear explicitly in problem (1.3).

In order to deal with these problems, new techniques emerged in the last 2 years (see for instance [8–10]). The key idea is to locally get control of the variation of a general shape functional which is to be minimized, by the variation of a particular functional which is easier to analyse, e.g. of energy type. As a consequence, minimality of the original functional will lead to a ‘partial’ minimality for the energy functional, which is much easier to handle, in the spirit of Alt and Caffarelli. Partial minimality is understood here in terms of particular deformations. This approach is formalized through the notions of shape sub- and supersolutions and allowed an interesting step forward in the proof of the global existence and weak regularity of the optimal shapes for spectral functionals like (1.3) (regularity means here Lipschitz regularity of the state function, finite perimeter of the optimal set, inner density, boundedness, etc.).

In this paper, we review and bring together all these notions, describe the main properties of the shape sub- and supersolutions and show how to use them to the analysis of general spectral functionals like (1.2).

2. Shape subsolutions

Let $\Omega \subseteq \mathbb{R}^d$ be a quasi-open set of finite measure. For the precise definition of a quasi-open set and of the Sobolev space $H^1_0(\Omega)$, we refer to [4]. Every open set is also quasi-open, so that all results apply to open sets and the reader may ignore the word ‘quasi’. We introduce the torsion energy
of $\Omega$, by $E(\Omega) = E_1(\Omega)$

$$E(\Omega) = \min_{u \in H^1_0(\Omega)} \frac{1}{2} \left| \nabla u \right|^2 \mathrm{d}x - \int u \mathrm{d}x.$$  

(2.1)

The unique minimizer of $E(\Omega)$ is denoted $w_\Omega$ and is called torsion function. The torsion function formally solves

$$w_\Omega \in H^1_0(\Omega) \quad \text{and} \quad -\Delta w_\Omega = 1 \quad \text{in} \quad H^1_0(\Omega).$$

The torsion function carries a lot of information about the spectrum. In particular, the following inequalities hold (see [8,11], respectively):

| if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^d$ are of finite measure, then
| $0 \leq \frac{1}{\lambda_k(\Omega_2)} - \frac{1}{\lambda_k(\Omega_1)} \leq c_k(d, \Omega_2) \int_{\Omega_2} (w_{\Omega_2} - w_{\Omega_1}) \mathrm{d}x$,  
| where the constant $c_k(d, \Omega_2)$ depends only on $\lambda_k(\Omega_2)$ and the dimension of the space.
| if $\Omega \subseteq \mathbb{R}^d$ is of finite measure, then

$$\frac{1}{\lambda_1(\Omega)} \leq \|w_\Omega\|_\infty \leq \frac{4 + 3d \log 2}{\lambda_1(\Omega)}.$$  

The Saint Venant inequality asserts that the minimizer of the torsion energy, among sets of prescribed measure, is the ball.

Let $c > 0$ be fixed. In order to understand problem (1.2), the main idea is to study the quasi-open sets which are partial minimizers for the functional $E(\cdot) + c |\cdot|$. Partial minimality refers to the competition with the quasi-open sets which are inside the domain. We start with a general definition.

**Definition 2.1.** Let $F : \mathcal{A} \to \mathbb{R}$ be a functional defined on a class $\mathcal{A}$ of measurable subsets of $\mathbb{R}^d$. We say that $\Omega^* \in \mathcal{A}$ is a subsolution for a shape functional $F$, if $F(\Omega^*) \leq F(\Omega)$ for every set $\Omega \in \mathcal{A}$, $\Omega \subseteq \Omega^*$.

In our case, we shall consider $\mathcal{A}$ to be the class of quasi-open subsets of $\mathbb{R}^d$ with finite measure and the shape functional to be the sum of the torsion energy and of the Lebesgue measure. Simply, a quasi-open set $\Omega^*$ is called energy subsolution (with constant $c$) if $\Omega^*$ is a subsolution for the functional $(\Omega \mapsto E(\Omega) + c |\Omega|)$, i.e.

$$E(\Omega^*) + c |\Omega^*| \leq E(\Omega) + c |\Omega|, \quad \forall \Omega \subseteq \Omega^*.$$  

(2.3)

The energy subsolution is called local, if it holds only for the quasi-open sets

$$\Omega \subseteq \Omega^* \quad \text{such that} \quad \|w_{\Omega^*} - w_{\Omega}\|_{L^1(\mathbb{R}^d)} = \int_{\Omega^*} (w_{\Omega^*} - w_{\Omega}) \mathrm{d}x < \varepsilon,$$

for some fixed $\varepsilon > 0$. We note that even if $\Omega^*$ is just a local energy subsolution, we can still test the suboptimality of $\Omega^*$ with the sets $\Omega = \Omega^* \setminus \overline{B}_r(x_0)$, $\Omega = \{w_{\Omega^*} > r\}$ and $\Omega = \Omega^* \setminus \{x \in \mathbb{R}^d : x_1 \geq t\}$, for $r$ small enough and $t$ large enough (see [12, §3.7.1]).

The family of shape subsolutions for the energy is very large, precisely one can construct a subsolution inside any set of prescribed measure $A$, just by solving

$$\min\{E(\Omega) + c |\Omega| : \Omega \subseteq A\}.$$  

(2.4)

The existence of a solution for (2.4) is trivial, and clearly the solution is also energy subsolution. One can observe that if the set $A$ is quasi-open and does not have interior points (up to adding sets of zero capacity), then the optimal set for the problem (2.4) may not have any interior points either (see [13] for the precise construction). As a consequence, one cannot expect that a quasi-open set that is only a subsolution has a priori any smoothness. In particular, it is not necessarily an open set. This observation is important, as the solutions of (1.2) are a priori quasi-open sets.
A second intuitive observation is the following. Assume that the set $\Omega^*$ is a smooth subsolution for the energy. Performing the shape derivative with respect to inner vector fields

$$V \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$$

such that $V.n \leq 0$ on $\partial \Omega^*$,

where $n$ is the outer normal derivative on $\partial \Omega^*$, one gets

$$-\int_{\partial \Omega^*} \frac{1}{2} |\nabla w_{\Omega^*}|^2 V.n \, dH^{d-1} + \int_{\partial \Omega^*} cV.n \, dH^{d-1} \geq 0.$$

This formally gives the information that for a shape subsolution, we have $|\nabla w_{\Omega^*}|^2 \geq 2c$ on $\partial \Omega^*$ which is related to an inner density property of the set $\Omega^*$ [13]. This connects the notion of shape subsolution to the classical notion of subsolution in free boundary problems, as introduced by Beurling [14] (see also [15]).

Here is the main result for subsolutions.

**Theorem 2.2.** Suppose that the quasi-open set $\Omega^* \subset \mathbb{R}^d$ is a local energy subsolution. Then

(i) $\Omega^*$ is a bounded set. Moreover, there exist constants $C, r_0 > 0$ depending on the measure of $\Omega^*$, $d, \epsilon$ and $c$ such that $\Omega^*$ can be covered with less than $C r^{-d-1}$ balls of radius $r$, for every $r \leq r_0$;

(ii) $\Omega^*$ has finite generalized perimeter in the sense of De Giorgi, and

$$\sqrt{\frac{c}{2}} \text{Per}(\Omega^*) \leq |\Omega^*|,$$

(2.5)

(iii) $\Omega^*$ is equivalent a.e. to a closed set, and $H^{d-1}$-a.e. to the set of points of density $1$ or $\frac{1}{2}$.

**Proof.** For a complete proof of this theorem, we refer to [8,13]. We point out here the main ideas.

As one can compare the set $\Omega^*$ only with inner domains, the following two kind of geometric perturbations can be used: $\Omega_r = \Omega^* \setminus B_r(x_0)$ and $\Omega_r = \{ x \in \Omega^* : w_{\Omega^*}(x) > r \}$, for every $x_0 \in \mathbb{R}^d$ and $r > 0$. The first geometric perturbation, inspired from Alt & Caffarelli [5] leads to the following result.

**Lemma 2.3.** Let $\Omega^* \subset \mathbb{R}^d$ be an energy subsolution. Then, there exist constants $C_0 > 0$ (depending only on the dimension $d$ and $c$) and $r_0 > 0$ (depending on the dimension $d$, the constant $c$ and on the constant $\epsilon$ following definition 2.1) such that for every $x_0 \in \mathbb{R}^d$ and every $0 < r < r_0$ the following implication holds:

$$\left\| w_{\Omega^*} \right\|_{L^\infty(B_r(x_0))} \leq C_0 r \Rightarrow (w_{\Omega^*} \equiv 0 \text{ on } B_{r/2}(x_0)).$$

(2.6)

With this result, the first point of theorem 2.2 is immediate. On the one hand, for $r = \delta$ (to be chosen later), the family of points $x_n$ for which $w(x_n) > C_0 \delta$ and at pairwise distance greater than $2\delta$ is finite, controlled by $\int w_{\Omega^*}$, and so via the Saint Venant inequality by $|\Omega^*|$. Indeed, the function

$$x \mapsto w_{\Omega^*}(x) + \frac{|x - x_n|^2}{2d}$$

being subharmonic in $\mathbb{R}^d$, one gets

$$C_0 \delta \leq w_{\Omega^*}(x_n) \leq \frac{1}{\omega_d \delta^d} \int_{B_r(x_n)} w_{\Omega^*}(x) + \frac{|x - x_n|^2}{2d} \, dx.$$

As the sum of $\int_{B_r(x_n)} w_{\Omega^*} \, dx$ over all balls $B_r(x_n)$ is bounded from above by the $L^1$ norm of the torsion function $\int w_{\Omega^*} \, dx$, one finds a bound on the number of balls for some suitable $\delta$ (see the precise account in [12]).

In order to prove the second point of theorem (2.2), we consider the perturbation $\Omega_r = \{ x \in \Omega^* : w_{\Omega^*}(x) > r \}$ and write inequality (2.3). As $(w_{\Omega^*} - r)^+$ is precisely the torsion function on the
Using the co-area formula, together with an average theorem, we have a constant $c$ such that for every solution is a shape subsolution for the energy, so it inherits the regularity properties from theorem 2.2. Let $\Omega \subseteq \Omega^*$ be a quasi-open set.

\[ \frac{1}{2} \int |\nabla w_{\Omega^*}|^2 \, dx - \int w_{\Omega^*} \, dx + c|\{ w_{\Omega^*} > 0 \}| \leq \frac{1}{2} \int |\nabla (w_{\Omega^*} - r)^+|^2 \, dx - \int (w_{\Omega^*} - r)^+ \, dx + c|\{ w_{\Omega^*}(x) > r \}|. \]

Consequently,

\[ \frac{1}{2} \int_{\{0 \leq w_{\Omega^*} \leq r\}} |\nabla w_{\Omega^*}|^2 \, dx + c|\{0 \leq w_{\Omega^*} \leq r\}| \leq \int_{\{0 \leq w_{\Omega^*} \leq r\}} w_{\Omega^*} \, dx + r|\{ w_{\Omega^*} > r \}| \leq r|\Omega^*|. \]

Estimating the left-hand side from below by the Cauchy–Schwartz inequality, we get

\[ \int_{\{0 \leq w_{\Omega^*} \leq r\}} |\nabla w_{\Omega^*}| \, dx \leq |\{ w_{\Omega^*} \leq r \}|^{1/2} \left( \int_{\{0 \leq w_{\Omega^*} \leq r\}} |\nabla w_{\Omega^*}|^2 \, dx \right)^{1/2} \leq r \sqrt{\frac{2}{c}} |\Omega^*|. \]

Using the co-area formula, together with an average theorem, we have

\[ \mathcal{H}^{d-1}(\Omega^* \{ w_{\Omega^*} > r_n \}) \leq \sqrt{\frac{2}{c}} |\Omega^*|, \]

for a sequence $r_n \to 0$. Passing to the limit and using the lower semi-continuity of the perimeter, we get

\[ \text{Per}(\Omega^*) \leq \sqrt{\frac{2}{c}} |\Omega^*|. \]

Above, $\mathcal{H}^{d-1}(\Omega^* \Omega^*)$ stands for the $(d - 1)$ Hausdorff measure of the reduced boundary.

In order to prove the last point of theorem (2.2), we note that for $\mathcal{H}^{d-1}$-almost all points in $\mathbb{R}^d$, the density of $\Omega^*$ equals to $1$, $0$ or $\frac{1}{2}$. Following proposition 3.12 in [13], there exists $\alpha > 0$ such that for every point $x$ where $|\{ w_{\Omega^*}(x) > 0 \} \cap B_r(x)| > 0$ holds for every $r > 0$

\[ \limsup_{r \to 0} \frac{|\{ w_{\Omega^*} > 0 \} \cap B_r(x_0)|}{|B_r|} \geq \alpha. \]

As a consequence, the family of points of density zero is an open set, hence $\Omega^*$ is a.e. equal to a closed set and the family of points of non-zero density is closed up to a set of $\mathcal{H}^{d-1}$ measure zero.

We can now use all the results on the energy subsolutions obtained above to deduce the analogous properties of the optimal sets of general spectral functionals. Let us fix the natural numbers $1 \leq i_1 < \cdots < i_k$. The following result was proved in [8] in the framework of (1.3).

**Theorem 2.4.** Let $F: \mathbb{R}^k \to \mathbb{R}$ be locally Lipschitz continuous and increasing in each variable. The following spectral optimization problem:

\[ \min \{ F(\lambda_{i_1}(\Omega), \ldots, \lambda_{i_k}(\Omega)) + |\Omega| : \Omega \subseteq \mathbb{R}^d \} \tag{2.7} \]

has at least one solution. Moreover, every solution is a shape subsolution for the energy, for a suitable constant $c > 0$.

**Proof.** We assume in a first instance the existence of a solution $\Omega^*$ for problem (2.7). We shall prove this fact in a second step. For now, we justify that $\Omega^*$ is a shape subsolution for the energy, so it inherits the regularity properties from theorem 2.2.
Following the notation from (2.2), we get
\[
0 \leq \frac{1}{\lambda_k(\Omega^*)} - \frac{1}{\lambda_k(\Omega)} \leq c_k(d, \Omega^*) \int_{\Omega^*} (w_{\Omega^*} - w_{\Omega}) \, dx.
\] (2.8)
Assume that \( \varepsilon > 0 \) is chosen such that
\[
\left[ \max_{j=1, \ldots, k} c_j(d, \Omega^*) \right] \varepsilon \leq \frac{1}{2\lambda_k(\Omega^*)}.
\]
Consequently,
\[
\lambda_j(\Omega^*) \leq \lambda_j(\Omega) \leq 2\lambda_j(\Omega^*),
\]
as soon as \( \int_{\Omega^*} (w_{\Omega^*} - w_{\Omega}) \, dx < \varepsilon \). Finally, for every \( j = 1, \ldots, k \), we have
\[
\lambda_j(\Omega) - \lambda_j(\Omega^*) \leq 4\lambda_j^2(\Omega^*) \max_{j=1, \ldots, k} c_j(E(\Omega) - E(\Omega^*)).
\]
From the optimality of \( \Omega^* \) in (2.7), we get
\[
|\Omega^*| - |\Omega| \leq F(\lambda_{i_1}(\Omega), \ldots, \lambda_{i_k}(\Omega)) - F(\lambda_{i_1}(\Omega^*), \ldots, \lambda_{i_k}(\Omega^*))
\]
\[
\leq \|\nabla F\|_{L^\infty} \sum_{j=1}^k (\lambda_j(\Omega) - \lambda_j(\Omega))
\]
\[
\leq \|\nabla F\|_{L^\infty} 4k\lambda_k^2(\Omega^*) \max_{j=1, \ldots, k} c_j(E(\Omega) - E(\Omega^*)).
\]
Denoting
\[
c = 4k\|\nabla F\|_{L^\infty} \lambda_k^2(\Omega^*) \max_{j=1, \ldots, k} c_j,
\]
we conclude that \( \Omega^* \) is a local energy subsolution, so theorem 2.2 holds.

In order to prove existence, let us take a minimizing sequence \((\Omega_n) \) for (2.7). We can assume that the measures of \( \Omega_n \) are uniformly bounded and that \( \lambda_j(\Omega_n) \to \alpha_j \). Then we consider the sequence of new minimization problems
\[
\min\{F(\lambda_{i_1}(\Omega), \ldots, \lambda_{i_k}(\Omega)) : |\Omega| \leq \Omega_n \}.
\] (2.9)
As \( \Omega_n \) are of finite measure, the existence of a solution \( \Omega_n^* \) for (2.9) follows from the Buttazzo–Dal Maso theorem [4]. Clearly, the sequence \((\Omega_n^*) \) is again a minimizing sequence for (2.9), but inherits all properties of subsolutions. In particular, they are equibounded up to translating the different connected components [12, \S 6.1], so that we can rely again on the Buttazzo–Dal Maso theorem to prove the existence of a solution.

**Remark 2.5.** If moreover \( F \) is bi-Lipschitz in each variable, i.e. there exists \( K > 0 \) such that
\[
\frac{1}{K} |x_i - y_i| \leq |F(x_1, \ldots, x_k) - F(y_1, \ldots, y_k)| \leq K \sum_{i=1}^k |x_i - y_i|,
\] (2.10)
for every \( x_1 \leq y_1, \ldots, x_k \leq y_k \), then the same conclusion holds for the spectral optimization problem
\[
\min\{F(\lambda_{i_1}(\Omega), \ldots, \lambda_{i_k}(\Omega)) : |\Omega| \leq \Omega_n \}.
\] (2.11)

### 3. Shape supersolutions and quasi-minimizers

**Definition 3.1.** Following the notations of definition 2.1, we say that \( \Omega^* \in \mathcal{A} \) is a shape supersolution for \( F \) if
\[
F(\Omega^*) \leq F(\Omega), \quad \text{for every } \Omega \in \mathcal{A} \text{ such that } \Omega \supset \Omega^*.
\]
In case \( F(\Omega) = E(\Omega) + c|\Omega| \), if \( \Omega^* \) is a smooth supersolution, then \( |\nabla w_{\Omega^*}|^2 \leq 2c \) on \( \partial \Omega^* \), connecting this notion with the classical one of supersolution for a free boundary problem [14,15].
Our main usage of a shape supersolution is based on the simple but useful observation that if \( \Omega^* \) is a supersolution for the functional \( F + G \) and if \( G \) is increasing with respect to the set inclusion, i.e.
\[
G(\Omega_1) \geq G(\Omega_2), \quad \text{for every } \Omega_2 \supset \Omega_1,
\]
then \( \Omega^* \) is also a shape supersolution for \( F \).

For instance, if \( \Omega^* \) is shape supersolution for
\[
\lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + |\Omega|,
\]
then it is also supersolution for every functional \( \lambda_i(\Omega) + |\Omega|, i = 1 \ldots k \). More general, if \( F \) in (1.2) is bi-Lipschitz (remark 2.5), then every shape supersolution for
\[
F(\lambda_i(\Omega), \ldots, \lambda_i(\Omega)) + |\Omega|
\]
is also shape supersolution for
\[
\lambda_i(\Omega) + K|\Omega|,
\]
for \( K \) depending on \( F \) (relation (2.10) above), and for every \( j = 1, \ldots, k \).

Contrary to the subsolutions which were used to get information on the optimal shape, the supersolutions will be useful to gather information on the state functions (here, the eigenfunctions of the optimal shape). The main tool is the following notion of quasi-minimality [9,16].

**Definition 3.2.** We say that a function \( u \in H^1(\mathbb{R}^d) \) is quasi-minimizer for the Dirichlet integral if
\[
\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^d} |\nabla (u + \varphi)|^2 \, dx + Cr^d, \quad \text{for every } r > 0,
\]
and every \( \varphi \in H^1(\mathbb{R}^d) \) with \( \text{diam}(\varphi \neq 0) \leq r \).

Following the ideas of Briançon et al. [6], it was proved in [9,12] that the quasi-minimizers which are eigenfunctions of their own support are globally Lipschitz. Indeed, we recall the following result [9,12,16].

**Theorem 3.3.** Suppose that the function \( u \in H^1(\mathbb{R}^d) \) is such that

— there exists \( \lambda \in \mathbb{R} \) such that
\[
-\Delta u = \lambda u \quad \text{weakly in } H^1_0(\{u \neq 0\}).
\]

— \( u \) is a quasi-minimizer in sense of (3.1).

Then, there is a constant \( M \) depending only on \( \lambda \) and \( C \) such that \( \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \leq M \).

In order to prove that the result above can apply to minimizers of (1.2), we restate a lemma which was proved in [9] and will allow us to handle the Rayleigh quotient for the perturbed domains. We shall analyse first the simpler problem (1.3), and then show how to use the supersolution property to pass from (1.3) to the general case (1.2).

**Lemma 3.4.** Let \( u_1, \ldots, u_k \in H^1(\mathbb{R}^d) \) be an orthonormal family in \( L^2(\mathbb{R}^d) \) such that the gradients \( \nabla u_1, \ldots, \nabla u_k \in L^2(\mathbb{R}^d; \mathbb{R}^d) \) form an orthogonal family in \( L^2(\mathbb{R}^d; \mathbb{R}^d) \), and \( u_k \) is the unique maximizer (up to sign changes on connected components) for the problem
\[
\Phi(u_k, \ldots, u_1) := \max \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 \, dx : u \in \text{span}(u_1, \ldots, u_k), \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\}.
\]

Then, there is a constant \( r > 0 \) such that for every map \( (-1,1) \ni t \mapsto u_k(t) \in H^1(\mathbb{R}^d) \) satisfying

— \( u_k(t) \) is continuous with respect to the strong \( H^1 \) topology, \( \sup_{t \in (-1,1)} \|u_k(t) - u_k\|_{H^1} \leq 1 \) and \( u_k(0) = u_k \);
— for every \( t \in (-1,1) \) the support of \( u_k(t) - u_k \) is contained in some ball \( B_r(x_0) \);
we have the estimate
\[
\Phi(u_k(t), u_{k-1}, \ldots, u_1) \leq \frac{\int_{\mathbb{R}^d} |\nabla u_k(t)|^2 \, dx + (1 + \|\nabla u_k\|^2) \int_{\mathbb{R}^d} |\nabla(u_k - u_k)|^2 \, dx}{\int_{\mathbb{R}^d} u_k(t)^2 \, dx - (1/2) \int_{\mathbb{R}^d} |\nabla u_k(t) - u_k|^2 \, dx}.
\]

**Remark 3.5.** Lemma 3.4 applies to the case \( u_k(t) = u_k + t\varphi \), where \( \varphi \in H^1(\mathbb{R}^d) \) is such that \( \int |\nabla \varphi|^2 \, dx = 1 \) and its support is contained in a ball of radius smaller than some value \( r > 0 \).

We can now use the above lemma to transform the optimality property of a set \( \Omega^\ast \) for (1.3) into an information on an eigenfunction \( u_k \in H^1_0(\Omega^\ast) \) corresponding to \( \lambda_k(\Omega^\ast) \). From a geometrical point of view, we shall use only the outer perturbations, i.e. the shape supersolution property.

Indeed, let \( \Omega^\ast \subset \mathbb{R}^d \) be a shape supersolution for the functional \( \lambda_k(\Omega) + |\Omega| \). If we have that \( \lambda_k(\Omega^\ast) > \lambda_{k-1}(\Omega^\ast) \), then we can apply lemma 3.4 to the eigenfunctions \( u_1, \ldots, u_k \in H^1_0(\Omega^\ast) \) and the function \( u_k(t) = u_k + t\varphi \), where \( \varphi \in H^1_0(B_r(x_0)) \) is supported on a ball \( B_r(x_0) \) of sufficiently small radius and \( \|\nabla \varphi\|_{L^2}^2 = 1 \). Thus, we get that for \( t = |B_r|^{1/2} \),
\[
\int_{\mathbb{R}^d} |\nabla u_k|^2 \, dx + |\Omega^\ast| = \lambda_k(\Omega^\ast) + |\Omega^\ast| \leq \lambda_k(\Omega^\ast \cup B_r(x_0)) + |\Omega^\ast \cup B_r(x_0)|
\]
\[
\leq \Phi(u_k + t\varphi, u_{k-1}, \ldots, u_1) + |\Omega^\ast \cup B_r(x_0)|
\]
\[
\leq \frac{\int_{\mathbb{R}^d} |\nabla(u_k + t\varphi)|^2 \, dx + (1 + \lambda_k(\Omega^\ast))(\int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dx)}{\int_{\mathbb{R}^d} u_k(t)^2 \, dx - \frac{t}{2} \int_{\mathbb{R}^d} |\nabla u_k(t) - u_k|^2 \, dx} + |\Omega^\ast \cup B_r(x_0)|,
\]
which, by the fact that \( \|\varphi\|_{L^2}^2 \leq r^2/\lambda_1(B_1) \|\nabla \varphi\|_{L^2}^2 \leq r^2/\lambda_1(B_1) \) gives
\[
\lambda_k(\Omega^\ast) \left( 1 + 2t \int_{\mathbb{R}^d} u_k \varphi \, dx \right) \leq \lambda_k(\Omega^\ast) + 2t \int_{\mathbb{R}^d} \nabla u_k \cdot \nabla \varphi \, dx + t^2(2 + \lambda_k(\Omega^\ast)) \|\nabla \varphi\|_{L^2}^2
\]
\[
+ \left( 1 + 2t \int_{\mathbb{R}^d} u_k \varphi \, dx \right) |B_r|\,
\]
\[
\leq \lambda_k(\Omega^\ast) + 2t \int_{\mathbb{R}^d} \nabla u_k \cdot \nabla \varphi \, dx + t^2(2 + \lambda_k(\Omega^\ast)) \|\nabla \varphi\|_{L^2}^2 + 2|B_r|,
\]
which gives
\[
t \int_{\mathbb{R}^d} (-\nabla u_k \cdot \nabla \varphi + \lambda_k(\Omega^\ast) u_k \varphi) \, dx \leq t^2(1 + \lambda_k(\Omega^\ast)) \|\nabla \varphi\|_{L^2}^2 + |B_r|.
\]
By the definition of \( t \), we get
\[
\int_{\mathbb{R}^d} (-\nabla u_k \cdot \nabla \varphi + \lambda_k(\Omega^\ast) u_k \varphi) \, dx \leq (2 + \lambda_k(\Omega^\ast)) \|\nabla \varphi\|_{L^2}^2 |B_r|^{1/2},
\]
(3.2)
that, as both sides are linear in \( \varphi \), holds for every \( \varphi \in H^1_0(B_r) \).

Using the Poincaré inequality for the term \( \int_{\mathbb{R}^d} \lambda_k(\Omega^\ast) u_k \varphi \, dx \) in the ball \( B_r \) and Cauchy–Schwarz, we deduce that \( u_k \) is a quasi-minimizer and so Lipschitz continuous, as a consequence of theorem 3.3.

**Remark 3.6.** Note that if \( \Omega^\ast \) was regular, then we can deduce the same Lipschitz bound for \( u_k \) in a direct manner from our (3.2):
\[
\int_{\partial \Omega^\ast} \frac{\partial u_k}{\partial n} \, d\mathcal{H}^{d-1} \leq (2 + \lambda_k(\Omega^\ast)) \|\nabla \varphi\|_{L^2} |B_r|^{1/2}.
\]
Let \( \varphi(x) = e^{1-d} \eta((x - x_0)/\epsilon) \) for a radially symmetric and positive function \( \eta \) satisfying
\[
\eta \in C_\infty_c(\mathbb{R}^d), \quad \eta = 1 \text{ on } B_1, \quad \eta = 0 \text{ on } \mathbb{R}^d \setminus B_2.
\]
As \( \varphi \in H^1_0(B_{2\epsilon}) \), we get that
\[
\left( \int_{B_{2\epsilon}(x_0)} |\nabla \varphi|^2 \, dx \right)^{1/2} |B_{2\epsilon}|^{1/2} = \left( \int_{B_{2\epsilon}} e^{-2d|\nabla \eta|^2 \left( \frac{x}{\epsilon} \right)} \, dx \right)^{1/2} |B_{2\epsilon}|^{1/2} = \|\nabla \eta\|_{L^2(\mathbb{R}^d)} |B_1|^{1/2},
\]
which in turn gives that passing to the limit as $\epsilon \to 0$, we get
\[
|\nabla u_k(x_0)| \leq C_d(2 + \lambda_k(\Omega^*)) \quad \text{for every } x_0 \in \partial \Omega^*,
\] (3.3)
where $C_d$ is a dimensional constant.

The boundary estimate (3.3) (that we deduced for smooth supersolutions $\Omega^* \subset \mathbb{R}^d$) can be extended inside the domain $\Omega^*$ due to the subharmonicity of the $P$-function $P \in C^\infty(\Omega^*)$
\[
P(x) = |\nabla u_k|^2 + \lambda_k(\Omega^*) u_k^2 - 2\lambda_k(\Omega^*)\|u_k\|_{L^\infty(\mathbb{R}^d)} w,
\] (3.4)
where $w \in H_0^1(\Omega^*)$ is the solution of the problem
\[-\Delta w = 1 \text{ in } \Omega^* \quad \text{and} \quad w = 0 \text{ on } \partial \Omega^*.
\]
We note that the bounds
\[
\|u_k\|_{L^\infty(\mathbb{R}^d)} \leq C_d \lambda_k(\Omega^*)^{d/4} \quad \text{and} \quad \|w\|_{L^\infty(\mathbb{R}^d)} \leq C_d|\Omega|^{2/d},
\] (3.5)
where proved, respectively, in [17,18], for a suitable dimensional constant $C_d$. The boundary estimate (3.3) together with the subharmonicity of $P(x)$ gives that $P(x)$ is bounded on the entire domain $\Omega^*$. On the other hand, the uniform boundedness of the second and the third terms in $P$, provided by (3.5), gives that
\[
\|\nabla u_k\|_{L^\infty(\mathbb{R}^d)} \leq C(k, |\Omega^*|, \lambda_k(\Omega^*)),
\] (3.6)
where $C(k, |\Omega^*|, \lambda_k(\Omega^*))$ is a constant depending only on $k$, the measure $|\Omega^*|$ and the eigenvalue $\lambda_k(\Omega^*)$.

We can now summarize the following result (see [9] for a detailed analysis and a complete proof).

**Theorem 3.7.** If $\Omega^*$ is a solution for (1.3), then $\Omega^*$ possesses a $k$th eigenfunction which is Lipschitz continuous.

**Proof.** The argument above was carried out under a very strong assumption, namely that the strict inequality
\[
\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)
\]
holds on the optimal set. Unfortunately, it is conjectured that this assumption is false at all optimal domains, so that the eigenvalue is expected to be always multiple. The strategy to attack this difficulty is to use an approximation procedure for $\lambda_k$ with a functional of the form $\epsilon \lambda_{k-1} + (1 - \epsilon)\lambda_k$ and the properties of their super solutions.

Assume that $\Omega^*$ is an optimal set for the problem
\[
\min\{|\lambda_k(\Omega)| + |\Omega| : \Omega \subset \mathbb{R}^d\}.
\]
If $\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)$, the previous argument works. Let us assume that
\[
\lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*) > \lambda_{k-2}(\Omega^*). \quad \text{(3.7)}
\]
We now consider the auxiliary problem
\[
\min\{|(1 - \epsilon)\lambda_k(\Omega) + \epsilon \lambda_{k-1}(\Omega) + 2|\Omega| : \Omega^* \subset \Omega \subset \mathbb{R}^d\},
\]
which has a solution $\Omega_\epsilon$. The existence of such a solution was proved in [19]. We consider two cases:
If \( \lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*) \) for some \( \epsilon \in (0, 1) \), then we have

\[
\lambda_k(\Omega^*) + 2|\Omega^*| = (1 - \epsilon)\lambda_k(\Omega^*) + \epsilon \lambda_{k-1}(\Omega^*) + 2|\Omega^*| \\
\leq (1 - \epsilon)\lambda_k(\Omega^*) + \epsilon \lambda_{k-1}(\Omega^*) + 2|\Omega^*| \\
\leq \lambda_k(\Omega^*) + 2|\Omega^*| \leq \lambda_k(\Omega^*) + |\Omega^*| + |\Omega^*|, 
\]

which implies \( \Omega^* = \Omega_\epsilon \). Thus, \( \Omega^* \) is a supersolution for \((1 - \epsilon)\lambda_k(\Omega) + \epsilon \lambda_{k-1}(\Omega) + 2|\Omega| \). Using the fact that we can eliminate the decreasing functionals, we get that \( \Omega^* \) is also a supersolution for \( \epsilon \lambda_{k-1}(\Omega) + 2|\Omega| \) and is such that \( \lambda_k(\Omega^*) > \lambda_k(\Omega^*) \). Thus, we have that there is an eigenfunction corresponding to the eigenvalue \( \lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*) \), which is Lipschitz on the entire space.

If we have \( \lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*) \) for every \( \epsilon \in (0, 1) \), then we simply eliminate the decreasing term \( (1 - \epsilon)\lambda_k(\Omega) + \epsilon \lambda_{k-1}(\Omega) \) from the functional obtaining that \( \Omega^* \) is a supersolution for \( \frac{1}{2}\lambda_k(\Omega) + 2|\Omega| \). Thus, there is a \( k \)th eigenfunction on \( \Omega^* \) which is Lipschitz continuous on the entire space \( \mathbb{R}^d \). As the Lipschitz constant is independent on \( \epsilon \), we can pass to the limit as \( \epsilon \to 0 \) obtaining on the limit an eigenfunction on \( \Omega^* \) which is still globally Lipschitz (for the details on the approximation argument, we refer to [9]).

If assumption (3.7) does not hold, then we assume

\[
\lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*) = \lambda_{k-2}(\Omega^*) > \lambda_{k-3}(\Omega^*),
\]

consider the auxiliary problem

\[
\min\{(1 - \epsilon_1)\lambda_k(\Omega) + \epsilon_1((1 - \epsilon_2)\lambda_{k-1} + \epsilon_2 \lambda_{k-2}(\Omega)) + 2|\Omega| : \Omega^* \subset \Omega \subset \mathbb{R}^d, \}
\]

and reason in the same way. We continue this process possibly up to the first eigenvalue, at which it stops giving the Lipschitz regularity.

**Remark 3.8.** If \( F \) is bi-Lipschitz in (1.2), then for every index \( i_j \), the optimal set \( \Omega^* \) in (1.2) is also a supersolution for \( \lambda_i(\Omega) + K|\Omega| \). Consequently, \( \Omega^* \) has a Lipschitz eigenfunction corresponding to \( \lambda_i(\Omega) \). For more details and precise results, we refer to [9].

4. Further remarks and open problems

(a) Supersolutions for the perimeter functional

Another shape optimization problem in which the notion of a shape supersolution has a crucial role concerns the spectral functionals with a perimeter penalization, an example being

\[
\min\{\lambda_k(\Omega) + \text{Per}(\Omega) : \Omega \subset \mathbb{R}^d, |\Omega| < +\infty\}. \tag{4.1}
\]

Using the monotonicity of the eigenvalue, every minimizer \( \Omega^* \) for the problem above is a supersolution for the perimeter functional so, in particular, it has a positive mean curvature in viscosity sense.

We point out that every bounded measurable set which is supersolution for the perimeter, is equivalent a.e. to an open set on which the torsion function is Lipschitz regular up to the boundary. Moreover, all points of the complement satisfy a uniform Lebesgue density estimate. The existence of an optimal set for (4.1) and its \( C^{1,\alpha} \) regularity (up to a set of dimension \( d - 8 \)) was proved in [10].

(b) Subsolutions in the class of simply connected sets

In §2, we presented the main results concerning a single subsolution for the energy functional \( E + c|\cdot| \) in the class of quasi-open sets of finite measure. The same results hold in the case when the admissible set \( \mathcal{A} \) is the class of the open simply connected sets in \( \mathbb{R}^2 \) (we precise that
in dimension two, an open simply connected set is referred to be an open set, not necessarily connected, whose complementary is connected). This class of open sets is very important as it is compact for the $\gamma$-convergence (see [4] for the precise definition). As a consequence, there is a large class of shape functionals involving the Dirichlet–Laplacian for which there are optimal sets in this class. Moreover, the optimal shapes are a priori open and the torsion function is $C^{0,\alpha}$ up to the boundary, as a consequence of the capacity density condition satisfied genuinely by simply connected sets [20].

Subsolutions in the class of simply connected sets satisfy the conclusion of theorem 2.2 as both perturbations

$$\Omega_t := \Omega^* \setminus \overline{B}_r(x_0) \text{ and } \Omega_t = \{w_{\Omega^*} > t\},$$

remain in the class of the simply connected open sets. Indeed, in the second case, we can easily prove that the level set $\Omega_t$ is simply connected. Suppose by absurd that the set $\mathbb{R}^d \setminus \Omega_t$ has a connected component $K$ that does not intersect the connected set $\mathbb{R}^d \setminus \Omega^*$. Then, we have that $K \subset \Omega^*$, $w \leq t$ on $K$ and moreover, by taking $t + \epsilon$ for $\epsilon > 0$ very small, we can suppose that $|K| > 0$. Now, as $w_{\Omega^*} \in H^1_0(\Omega^*) \cap H^1_{loc}(\Omega^*)$ is the unique minimizer of the functional

$$w \mapsto \int_{\mathbb{R}^d} \left(\frac{|
abla w|^2}{2} - w\right) \, dx,$$

in $H^1_0(\Omega^*)$, we can test its optimality with the function $w_t \in H^1_0(\Omega^*)$ defined as

$$w_t = w_{\Omega^*} \text{ on } \mathbb{R}^d \setminus K \text{ and } w_t = t \text{ on } K,$$

obtaining that $w_t = w_{\Omega^*}$ and so $w_{\Omega^*} \equiv t$ on $K$. On the other hand, as $K$ is of positive measure, we have $\Delta w_{\Omega^*} = 0$ on $K$, which is a contradiction with the fact that $-\Delta w_{\Omega^*} = 1$ on $\Omega^*$.

As a consequence of the above observations, we obtain the following result.

**Theorem 4.1.** Suppose that the open simply connected set $\Omega^* \subset \mathbb{R}^d$ of finite measure is a (local) subsolution for the functional $E + c \cdot |$ in the class of open simply connected sets. Then

(i) $\Omega^*$ is a bounded set. Moreover, there exist constants $C, r_0 > 0$ depending on the measure of $\Omega$, $d, \epsilon$ and $c$ such that $\Omega$ can be covered with less than $C r^{-d-1}$ balls of radius $r$, for every $r \leq r_0$;

(ii) $\Omega^*$ has finite generalized perimeter in the sense of De Giorgi, and

$$\sqrt{\frac{c}{2}} \text{Per}(\Omega^*) \leq |\Omega^*|.$$

(c) **Supersolutions for functionals involving Schrödinger operators**

One can consider spectral functionals involving the eigenvalues $\lambda^V_k(\Omega)$ of the Schrödinger operator $-\Delta + V$ on $\Omega$ with Dirichlet boundary conditions on $\partial \Omega$. Provided a suitable relationship between $\Omega$ and $V$ (e.g. $\Omega$ is of finite measure and $V$ is arbitrary, or $\Omega$ is arbitrary and $V$ has a suitable behaviour at infinity), the spectrum consists only on eigenvalues. The study of the spectral optimization problem

$$\min \left\{ F(\lambda^V_k(\Omega), \ldots, \lambda^V_N(\Omega)) + \int_\Omega h(x) \, dx : \Omega \subset \mathbb{R}^d \right\},$$

for fixed positive functions $V : \mathbb{R}^d \to \mathbb{R}^+$ and $h : \mathbb{R}^d \to \mathbb{R}^+$, naturally reduces to the study of the subsolutions to the functional

$$\Omega \mapsto E^V(\Omega) + \int_\Omega h(x) \, dx,$$

where the energy $E^V(\Omega)$ is given by

$$E^V(\Omega) = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} Vu^2 \, dx - \int_{\mathbb{R}^d} u \, dx : u \in H^1_0(\Omega \cap B_R), R > 0 \right\}.$$
In this case, assuming suitable growth conditions on \( V \) and \( h \), one can still deduce the boundedness of a subsolutions \( \Omega^* \) by using test sets obtained by cutting away a half-space from \( \Omega^* \) ([21], [12, §4.5]).

**Lemma 4.2.** Let \( \Omega^* \) be a local subsolution for the functional \( E^V(\Omega) + m \int \Omega h(x) \, dx \), where \( m > 0 \) and \( V : \mathbb{R}^d \to [0, +\infty] \) and \( h : \mathbb{R}^d \to [0, +\infty] \) are given measurable functions such that

\[
E^V(\Omega^*) > -\infty, \quad \int_{\Omega^*} h(x) \, dx < +\infty \quad \text{and} \quad h \geq V^{-\alpha},
\]

for some \( \alpha \in [0, 1) \), then \( \Omega^* \) is a bounded set.

**(d) Open problems**

For \( c > 0 \), consider the energy functional \( E(\Omega) + c|\Omega| \) defined on the family of quasi-open sets of finite measure in \( \mathbb{R}^d \). Consider a quasi-open set \( \Omega^* \subset \mathbb{R}^d \) which is

— a subsolution for the functional \( E(\Omega) + c|\Omega| \), for some \( c > 0 \); and
— a supersolution for the functional \( E(\Omega) + C|\Omega| \), for some \( C \geq c \).

If \( \Omega^* \) was a regular set, then the above condition would imply that

\[
\sqrt{c} \leq |\nabla w_\Omega| \leq \sqrt{C} \quad \text{on } \partial \Omega^*,
\]

where \( w_\Omega \in H^1_0(\Omega) \) is the torsion function.

On the other hand, if \( \Omega^* \) is just a quasi-open set, then repeating the arguments from [5], we obtain that the conditions above will provide us with the following primary regularity properties of \( \Omega^* \):

— \( \Omega^* \) is a bounded open set of finite perimeter;
— the torsion function \( w_{\Omega} \) is Lipschitz continuous on \( \mathbb{R}^d \);
— if \( x_0 \in \partial \Omega^* \), then we have the following two-side density estimate for (small) balls centred in \( x_0 \):

\[
\delta \leq \frac{|B_r(x_0) \cap \Omega^*|}{|B_r|} \leq 1 - \delta,
\]

where \( \delta \) is a constant depending on \( c \) and \( C \); and
— if \( x_0 \in \partial \Omega^* \), then \( \Omega^* \) is Alhfors regular, i.e. there is a constant \( \epsilon > 0 \) such that

\[
\epsilon r^{d-1} \leq H^{d-1}(B_r(x_0) \cap \partial \Omega^*) \leq \frac{1}{\epsilon} r^{d-1}.
\]

The further arguments in [5] depend strongly on the assumption that the gradient is not only bounded from above and below on the boundary but is also continuous in some weak sense. This assumption does not hold \textit{a priori} for a set \( \Omega^* \) as above. Therefore, we raise the following question.

**Problem 4.3.** If the (open) set \( \Omega^* \) is a subsolution for the functional \( E + c|\cdot| \) and a supersolution for \( E + C|\cdot| \), is it true that the boundary \( \partial \Omega^* \) is locally the graph of a Lipschitz function?

A second open problem concerns the regularity of the following shape optimization problem involving the first eigenvalue of the buckling problem. For every simply connected open set \( \Omega \) of finite measure in \( \mathbb{R}^2 \), we define

\[
\Lambda(\Omega) = \min \left\{ \frac{\int_{\Omega} |\Delta u|^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} : u \in H^2_0(\Omega), \ u \neq 0 \right\}.
\]

One considers the problem

\[
\min\{\Lambda(\Omega) + |\Omega| : \Omega \subseteq \mathbb{R}^2, \ \Omega \text{ simply connected}\}. \quad (4.2)
\]
It has been conjectured since 1951 that the solution of this problem is the ball (see [22] for an overview of open problems of spectral isoperimetric type). Following an idea of Wills and Weinberger [23], a way to prove this fact is to prove the existence of a smooth set $Ω^*$ solving (4.2). We refer to [24] for a proof of the existence, but still the smoothness question remains unsolved.

**Problem 4.4.** Prove the existence of a smooth solution for (4.2).

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