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Quantum generalized observables framework for psychological data: a case of preference reversals in US elections

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Politics is regarded as a vital area of public choice theory, and it is strongly relying on the assumptions of voters' rationality and as such, stability of preferences. However, recent opinion polls and real election outcomes in the USA have shown that voters often engage in 'ticket splitting', by exhibiting contrasting party support in Congressional and Presidential elections (cf. Khrennikova 2014 *Phys. Scripta* **T163**, 014010 (doi:10.1088/0031-8949/2014/T163/014010); Khrennikova & Haven 2016 *Phil. Trans. R. Soc. A* **374**, 20150106 (doi:10.1098/rsta.2015.0106); Smith *et al.* 1999 *Am. J. Polit. Sci.* **43**, 737–764 (doi:10.2307/2991833)). Such types of preference reversals cannot be mathematically captured via the formula of total probability, thus showing that voters' decision making is at variance with the classical probabilistic information processing framework. In recent work, we have shown that quantum probability describes well the violation of Bayesian rationality in statistical data of voting in US elections, through the so-called *interference effects* of probability amplitudes. This paper is proposing a novel generalized observables framework of voting behaviour, by using the statistical data collected and analysed in previous studies by Khrennikova (Khrennikova 2015 *Lect. Notes Comput. Sci.* **8951**, 196–209) and Khrennikova & Haven (Khrennikova & Haven 2016 *Phil. Trans. R. Soc. A* **374**, 20150106 (doi:10.1098/rsta.2015.0106)). This framework aims to overcome the main problems associated with the quantum probabilistic representation of psychological data, namely the *non-double stochasticity* of transition probability matrices. We develop a simplified construction of generalized

positive operator valued measures by formulating special non-orthonormal bases with respect to these operators.

This article is part of the themed issue 'Second quantum revolution: foundational questions'.

1. Introduction

Decision theories under uncertainty and risk (expected utility theory under risk by von Neumann [1], subjective expected utility under uncertainty by Savage [2]) are applied as key building blocks in modern economic and finance models, as well as in public choice theory. In recent decades, starting from the advent of behavioural science and its penetration into the traditional domains of economics and finance, expected utility theories encountered a wave of experimental studies that challenged their axiomatic foundations. One of which is the rational mode of information processing and decision formation, which rests upon the canonical formulation of Kolmogorovian probability theory.

One of the core axioms of consequential reasoning (i.e. following classical probability that implies the Bayesian mode of updating of new information) namely Savage's [2] *Sure Thing Principle* was shown to be not satisfied in DM's (decision maker's) preference formations. More specifically, the DM's choice of frequency for some option A without any certain information given on a conditional event B , and its negation, B' , was very often below the conditional frequencies $A|B$ and $A|B'$, as well as below the total probability expressed via the disjunction of these conditional choice outcomes (given by the formula of total probability [3]). This type of decision-making fallacy is also known as *disjunction effect*, which incorporates both sub- and super-additivity of disjunctions in the formula of total probability. Findings showed that a disjunction effect was exhibited in a variety of decision-making contexts, involving both objective and subjective risk (uncertainty), related to preferences for monetary pay-offs [4–8], as well as voting preferences [9–11].

Many of the applications related to the above-discussed probabilistic fallacy, emerging under uncertainty and risk, were explained by generalized models of quantum probability that relaxes the additivity and distributivity axioms of classical probability theory (cf. [4,10,12–20] and others). The usage of the generalized quantum framework allowed one to capture the observed probability sub-additivity effect in the behaviour of decision makers via the so-called interference term (emerging due to interference of probability amplitudes [1]).

We can witness that the collection of decision-making situations where a disjunction effect was detected is quite broad, and the emergence of non-classicality of human reasoning could be well presented through the geometric properties of Hilbert space and decision-making projectors therein. However, quantum probability models faced a major constraint, in terms of applicability of the 'genuine' quantum formalism. The so-called *transition probability matrices* for the conditional probabilities collected in decision-making experiments exhibit stochasticity (due to the mutual exclusivity of choices in each context), but violate double stochasticity (a requirement for usage of Hermitian operators), to represent random variables in a quantum framework.¹ A body of the literature approached the non-double stochasticity constraints via the so-called generalized versions of Hermitian operators (positive operator valued measures, POVMs; cf. [22,23]), where the latter work explores the possible origins of non-additivity of transition probabilities for statistics in decision making and reasoning.

In quantum probability models, observables (such as questions and decision-making tasks) are represented by quantum observables. In applied models, it was shown to be problematic to find an appropriate mathematical form of operator representation of psychological observables (that

¹Some authors approached this constraint by increasing the dimension of state space representation of the observables (cf. [6,21]). In these models, degenerate spectra can be used to represent eigenvalues by corresponding bases that are modelled as subspaces of larger dimensions. We also remark that there is one exception, where one dataset, collected in the so-called 'Categorization decision experiment', is obeying double stochasticity (cf. [4, p. 236]).

describes the experimental statistics), due the above discussed violation of double stochasticity for projective measurements.² If the matrix of transition probabilities would be doubly stochastic, then psychological observables could be described by the conventional quantum probability framework, through the usage of Hermitian operators (cf. a mathematical representation of such a measurement scheme in §4, equations (4.8)–(4.13)).

However, for a non-doubly stochastic matrix, it is impossible to represent statistical data by Hermitian operators (with non-degenerate spectra). One has to proceed with generalized quantum observables given by POVMs. Unfortunately, there is no specified algorithm on construction of a POVM observable, corresponding to an arbitrary collection of statistical data (in our analysis and in related studies, we consider probabilities for two incompatible observables in combination with a set of transition probabilities). Moreover, to our knowledge, no previous works focused on exploring whether, for some arbitrary statistics, an operational POVM representation with the conventional Born rule would be possible at all. In this paper, we suggest a quantum-like (via a generalization of the constraints imposed on the classical POVM quantum measurement scheme) representation for statistical data on voting preferences, with a non-doubly stochastic matrix of transition probabilities. To be able to describe precisely the decision-making statistics, a generalization of the conventional POVM structure is proposed. We relax the condition of additivity up to the unit operator in equation (3.1), i.e. for the generalized projectors Q_k we can have $\sum_k Q_k \neq I$.

To operate with such a *generalization of quantum observables* beyond the standard POVM formalism, we proceed with a corresponding generalization of the Born rule, in equation (3.5), where the presence of the denominator differentiates it from the classical Born rule, given in equation (3.2).

The devised paradigm has a straightforward geometric interpretation. In the presented general framework, the eigenstates of our generalized observables form *non-orthogonal bases*. Orthogonal bases correspond to the special case of double stochasticity in our model and, hence, a possibility of Hermitian operators representation. We also provide a numerical computation of the devised generalized quantum operators, applied to the voting statistics collected in previous studies (cf. §6).

2. A short introduction to politics and voting theory

Politics is regarded as a vital area of social science and relies strongly on the assumption of voters' rationality, implying a stability of preferences. People would naturally follow the same principles, i.e. the axioms of rationality, in their political decisions as well as in other situations, such as investment decisions. The rational choice paradigm of modern decision theories was naturally conveyed into decision-making domains other than economics, namely political decisions, in particular voting theory. Political decision making is a special sphere, where humans have to make decisions with far-reaching implications for themselves and society as a whole. The types of decisions can involve ballot casting in different types of elections from local to governmental. Similarly, on the party level, the involved parties as political entities have the responsibility to strategically plan their political actions, taking into consideration all possible consequences. Political decisions are by their nature not as specific as the lottery choices in von Neumann–Morgenstern (vNM) expected utility theories. It can be difficult to associate political decisions with some concrete expected utilities derived from monetary pay-offs. At the same time, some non-monetary benefits can always be attached to the outcomes of different policies that are

²We remark that the conventional projection postulate quantum measurement scheme, based on the application of Hermitian operators to depict observables, demands that the row probabilities and column probabilities in a transition probability matrix (i.e. probabilities of transition from one orthonormal basis to the other) are equal to one if one proceeds with non-degenerate spectra observables. This is due to the need to preserve the unit length requirement of the state vector ψ . When faced with violations of the double stochasticity rule, in the context of state space representation of psychological variables, one could proceed with the usage of Hermitian matrices of size larger than 2×2 , by introducing observables with degenerate spectra. Effectively, a larger dimension representation for the same number of eigenvalues would be used. Another solution is to relax some constraints imposed on the Hermitian operators. This is done by introducing POVMs; cf. detailed mathematical exposition in Khrennikov & Basieva [23] and Brandt [24].

exercised at the governmental, as well as at a local level. A large body of research in decision theory operates on the assumption that the domain of the utility function can also be used to derive personal utility over a range of consequences beyond monetary outcomes [25].

(a) Some features of the US political system

In the USA, as we mention in Khrennikova [9] (see p. 2 for much more details), voters typically keep preferences stable, in the sense that they will keep supporting the same political party both in the White House and in (at least one of) the Houses of Congress elections. See [26–28] for studies in political science on this very topic. In line with the postulates of vNM utility theories, voters would naturally choose the same party in both types of elections, obeying principles of completeness and invariance of choices. Over the last 40 years, the situation of power distribution in US politics began to change and voters started to seek to divide political power, at least based on the results of election campaigns that led to the domination of Congress and the White House by different political parties. Many studies, including the study by Smith *et al.* [28], associated the emergence of such bipartisan preferences with a voters' tendency to *condition* their choices for Congress upon the President's party (and vice versa) to achieve some personal goals. At the same time, the question emerged whether such types of preference reversals are either a strategic act, or a type of heuristic thinking (cf. [29]). As such, voters may have a so-called preference 'non-separability' for different election contests, but at the same time they do not follow a classical mode of information processing.

(b) Disjunction effect and preference reversals in US voting statistics

A cascade of behavioural studies focused on the emergence of preference reversals exhibited by US citizens in the process of ballot casting, where the study of Smith *et al.* [28] explicitly focused on collecting evidence on preference reversals in Congress elections, related to obtaining certain information on the outcomes of Presidential elections. Smith *et al.* [28] showed that voters are highly influenced by the informational context; in particular, the respondents of their opinion poll study strongly relate the outcomes of the Presidential elections to their subsequent decisions, by altering their preferences in favour of another party in the context of Congressional elections. In a related study, Khrennikova & Haven [11] collected voting statistics from actual US elections (period 2008–2014), in order to explore whether US voters exhibit conditional preferences in the Congress elections. With the aid of an additional large-scale opinion poll study [30], the authors analysed voting frequencies and ascertained the existence of a trend among US voters to switch their partisanship support in the Congress elections.

Furthermore, in [10,11,31] the authors carried out a so-called theoretical analysis of the statistical data, to observe whether non-classicality of voters' information processing from the viewpoint of classical probability theory holds. The analysis of the collected data demonstrated violation of the *Formula of Total Probability* [3]. The authors proposed a quantum probability framework, seeking to overcome the non-commutativity of question-observables, acting upon voters' decision-making states in the process of ballot casting. Interference terms, expressing the so-called *coefficients of non-commutativity* were computed, and their psychological interpretation was discussed in Khrennikova & Haven [31].

In §2c, we will extend the analysis of voting statistics, by computing the average unconditional and conditional preferences from the studies performed by Smith *et al.* [28] and Khrennikova & Haven [11], and performing a statistical analysis to ascertain the degree of non-classicality of statistical data.³ In §2c(i), we also carry out a theoretical analysis of the summary statistics, with respect to ascertaining the degree of non-commutativity (via the interference term) presenting the

³We are aware of the limitations related to merging and averaging statistics, obtained from different cohorts of voters and periods. The averaging of results is due to the endeavour to propose a general framework (we name it the *generalized quantum observables framework*) for a representative US voter (or any voter who operates in a two-party political system). We adopt a similar approach on data averaging as used by Busemeyer & Bruza [4] and Pothos & Busemeyer [6], for generalizing the results from different related experiments and studies.

Table 1. Summary of the conditional and unconditional preferences of voters in US Congress elections.

| preference statistics | baseline | | Cond. Pres.=D | | Cond. Pres.=R | | Pres.=(D, R) | |
|--------------------------|----------|-------|---------------|-------|---------------|-------|--------------|-------|
| Smith <i>et al.</i> [28] | C_d | 0.44 | $C_d P_d$ | 0.336 | $C_d P_r$ | 0.43 | P_d | 0.5 |
| | C_r | 0.56 | $C_r P_d$ | 0.664 | $C_r P_r$ | 0.57 | P_r | 0.5 |
| Khrennikova & Haven [11] | C_d | 0.47 | $C_d P_d$ | 0.915 | $C_d P_r$ | 0.073 | P_d | 0.537 |
| | C_r | 0.53 | $C_r P_d$ | 0.085 | $C_r P_r$ | 0.927 | P_r | 0.463 |
| average results | C_d | 0.455 | $C_d P_d$ | 0.625 | $C_d P_r$ | 0.25 | P_d | 0.518 |
| | C_r | 0.545 | $C_r P_d$ | 0.375 | $C_r P_r$ | 0.75 | P_r | 0.482 |

matrix of transition probabilities. In §§2d–6, we devise a two-observables quantum-like model, by deriving a generalized observables scheme that could be potentially adopted to other types of statistics in decision making.

(c) Classical probabilistic scheme: a case of non-additivity of voters' preferences

The core framework to represent the classical probabilistic assessment of marginal probability by a DM (with respect to objective as well as subjective probabilities) is the formula of total probability (henceforth, FTP), resting upon Bayesian conditional probabilities, given some disjoint events (also known as states of the world); see Kolmogorov [3]. In the analysis presented in this paper, the states of the world are corresponding to $s_1 = P_d$ (Democratic President elected) and $s_2 = P_r$ (Republican President elected).⁴ By representing the Congressional and Presidential elections via dichotomous random variables (C and P), the total probability to prefer a Congress dominated by Democrats can be defined via the FTP as⁵

$$p(C_d) = p(P_d)p(C_d|P_d) + p(P_r)p(C_d|P_r). \quad (2.1)$$

In a similar way, the total probability p_{tot} for (C_r) can be expressed. If the FTP in equation (2.1) does not hold, then the total probability of the disjunctions on the right-hand side of the formula does not equal the baseline probability in the left-hand side and, thus, some of the voters do not follow the classical probabilistic scheme in their information update.

Table 1 summarizes the average unconditional and conditional statistics from the studies of Smith *et al.* [28] and Khrennikova & Haven [11]. By embedding the frequencies from the last row (average results) in table 1 into the FTP from equation (2.1), we can ascertain whether non-additivity of voters' conditional preferences exists.⁶ With respect to the C_d outcome, we obtain a baseline voting preference from the average results in table 1, given by $p(C_d) = 0.455$. From the FTP, we get a total probability denoted as $p_{\text{tot}}(C_d)$:

$$0.625 \times 0.518 + 0.25 \times 0.482 = 0.44 < 0.455. \quad (2.2)$$

With respect to (C_r), we get a baseline probability, $p(C_r) = 0.545$, and the FTP results as follows:

$$0.375 \times 0.518 + 0.75 \times 0.482 = 0.56 > 0.545. \quad (2.3)$$

⁴DM's assessment of disjunctions of conditional probabilities associated with realization of some choices (i.e. conditioned upon the probability of $s_1 = P_d$ and corresponding consequences, respective $s_2 = P_r$) should be *additive* with respect to the baseline probability of that choice. Such an equality would hold if a DM can process information in a classical probabilistic mode. As we have discussed in §2a, the voters conditioning their preferences to split their Congressional and Presidential preferences ought to follow a Bayesian updating procedure.

⁵Democrats are denoted by an index d and Republicans by r.

⁶The marginal probabilities for outcomes $P = (P_d, P_r)$ were assessed based on the frequency of being in the state P_d and P_r , respectively. In [28], the samples of voters for which a random variable P (Presidential elections) was measured were equal, thus $p(P_d) = p(P_r) = 0.5$. In [11], the statistics were extracted from actual US elections and thus $p(P_d), p(P_r)$ represent the frequency of being the part of the electorate who voted for a Democrat and Republican President, respectively, with $p(P_d) = 0.537$ and $p(P_r) = 0.463$. We also removed from the samples all the voters who abstained from voting for either the Democrat or Republican Congress, both in unconditional and conditional settings.

We have witnessed that the statistics above indicate a minor sub-additivity of the disjunctions with respect to the voting for Democratic Congress and super-additivity in respect to voting for a Republican Congress.⁷

(i) Transition probabilities and interference effect

From a quantum point of view, the violation of the FTP has its origins in the interference of probabilities, when they are not observed (e.g. through a question measurement). In a political context, the cognitive state of voters can be said to be in a superposition of different beliefs, with respect to the outcomes of the Presidential elections.

We depict the indeterminacy of voters' decision states, with respect to the Presidential elections variable, through an extension of equation (2.1) with the so-called *interference term* defined as

$$p(C_d) = p(P_d)p(C_d|P_d) + p(P_r)p(C_d|P_r) + 2\lambda_d \sqrt{p(P_d)p(C_d|P_d)p(P_r)p(C_d|P_r)}. \quad (2.4)$$

The probability interference coefficient is represented here in a general form through λ_d that can be a conventional trigonometric cos interference, as well as a more exotic hyperbolic cosh interference, when $\lambda > 1$ (cf. [18]).

In the analysed statistics from table 1, we have computed the following values for the interference coefficients: with respect to the C_d coordinate, we have $\cos \theta_d = 0.0275$ and with respect to the C_r , $\cos \theta_r = -0.02028$. Hence, we observe a positive interference of voters' beliefs (of a small magnitude) with respect to the Democrat Congress outcome, and a minor negative interference with respect to the Republican Congress choice. The matrix of transition probabilities (M) which allows one to reconstruct voters' belief state with respect to the basis state C_d and C_r , respectively, has the form

$$M = \begin{bmatrix} p(C_d|P_d) & p(C_d|P_r) \\ p(C_r|P_d) & p(C_r|P_r) \end{bmatrix} = \begin{bmatrix} 0.625 & 0.25 \\ 0.375 & 0.75 \end{bmatrix}.$$

We can witness that the matrix of transition probabilities is not satisfying double stochasticity. By definition double stochasticity would imply that $p(C_d|P_d) = 1 - p(C_r|P_d) = p(C_r|P_r)$; $p(C_d|P_r) = 1 - p(C_r|P_r) = p(C_r|P_d)$. Yet, we can witness from M that $p(C_d|P_d) \neq p(C_r|P_r)$, and similarly, $p(C_d|P_r) \neq p(C_r|P_d)$.⁸

Mathematical representation of observables in Hilbert state space requires the usage of orthonormal bases associated with state vectors that are transformed by Hermitian operators. Orthonormality (orthogonal and normalized by one state vector) is the key property of state representation of quantum observables. For two Hermitian operators A and B , associated with the observables for Presidential and Congress elections that act on the voters' state vector in a two-dimensional Hilbert space with non-degenerate spectra, one can express the state vector with respect to the B observable:

$$\psi = \psi_1 e_1 + \psi_2 e_2$$

with orthonormal, (e_1, e_2) basis. With respect to the A observable, we depict the state vector

$$\psi = c_1 f_1 + c_2 f_2$$

with corresponding basis, (f_1, f_2) . Owing to the normalization condition on the state vector ψ , one always obtains additivity to unity of the complex amplitudes that give the baseline probabilities: $|\langle \psi | e_1 \rangle|^2 + |\langle \psi | e_2 \rangle|^2 = 1$; and in the same vein in respect of the A observable the

⁷We see that the violation of the FTP is minor. At the same time we note that the number of respondents in the sample from the study [30] that we used in our analysis is reported to be quite large, $N = 45000$. For the baseline question outcome C_d , the statistics were obtained from the actual electorate that voted for the Congress. A χ^2 -test for goodness of fit (with $p(C_d)$ as the expected frequency) provided $\chi^2_{(1)} = 40.831$ with $p < 0.01$.

⁸To observe the deviations from double stochasticity, we performed a Z-test for proportions, to check whether the discrepancy between the above conditional probabilities that are supposed to be equal in a double stochastic matrix (0.625 and 0.75, respective 0.25 and 0.375) is due to chance. The results are indicating that the observed (conditional) frequency 0.625 is significantly lower than 0.75 with $Z = 38.763$, $p < 0.01$ (the same results for the 0.375 frequency being significantly above 0.25). Hence, we are motivated to proceed with the usage of non-Hermitian operators, to capture the statistical data.

inner product gives normalization: $|\langle \psi | f_1 \rangle|^2 + |\langle \psi | f_2 \rangle|^2 = 1$. After the voters' belief state interacts with the A observable (related to the Democrat and Republican Presidencies), the state ψ is updated in respect of the basis (f_1, f_2) . Hence, due to orthogonality and normalization by one of the basis vectors

$$\|f_1\|^2 = |\langle f_1 | e_1 \rangle|^2 + |\langle f_1 | e_2 \rangle|^2 = p(C_d | P_d) + p(C_r | P_d) = 1$$

and

$$\|f_2\|^2 = |\langle f_2 | e_1 \rangle|^2 + |\langle f_2 | e_2 \rangle|^2 = p(C_d | P_r) + p(C_r | P_r) = 1.$$

In the same way, the sum of the squared coordinates for the (e_1, e_2) basis can be written if (f_1, f_2) are orthogonal

$$\|e_1\|^2 = |\langle f_1 | e_1 \rangle|^2 + |\langle f_2 | e_1 \rangle|^2 = p(C_d | P_d) + p(C_d | P_r) = 1$$

and

$$\|e_2\|^2 = |\langle f_1 | e_2 \rangle|^2 + |\langle f_2 | e_2 \rangle|^2 = p(C_r | P_d) + p(C_r | P_r) = 1.$$

Double stochasticity is apparently violated in our statistical data, hence the basis for the A observable does not obey orthogonality, i.e. (f_1, f_2) are not mutually orthogonal.

We remark that the roots of the above violations in cognitive data are yet to be discovered (e.g. by exploring decision-making contexts, where this violation diminishes).⁹ Possible interpretations of the above violation of additivity in the matrix M are suggested in Haven & Khrennikov [16], mentioning the possible unconventionality of psychological data that violate both the probabilistic rules of classical probability theory and conventional quantum mechanical rule, imposed on the operator projectors. In the second instance, the measurements on cognitive states of DMs may involve some hidden parameters (additional information about the state of the DM) that cannot be captured through the usual von Neuman–Lüders projective measurement, unless operators with non-degenerate spectra (with larger than one-dimensional eigensubspaces) are applied.

To be able to contain the simplicity of a two-dimensional quantum probabilistic model, we propose to represent the psychological observables via another class of operators that relaxes the orthogonality constraints of Hermitian projectors. We proceed with the presentation of a simple generalization scheme in §3.

(d) Generalized framework of voters' decision making

Based on the example of decision-making of voters who are faced with two decisions, the following random variables are measured: (i) the first decision-making question is related to Congressional elections and (ii) the second variable is associated with a decision on the Presidential election outcome.

We devise two dichotomous observables, $A = \alpha_1, \alpha_2$ and $B = \beta_1, \beta_2$, that represent these random variables, associated with the outcomes of the two election contests, Presidential and Congressional, as discussed above. To simplify the model construction, we restrict the representation to two possible values associated with each observable, and this is suitable for depicting the casting of ballots in a two-party political system such as in the USA. The model can also be generalized to other decision-making settings. The decision-making statistics can be

⁹The same problem persists for violations of the classical probabilistic scheme, via the emergence of various probabilistic fallacies, including the disjunction effect explored here. Quantum probability allows one to use an alternative probabilistic framework that overcomes the additivity constants of classical probability theory. Yet, a more structured analysis of cognitive processes behind the violations of the FTP (and respective magnitudes of these violations) would enlighten further the 'non-consequential reasoning' paradigm [7] and the 'incompatible decision observables' paradigm [4,16].

presented in a general form via frequencies, approximated by probabilities:

$$p_j^b = P(B = \beta_j), \quad p_j^a = P(A = \alpha_j) \quad \text{and} \quad p_{ij} = P(B = \beta_j | A = \alpha_i). \quad (2.5)$$

The probabilities on the right-hand side are *transition probabilities* that we depicted in a matrix of transition probabilities, in §6.¹⁰ We proceed with the simplified notations from equation (2.5) throughout the generalized framework construction. We represent the psychological data via a complex probability amplitude, and observables by special generalized operators acting in a (two-dimensional) complex Hilbert space.

We rewrite the FTP with an *interference term of the trigonometric type* with the aid of the above (generalized) notations:

$$p_j^b = p_1^a p_{1j} + p_2^a p_{2j} + 2 \cos \theta_j \sqrt{p_1^a p_{1j} p_2^a p_{2j}}. \quad (2.6)$$

We now use the elementary formula

$$D = a + b + 2\sqrt{ab} \cos \phi = |\sqrt{a} + e^{i\phi} \sqrt{b}|^2$$

for real numbers $a, b > 0, \phi \in [0, 2\pi]$ to introduce a complex amplitude corresponding to the FTP (2.6):

$$\psi_j = \sqrt{p_1^a p_{1j}} + e^{i\theta_j} \sqrt{p_2^a p_{2j}}, \quad j = 1, 2. \quad (2.7)$$

This amplitude can be used to represent the probability p_j^b as the square of the complex amplitude (Born's rule):

$$p_j^b = |\psi_j|^2. \quad (2.8)$$

By formula (2.7), the probabilistic data are represented by the unit normalization of the vector of the two-dimensional complex Hilbert space:

$$\psi = \psi_1 e_1 + \psi_2 e_2, \quad (2.9)$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.10)$$

Born's rule for complex amplitudes (2.8) can be rewritten in the following form:

$$p_j^b = |\langle \psi, e_j \rangle|^2.$$

The B -observable (corresponding to Congress election decision task) we could potentially represent by the Hermitian operator

$$B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad (2.11)$$

which is diagonal with respect to the basis (e_1, e_2) .

Next, we proceed by constructing an operator representation for the A -observable (representing the random variable associated with the Presidential elections). The matrix of transition probabilities (p_{ij}) does not obey the double stochasticity assumption, required for the usage of Hermitian operators. To proceed to the operator representation of the A -observable, we represent the pure state ψ as

$$\psi = \sqrt{p_1^a} f_1 + \sqrt{p_2^a} f_2, \quad (2.12)$$

where

$$f_1 = \begin{pmatrix} \sqrt{p_{11}} \\ \sqrt{p_{12}} \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} e^{i\theta_1} \sqrt{p_{21}} \\ e^{i\theta_2} \sqrt{p_{22}} \end{pmatrix}. \quad (2.13)$$

If the vectors (f_1, f_2) were orthonormal, the A -observable could be represented by a Hermitian operator, diagonal in the basis (f_1, f_2) . As the latter is not orthonormal, the representation by a

¹⁰When applied to the voting statistics, p_{ij} are associated with the conditional probabilities $p(C_d | P_d)$; $p(C_d | P_r)$; $p(C_r | P_d)$; $p(C_r | P_r)$.

Hermitian operator is impossible. We remark that the vectors $f_j, j = 1, 2$ are always normalized by one:

$$\|f_1\|^2 = p_{11} + p_{12} = P(B = \beta_1 | A = \alpha_1) + P(B = \beta_2 | A = \alpha_1) = 1$$

and

$$\|f_2\|^2 = p_{21} + p_{22} = P(B = \beta_1 | A = \alpha_2) + P(B = \beta_2 | A = \alpha_2) = 1.$$

This is the property of stochasticity of the matrix of transition probabilities.¹¹

3. Generalized quantum(-like) observables

Definition 3.1. A POVM is a family of linear operators $A = (Q_k)$ such that each Q_k is Hermitian and positive semidefinite and the following normalization takes place:

$$Q \equiv \sum_k Q_k = I. \quad (3.1)$$

Each POVM represents a quantum observable (denoted by the same letter) and, for a quantum state given by a density operator ρ , the probabilities of its values are defined by the following generalization of the Born rule:

$$p(A = \alpha_k) = \text{Tr } \rho Q_k. \quad (3.2)$$

Here, (α_k) are the values of A . The probabilities satisfy the condition of normalization by one:¹²

$$\sum_k p(A = \alpha_k) = 1. \quad (3.3)$$

We note that the class of quantum observables represented by POVMs generalizes the originally considered (by von Neumann–Lüders and Dirac) class of quantum observables given by Hermitian operators. If we consider, in a finite-dimensional space, a Hermitian operator A and its orthogonal projectors Q_k onto eigenspaces H_k , corresponding to its eigenvalues α_k , then Q_k is the POVM (in the infinite-dimensional case, the mathematics is more complicated, because there exist operators with a continuous spectrum).

Further developments in quantum measurement theory (e.g. [32–34]) demonstrated that this type of projection POVMs was too restricted, and it was essentially extended. This historical remark is very important for the future applications of the quantum measurement framework (and its generalizations) to psychological statistical data. In quantum measurement, scientists pragmatically extended the class of quantum observables—to serve new problems, especially in the domain of quantum information theory.

In a similar vein, in applications to decision making and cognition, some problems can be described by such generalized observables better than by the standard POVMs.¹³ We extend definition 3.1, by relaxing the normalization condition (3.1); hence we only assume that

$$Q \equiv \sum_k Q_k \neq 0. \quad (3.4)$$

We call such an observable $A = (Q_k)$, where the Q_k are Hermitian and positive semidefinite and (3.4) holds, a *generalized observable*. In the light of the relaxation of (3.1)–(3.4), the definition of probabilities (3.2) also has to be modified:

$$p(A = \alpha_k) = \frac{\text{Tr } \rho Q_k}{\text{Tr } \rho Q}. \quad (3.5)$$

Such probabilities also satisfy condition (3.3).

¹¹In the data on voting statistics, we can observe that the additivity of columns in the matrix of transition probabilities, $p(C_d|P_d) + p(C_r|P_d) = 1$ and $p(C_d|P_r) + p(C_r|P_r) = 1$, is preserved.

¹²This is a consequence of equality (3.1). By multiplying both of its sides by ρ , we obtain the equality $\sum_k \rho Q_k = \rho$. Next, the trace of both sides is taken, knowing that $\text{Tr } \rho = 1$.

¹³We remark that some authors, e.g. Aerts *et al.* [35], discussed the need of extension of the quantum measurement formalism—to serve better the applications to decision making and proposed to leave the Hilbert state space representation of psychological observables. Thus, the present generalization is a natural step in this direction.

Now, we consider a special class of generalized observables given by *non-orthogonal one-dimensional* projectors. We restrict our representation to a two-dimensional state space. Let H be a two-dimensional complex Hilbert space with some basis (f_1, f_2) which is in *general non-orthogonal*, i.e. $\langle f_1, f_2 \rangle$ need not be equal to zero. At the same time, we assume that vectors are normalized:

$$\|f_j\|^2 = \langle f_j, f_j \rangle = 1. \quad (3.6)$$

Each vector $\phi \in H$ can be expanded with respect to this basis:

$$\phi = c_1 f_1 + c_2 f_2. \quad (3.7)$$

We introduce the linear operators $C_j \phi = c_j f_j$. These are projectors on the vectors f_j , but in general they are not orthogonal. We remark again that each orthogonal projector C is Hermitian and *idempotent*, i.e.

$$C^2 = C.$$

The latter holds even for C_j . We have

$$C_j^2 \phi = C_j(c_j f_j) = c_j C_j(f_j) = c_j f_j = C_j \phi, \phi \in H. \quad (3.8)$$

However, if the basis is not orthonormal, then C_j is not Hermitian. We set

$$Q_j = C_j^* C_j. \quad (3.9)$$

This operator is Hermitian and positive semidefinite. We have

$$\begin{aligned} \langle Q_j \phi, \phi \rangle &= \langle C_j^* C_j \phi, \phi \rangle = \langle C_j \phi, C_j \phi \rangle \\ &= \langle \phi, C_j^* C_j \phi \rangle = \langle \phi, Q_j \phi \rangle, \phi \in H. \end{aligned} \quad (3.10)$$

Thus $Q_j^* = Q_j$. We also have

$$\langle Q_j \phi, \phi \rangle = \langle C_j \phi, C_j \phi \rangle = \|C_j \phi\|^2 \geq 0, \phi \in H. \quad (3.11)$$

Thus $Q_j \geq 0$.

If the basis (f_j) is orthonormal, then C_j is the orthogonal projector. Here, $C_j^* = C_j$ and $C_j^2 = C_j$ and hence $Q_j = C_j^* C_j = C_j^2 = C_j$. This case is essentially simpler mathematically. However, it is too restrictive for our data.

Suppose now that ψ be a pure state and ρ_ψ be the corresponding density operator; the orthogonal projector on ψ : $\rho_\psi \phi = \langle \psi, \phi \rangle \psi$. Suppose that in the (f_1, f_2) basis, the vector ψ is expanded as

$$\psi = \sqrt{p_1} f_1 + \sqrt{p_2} f_2, p_j \geq 0, p_1 + p_2 = 1 \quad (3.12)$$

(cf. §2d), where the quantum-like representation algorithm produces the following state and basis vectors:

$$\psi = \sqrt{p_1^a} f_1 + \sqrt{p_2^a} f_2, \quad (3.13)$$

where

$$f_1 = \begin{pmatrix} \sqrt{p_{11}} \\ \sqrt{p_{12}} \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} e^{i\theta_1} \sqrt{p_{21}} \\ e^{i\theta_2} \sqrt{p_{22}} \end{pmatrix}. \quad (3.14)$$

We recall that from the properties of the trace it is possible to derive that, for a density operator ρ_ψ corresponding to the pure state ψ and any Hermitian operator O , the following equality holds:

$$\text{Tr } \rho_\psi O = \langle O \psi, \psi \rangle. \quad (3.15)$$

Thus, we obtain

$$\text{Tr } \rho_\psi Q_j = \langle Q_j \psi, \psi \rangle = \langle C_j \psi, C_j \psi \rangle = p_j \langle f_j, f_j \rangle = p_j \quad (3.16)$$

(in the last step we used the normalization condition (3.6)). We also remark that, although in general $Q = Q_1 + Q_2 \neq I$, for the state ρ_ψ corresponding to the pure state ψ of the form (3.12),

we have

$$\text{Tr } \rho_\psi Q = \langle Q_1 \psi, \psi \rangle + \langle Q_2 \psi, \psi \rangle = p_1 + p_2 = 1. \quad (3.17)$$

For such a pure state, the normalization constant in the denominator of (3.5) is trivial, and it equals one.¹⁴ For the pure state ψ of the form (3.12), the probabilities given by (3.5) are reduced to the ones given by (3.16):

$$p(A = \alpha_j) = \frac{\text{Tr } \rho_\psi Q_k}{\text{Tr } \rho Q} = p_j. \quad (3.18)$$

Thus, probabilities for the A -observation are encoded in the state ψ ; see (3.13).

4. Matrix representations

We now consider a matrix representation of the operators C_j . Consider the canonical basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.1)$$

Vectors $(f_j, j = 1, 2)$ have the coordinate representation

$$f_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \quad (4.2)$$

Thus

$$f_1 = k_1 e_1 + k_2 e_2 \quad (4.3)$$

and

$$f_2 = d_1 e_1 + d_2 e_2. \quad (4.4)$$

By a scalar multiplication of these equations by the basis vectors $(e_i, i = 1, 2)$ and using their orthonormality, we obtain that $k_i = \langle f_1, e_i \rangle$ and $d_i = \langle f_2, e_i \rangle$. Then, by solving the system of two linear equations (4.3) and (4.4), we obtain

$$e_1 = \frac{d_2}{k_1 d_2 - k_2 d_1} f_1 - \frac{k_2}{k_1 d_2 - k_2 d_1} f_2 \quad (4.5)$$

and

$$e_2 = -\frac{d_1}{k_1 d_2 - k_2 d_1} f_1 + \frac{k_1}{k_1 d_2 - k_2 d_1} f_2. \quad (4.6)$$

We recall that the columns of the matrix of the operator C_j in the basis (e_i) are given by the coordinates of the images of these basis vectors. We have

$$C_1 e_1 = \frac{d_2}{k_1 d_2 - k_2 d_1} f_1 = \frac{d_2}{k_1 d_2 - k_2 d_1} (k_1 e_1 + k_2 e_2)$$

and

$$C_1 e_2 = -\frac{d_1}{k_1 d_2 - k_2 d_1} f_1 = -\frac{d_1}{k_1 d_2 - k_2 d_1} (k_1 e_1 + k_2 e_2).$$

Thus,

$$C_1 = \frac{1}{k_1 d_2 - k_2 d_1} \begin{pmatrix} k_1 d_2 & -k_1 d_1 \\ k_2 d_2 & -k_2 d_1 \end{pmatrix}. \quad (4.7)$$

We can check that by multiplying this matrix to itself, $C_1^2 = C_1$. In general, this matrix is not Hermitian. However, it becomes Hermitian in the case of the orthonormal basis $(f_j, j = 1, 2)$.

¹⁴We remark that, for an arbitrary pure state, the sum of squared coefficients for the basis (f_1, f_2) need not be equal to one—if this basis is not orthonormal.

Consider an example:

$$f_1 = \frac{1}{\sqrt{2}}(e_1 + e_2) \quad (4.8)$$

and

$$f_2 = \frac{1}{\sqrt{2}}(e_1 - e_2). \quad (4.9)$$

Here, $k_1 = k_2 = d_1 = -d_2 = 1/\sqrt{2}$ and

$$C_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (4.10)$$

which corresponds to the orthogonal projector on the vector f_1 .

In the same way we find

$$C_2 e_1 = -\frac{k_2}{k_1 d_2 - k_2 d_1} (d_1 e_1 + d_2 e_2)$$

and

$$C_2 e_2 = \frac{k_1}{k_1 d_2 - k_2 d_1} (d_1 e_1 + d_2 e_2).$$

Hence,

$$C_2 = \frac{1}{k_1 d_2 - k_2 d_1} \begin{pmatrix} -k_2 d_1 & k_1 d_1 \\ -k_2 d_2 & k_1 d_2 \end{pmatrix}. \quad (4.11)$$

For the orthonormal basis (4.3) and (4.4), where $k_1 = k_2 = d_1 = -d_2 = 1/\sqrt{2}$, we have

$$C_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (4.12)$$

We remark that in this example $Q_j = C_j$ (because they are orthogonal projectors), and we have

$$Q = Q_1 + Q_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.13)$$

Hence, equality (3.1) holds.

We now find the components of the generalized quantum observable A corresponding to the basis (f_j) . We obtain

$$Q_1 = C_1^* C_1 = \frac{|k_1|^2 + |k_2|^2}{|k_1 d_2 - k_2 d_1|^2} \begin{pmatrix} |d_2|^2 & -d_1 \bar{d}_2 \\ -\bar{d}_1 d_2 & |d_1|^2 \end{pmatrix} \quad (4.14)$$

and

$$Q_2 = C_2^* C_2 = \frac{|d_1|^2 + |d_2|^2}{|k_1 d_2 - k_2 d_1|^2} \begin{pmatrix} |k_2|^2 & -k_1 \bar{k}_2 \\ -\bar{k}_1 k_2 & |k_1|^2 \end{pmatrix}. \quad (4.15)$$

The operators are Hermitian and positively defined, but in general they do not sum up to the unit matrix.

5. Generalized observables construction

We remind the reader that the quantum-like representation algorithm produces the following definition of the voters' DM state:

$$\psi = \sqrt{p_1^a} f_1 + \sqrt{p_2^a} f_2, \quad (5.1)$$

where

$$f_1 = \begin{pmatrix} \sqrt{p_{11}} \\ \sqrt{p_{12}} \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} e^{i\theta_1} \sqrt{p_{21}} \\ e^{i\theta_2} \sqrt{p_{22}} \end{pmatrix}. \quad (5.2)$$

The above system of vectors can be orthonormal only if the matrix of transition probabilities is *doubly stochastic*. We also remark (see [16] for details) that, in this case, the phases are constrained

by the equality

$$\theta_1 - \theta_2 = \pi. \quad (5.3)$$

However, the transition probability matrix (M) that we computed in §2c(i) is not satisfying double stochasticity. Expansion (4.3) and (4.4) has the following probabilistic form:

$$f_1 = \sqrt{p_{11}}e_1 + \sqrt{p_{12}}e_2 \quad (5.4)$$

and

$$f_2 = e^{i\theta_1} \sqrt{p_{21}}e_1 + e^{i\theta_2} \sqrt{p_{22}}e_2. \quad (5.5)$$

Hence,

$$k_1 = \sqrt{p_{11}}, \quad k_2 = \sqrt{p_{12}}, \quad d_1 = e^{i\theta_1} \sqrt{p_{21}} \quad \text{and} \quad d_2 = e^{i\theta_2} \sqrt{p_{22}}. \quad (5.6)$$

The operators C_j can be expressed in the probabilistic terms¹⁵

$$C_1 = \frac{1}{\sqrt{p_{11}p_{22}} e^{i\theta_2} - \sqrt{p_{12}p_{21}} e^{i\theta_1}} \begin{pmatrix} \sqrt{p_{11}p_{22}} e^{i\theta_2} & -\sqrt{p_{11}p_{21}} e^{i\theta_1} \\ \sqrt{p_{12}p_{22}} e^{i\theta_2} & -\sqrt{p_{12}p_{21}} e^{i\theta_1} \end{pmatrix} \quad (5.11)$$

$$= \frac{1}{\sqrt{p_{11}p_{22}} - \sqrt{p_{12}p_{21}} e^{i\Delta_{12}}} \begin{pmatrix} \sqrt{p_{11}p_{22}} & -\sqrt{p_{11}p_{21}} e^{i\Delta_{12}} \\ \sqrt{p_{12}p_{22}} & -\sqrt{p_{12}p_{21}} e^{i\Delta_{12}} \end{pmatrix}, \quad (5.12)$$

where the difference between the phases $\Delta_{12} = (\theta_1 - \theta_2)$.

$$C_2 = \frac{1}{\sqrt{p_{11}p_{22}} e^{i\theta_2} - \sqrt{p_{12}p_{21}} e^{i\theta_1}} \begin{pmatrix} -\sqrt{p_{12}p_{21}} e^{i\theta_1} & \sqrt{p_{11}p_{21}} e^{i\theta_1} \\ -\sqrt{p_{12}p_{22}} e^{i\theta_2} & \sqrt{p_{11}p_{22}} e^{i\theta_2} \end{pmatrix} \quad (5.13)$$

$$= \frac{1}{\sqrt{p_{11}p_{22}} - \sqrt{p_{12}p_{21}} e^{i\Delta_{12}}} \begin{pmatrix} -\sqrt{p_{12}p_{21}} e^{i\Delta_{12}} & \sqrt{p_{11}p_{21}} e^{i\Delta_{12}} \\ -\sqrt{p_{12}p_{22}} & \sqrt{p_{11}p_{22}} \end{pmatrix}. \quad (5.14)$$

As we stressed, the transition probabilities on voting preferences (and in fact data from psychological studies; cf. [16]) are not doubly stochastic. Next, we find the components Q_j of the generalized quantum observable $B = (Q_1, Q_2)$. From expressions (4.14) and (4.15) and by taking into account the probabilistic form of the coefficients (5.6), we obtain

$$Q_1 = \frac{1}{\mathcal{K}} \begin{pmatrix} p_{22} & -\sqrt{p_{21}p_{22}} e^{i\Delta_{12}} \\ -\sqrt{p_{21}p_{22}} e^{-i\Delta_{12}} & p_{21} \end{pmatrix}, \quad (5.15)$$

where

$$\mathcal{K} = |\sqrt{p_{11}p_{22}} - \sqrt{p_{12}p_{21}} e^{i\Delta_{12}}|^2 = p_{11}p_{22} + p_{12}p_{21} - 2\sqrt{p_{11}p_{22}p_{12}p_{21}} \cos \Delta_{12}. \quad (5.16)$$

To calculate the absolute value of the sum, one can use the formula

$$|a - b e^{i\Delta}|^2 = a^2 + b^2 - 2ab \cos \Delta,$$

¹⁵In the case of the double stochastic matrix of transition probabilities, the above expressions can be essentially simplified. Double stochasticity implies that $p_{11} = p_{22} = p, p_{12} = p_{21} = 1 - p$, where $0 \leq p \leq 1$. This is the solution of the system of equations:

$$p_{11} + p_{12} = p(B = \beta_1 | a = \alpha_1) + p(B = \beta_2 | a = \alpha_1) = 1, \quad i = 1, 2 \quad (5.7)$$

(this is due to single stochasticity of the matrix) and

$$p_{1i} + p_{2i} = p(B = \beta_i | a = \alpha_1) + p(B = \beta_i | a = \alpha_2) = 1, \quad i = 1, 2 \quad (5.8)$$

(this is double stochasticity). From the above expressions for C_j and by taking into account that in the case of double stochasticity $\theta_1 - \theta_2 = \pi$, we get

$$C_1 = \begin{pmatrix} p & \sqrt{p(1-p)} \\ \sqrt{p(1-p)} & (1-p) \end{pmatrix} \quad (5.9)$$

and

$$C_2 = \begin{pmatrix} (1-p) & -\sqrt{p(1-p)} \\ -\sqrt{p(1-p)} & p \end{pmatrix}. \quad (5.10)$$

Here, comparing with the above example with the concrete orthonormal basis, (4.3), (4.4), $C_j^* = C_j$ and $C_j^2 = C_j$, i.e. $Q_j = C_j$.

where a, b are positive real numbers. In the same way,

$$Q_2 = \frac{1}{\mathcal{K}} \begin{pmatrix} p_{12} & -\sqrt{p_{11}p_{12}} \\ -\sqrt{p_{11}p_{12}} & p_{11} \end{pmatrix}. \quad (5.17)$$

We are reminded (see §2d) that, in the basis (e_1, e_2) , the reconstructed DM state ψ can be represented as¹⁶

$$\psi = (\sqrt{p_1 p_{11}} + e^{i\theta_1} \sqrt{p_2 p_{21}})e_1 + (\sqrt{p_1 p_{12}} + e^{i\theta_2} \sqrt{p_2 p_{22}})e_2. \quad (5.18)$$

We now can find the vector $Q_k \psi$, $k = 1, 2$ and calculate its scalar product with ψ .

For the density operator ρ_ψ corresponding to the state ψ , we have

$$\text{Tr } \rho_\psi Q_2 = \langle Q_2 \psi, \psi \rangle = p_k. \quad (5.19)$$

6. Numerical representation

We present the numerical representation of the DM's state with respect to the Congressional and Presidential elections. By using the matrix of transition probabilities from §2c(i) (we use different notations for conditional probabilities, to adhere to the definition in (2.6) and subsequent developments),

$$M = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.625 & 0.25 \\ 0.375 & 0.75 \end{bmatrix},$$

we were able to find the complex coordinates ψ_1 and ψ_2 (cf. equation (2.9)) and reconstruct the DM's state ψ in the basis (e_1, e_2) by using the generalized Born rule (see equation (5.18)):

$$\psi = \psi_1 e_1 + \psi_2 e_2 = (0.5785 + 0.3470i)e_1 + (0.4285 + 0.6012i)e_2. \quad (6.1)$$

Hence, we obtained $p_1^b = |\psi_1|^2 = 0.455$ and $p_2^b = |\psi_2|^2 = 0.545$. Next, we can define numerically the generalized operator projectors, derived in equations (5.15)–(5.17). We compute the constant $\mathcal{K} = 0.1437$ via the use of equation (5.16):

$$Q_1 = \frac{1}{\mathcal{K}} \begin{pmatrix} 0.75 & -0.43 + 0.0198i \\ -0.43 - 0.0198i & 0.25 \end{pmatrix}. \quad (6.2)$$

We also use that $\Delta_{12} = (\theta_1 - \theta_2) = 1.544 \text{ rad} - 1.59 \text{ rad} = -0.0457 \text{ rad}$.¹⁷ The projector Q_2 is defined numerically:

$$Q_2 = \frac{1}{\mathcal{K}} \begin{pmatrix} 0.375 & -0.4841 \\ -0.4841 & 0.625 \end{pmatrix}. \quad (6.3)$$

By knowing the projectors Q_1 and Q_2 , we can use equation (5.19) to obtain the marginal probabilities p_1 and p_2 for observing the density operator ρ_ψ , corresponding to the state ψ , in eigenstates (α_k) , which, in our example, correspond to P_d and P_r .

Remark on the cognitive meaning of generalized observables:

In quantum physics POVMs are typically introduced through the theory of open quantum systems (cf. [32]). A measurement on a system S represented by a POVM, Q , can be represented also as the composition of unitary evolution U_t of a compound system, containing S coupled with some environment E (e.g. the measurement apparatus) and the standard von Neumann measurement of the projection type. The probability conservation condition, $Q = \sum_k Q_k = I$, is a simple consequence of the unitary property of the dynamics U_t of the compound system, $S \otimes E$.

We remind the reader that in the presented measurement framework, one can obtain $Q \neq I$. From the viewpoint of the theory of open quantum systems, this means that the dynamics of the compound system $S \otimes E$ is *non-unitary*. Hence, such a dynamics U_t (generating generalized POVM with $Q \neq I$) is not a part of the standard quantum formalism. Therefore, by extending

¹⁶To shorten the notation, we set $p_1 = p_1^d, p_2 = p_2^d$.

¹⁷We remind the reader that θ_1 corresponds to the phase for the ψ_1 coordinate and θ_2 to that of the ψ_2 coordinate.

the class of POVMs, effectively we generalize even the dynamical counterpart of the quantum formalism. Although from the viewpoint of traditional quantum formalism such a generalization might be considered as being too radical, for applications outside of physics it might be justified. For instance, we point to the dynamical systems driven by non-Hermitian operators (and hence leading to non-unitary evolution) that have already been considered in quantum modelling of financial processes by Baaquie [36].

We aim to explore further the generalized operator formalism for modelling psychological data, as well as developing a psychological interpretation for the applied parameters in ongoing studies.

7. Concluding remarks

In this study, we proposed a method for an operator representation of observables, which is a natural extension of the operational calculus of POVMs. We applied this method to construct a quantum-like representation of observables, corresponding to voters' decision-making process when they have to make up their preferences for the political parties in Congressional and Presidential elections, with the aid of statistics collected in Khrennikova & Haven [11] and Smith *et al.* [28].

Data accessibility. Supporting data were extracted from Khrennikova & Haven [11], Smith *et al.* [28] and *The New York Times* [30].

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