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A review of the theory of static and quasi-static frictional contact problems in elasticity

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This paper gives a review of mathematical results of existence and uniqueness of solutions to problems of linear elasticity involving friction. Static, steady sliding and quasi-static problems are discussed. Both the case of a continuum body and that of a space-discrete one are treated. The intention is to describe the state of the art for these problems.

Keywords: Coulomb friction; unilateral contact; linear elasticity; quasi-static problem; existence; uniqueness

1. Introduction

The present paper reviews known results concerning existence and uniqueness of solutions for the problem of a linearly elastic body in frictional contact with a rigid obstacle. The unilateral contact problem without friction was given its basic formulation and was shown to have a unique solution by Signorini; Fichera (1964, 1972*a*) proved the existence of a solution in the function space of finite energies using a formulation as a quadratic minimization problem. However, when friction is included, the properties of the problem change dramatically: firstly, any physically reasonable friction model has an evolutionary character, i.e. time enters the formulation; secondly, friction models are non-self-adjoint and the problem is no longer a minimization problem. In Duvaut & Lions (1972, 1976), where the first mathematical treatment of the friction problem in linear elasticity was given, both of these facts were ignored or suppressed in order to get a tractable mathematical problem. In fact, as described in this paper, the full problem is still only partly solved.

This review is limited to problems not too far removed from the frictionless Signorini–Fichera problem. That is, we limit ourselves to the quasi-static problem where inertia terms are assumed negligible and to problems derivable by assumptions or approximations from the quasi-static problem. Both the case of a continuum body and that of a space-discrete one are treated. We mostly restrict our attention to so-called coercive problems, where some part of the boundary of the elastic body has a prescribed motion, excluding rigid-body motions. Most results valid for the coercive case have non-coercive generalizations. How to obtain these generalizations is not always obvious. Further we do not treat dynamic problems or problems including thermal effects, such as those extensively treated by Shillor and co-workers.

The mathematical analysis of frictional contact problems is very different when the coefficient of friction, μ, is small compared to when it is large. It is well known that for large μ one cannot expect any general existence or uniqueness results for quasi-static problems, not even for the simplest possible case with a single particle system with two degrees of freedom (DOFs). A counterexample is presented in § 2. In the present article we mostly review existence and uniqueness results which can be obtained under the assumption that μ is sufficiently small. The case with large coefficient of friction has been studied extensively for simple discrete systems by Martins and co-workers (see, for example, Martins *et al*. 1994, 1995). Furthermore, we consider only the case of small deformations and assume that the equations of linearized elasticity can be used.

Even with these limitations our exposition is surely not completely exhaustive. We believe, however, that the most important works are covered and that the state of the art is correctly described.

The rest of the article is organized as follows. In § 2 we give the above mentioned counterexample. In § 3 we give classical formulations of the quasi-static problem for a continuous elastic system, including the so-called normal compliance approach. By classical we mean partial differential equations and boundary contact conditions, assuming sufficient regularity of displacement and stress fields. In § 4 we review a static problem reformulated as a variational problem in an appropriate function space. Section 5 deals with a steady sliding problem, which may be non-coercive. Section 6 contains an account of the quasi-static friction evolution problem, including a review of recent results of our own for the case with Signorini contact conditions. In § 7 we specialize to the space-discretized problems, for which the mathematical analysis is quite different. Contrary to the case with a continuous elastic system, it is here possible to obtain some rather general uniqueness results, provided that one imposes some condition of time-regularity for the exterior force field. In § 8 there are some concluding remarks. Finally, in Appendix A we have collected some notation and definitions.

2. Elementary example of non-uniqueness and non-existence

It has been known for more than a century, through an example by Paul Painlevé, that problems of rigid-body dynamics with Coulomb friction may apparently not have solutions, or, if solutions exist, they may be non-unique. These facts have been debated and investigated up to this day (see, for example, Stewart 1997). What, on the other hand, seemingly remained unknown until 1980 is that similar difficulties exist in static or quasi-static problems. Such problems may admittedly appear to be of a less fundamental nature than dynamic problems, but they have always been of great practical value in engineering practice, in recent years particularly in connection with the finite-element method. Janovský (1980, 1981) was the first to discover that a discrete abstract version of the static friction problem of Duvaut & Lions (1976) may have non-unique solutions. Klarbring (1987, 1990*a*, *b*) showed that non-uniqueness may also appear in the quasi-static problem and, furthermore, that non-existence of continuous solutions is possible. Similar non-existence is not found in the static problem treated by Janovský. A concrete mechanical example consisting of two connected bars was also given in Klarbring (1987, 1990*a*, *b*). Below we will present an example which shows how right-hand derivatives of the displacements and contact forces are non-unique and non-existent for particular different loading directions.

Consider a structure, condensed into a single contact node, in two-dimensional contact with a rigid obstacle. The external load and previous evolution of the structure is such that the normal component of the contact force $p_{\mathrm{N}} < 0$ satisfies $p_{\mathrm{T}} = \mu p_{\mathrm{N}}$, where p_{T} is the tangential component of the contact force and μ is the friction coefficient. That is, the contact force state of the node is on the boundary of the friction cone and two qualitatively different further continuous evolutions are possible:

(i) the node may evolve into a stick state, which means that

$$\dot{p}_{\mathrm{T}} \geqslant \mu \dot{p}_{\mathrm{N}} \quad \text{and} \quad \dot{w}_{\mathrm{T}} = 0, \tag{2.1}$$

where $\dot{p}_{\mathrm{T}}$, $\dot{p}_{\mathrm{N}}$ and $\dot{w}_{\mathrm{T}}$ are right-hand time derivatives of the contact forces and the tangential displacement; or

(ii) the node may evolve into a slip state, which means that

$$\dot{p}_{\mathrm{T}} = \mu \dot{p}_{\mathrm{N}} \quad \text{and} \quad \dot{w}_{\mathrm{T}} \geqslant 0. \tag{2.2}$$

The previously introduced time derivatives $\dot{p}_{\mathrm{T}}$, $\dot{p}_{\mathrm{N}}$ and $\dot{w}_{\mathrm{T}}$, and the time derivative of the normal displacement, $\dot{w}_{\mathrm{N}}$, are linearly coupled through a stiffness equation:

$$\begin{bmatrix} k_{\mathrm{NN}} & k_{\mathrm{NT}} \\ k_{\mathrm{NT}} & k_{\mathrm{TT}} \end{bmatrix} \begin{bmatrix} \dot{w}_{\mathrm{N}} \\ \dot{w}_{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \dot{p}_{\mathrm{N}} \\ \dot{p}_{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \dot{f}_{\mathrm{N}} \\ \dot{f}_{\mathrm{T}} \end{bmatrix}, \tag{2.3}$$

where k_{NN}, etc., are real numbers such that the system matrix is positive definite and $\dot{f}_{\mathrm{N}}$ and $\dot{f}_{\mathrm{T}}$ are time derivatives of the external forces. Since $p_{\mathrm{N}} < 0$, it holds that $\dot{w}_{\mathrm{N}} = 0$, so we may conclude from (2.3), by multiplying the first equation by μ and subtracting, that

$$(-\mu k_{\mathrm{NT}} + k_{\mathrm{TT}})\dot{w}_{\mathrm{T}} = -\mu \dot{p}_{\mathrm{N}} + \dot{p}_{\mathrm{T}} - \mu \dot{f}_{\mathrm{N}} + \dot{f}_{\mathrm{T}}. \tag{2.4}$$

From (2.1) and (2.4) it follows that if the node goes into a *stick state*, then

$$\mu \dot{f}_{\mathrm{N}} \geqslant \dot{f}_{\mathrm{T}}.$$

From (2.2) and (2.4) one finds that if the node goes into a *slip state*, the condition on derivatives of external forces depends on the sign of the 'effective stiffness' in (2.4):

$$\begin{aligned} &\text{if } -\mu k_{\mathrm{NT}} + k_{\mathrm{TT}} > 0, \quad \text{then } \mu \dot{f}_{\mathrm{N}} \leqslant \dot{f}_{\mathrm{T}}, \\ &\text{if } -\mu k_{\mathrm{NT}} + k_{\mathrm{TT}} = 0, \quad \text{then } \mu \dot{f}_{\mathrm{N}} = \dot{f}_{\mathrm{T}} \\ &\text{if } -\mu k_{\mathrm{NT}} + k_{\mathrm{TT}} < 0, \quad \text{then } \mu \dot{f}_{\mathrm{N}} \geqslant \dot{f}_{\mathrm{T}}. \end{aligned}$$

We conclude the following for the problem of determining the right-hand time derivative of contact forces and displacements when given a rate of change of the external forces:

(i) if $-\mu k_{\mathrm{NT}} + k_{\mathrm{TT}} > 0$, then there exists a unique solution; and

(ii) if $-\mu k_{\mathrm{NT}} + k_{\mathrm{TT}} \leqslant 0$, then, depending on the direction of change of external loading, there does not exist any solution or there are multiple solutions.

Note that $\mu > 0$ and $k_{\mathrm{TT}} > 0$, while k_{NT} may have any sign. This latter constant represents a stiffness coupling between normal and tangential degrees of freedom, so, clearly, for large such couplings we may experience non-uniqueness or non-existence. The non-existence of right-hand time derivatives means that the quasi-static problem cannot have any solutions that are continuous in time.

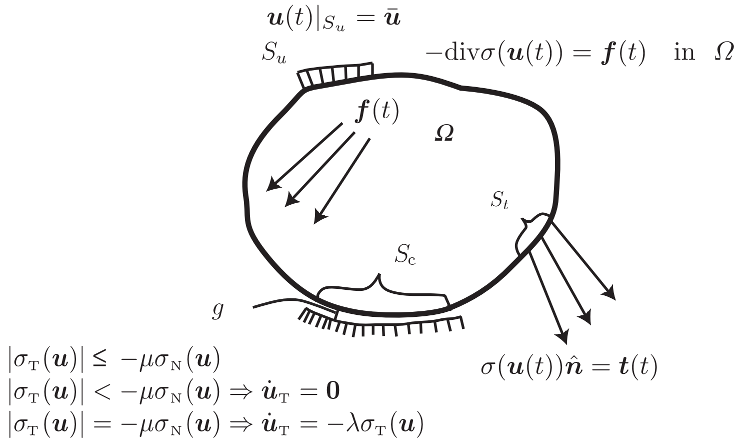

Figure 1. An elastic body in contact with a rigid obstacle.

3. Classical formulation of the quasi-static frictional contact problem

We will give a classical formulation of the quasi-static contact problem with Coulomb friction. A linearly elastic body which may come into contact with a fixed rigid obstacle is considered. Two mathematical formulations of the physical conditions of impenetrability and non-adhesion are given. The first one is the Signorini–Fichera complementarity condition. The second one is the so-called normal compliance law. The classical quasi-static formulation is obtained from the dynamic formulation by neglecting inertial terms, which means that we are looking at a sequence of equilibrium states.

The linearly elastic body occupies a bounded Lipschitz domain Ω in $\mathbb{R}^3$ (or $\mathbb{R}^2$). The body is subjected to body forces $\boldsymbol{f} = (f_1, f_2, f_3)$ and to prescribed tractions $\boldsymbol{t} = (t_1, t_2, t_3)$ and displacements $\bar{\boldsymbol{u}}$ on the parts S_t and S_u of the boundary $\partial\Omega$, respectively. The potential contact boundary is S_c. Furthermore, S_c, S_t and S_u will be mutually disjoint, relatively open subsets of $\partial\Omega$. The following classical equations of linear elasticity are valid:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$\sigma_{ij} = a_{ijkl}\frac{\partial u_k}{\partial x_l} \quad \text{in } \Omega, \tag{3.2}$$

$$\sigma_{ij} n_j = t_i \quad \text{on } S_t. \tag{3.3}$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{on } S_u. \tag{3.4}$$

Here $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector, $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ is the stress tensor, and n_j are the components of the outward unit normal vector. The elasticity coefficients a_{ijkl} satisfy the usual symmetry and ellipticity conditions. Furthermore, $i, j, k, l = 1, 2, 3$, the summation convention is used and $(0, x_1, x_2, x_3)$ is the Cartesian reference frame.

To state the laws of contact and friction we decompose the displacement and traction vectors on S_c into normal and tangential components:

$$\sigma_\mathrm{N} = \sigma_{ij} n_i n_j, \qquad \sigma_{\mathrm{T}i} = \sigma_{ij} n_j - \sigma_\mathrm{N} n_i,$$
$$u_\mathrm{N} = u_i n_i, \qquad u_{\mathrm{T}i} = u_i - u_\mathrm{N} n_i.$$

The classical Signorini–Fichera contact law may now be stated as

$$\sigma_{\mathrm{N}} \leqslant 0, \quad u_{\mathrm{N}} - g \leqslant 0, \quad \sigma_{\mathrm{N}}(u_{\mathrm{N}} - g) = 0 \quad \text{on } S_{\mathrm{c}}, \tag{3.5}$$

where g is the initial gap between the body and the rigid support. Note that there is no sign restriction for g.

The friction law is that of Coulomb, which can be written as

$$\left.\begin{aligned} &|\boldsymbol{\sigma}_{\mathrm{T}}| \leqslant -\mu\sigma_{\mathrm{N}}, \\ |\boldsymbol{\sigma}_{\mathrm{T}}| < -\mu\sigma_{\mathrm{N}} \Rightarrow \dot{\boldsymbol{u}}_{\mathrm{T}} = 0, \quad &0 < |\boldsymbol{\sigma}_{\mathrm{T}}| = -\mu\sigma_{\mathrm{N}} \Rightarrow \dot{\boldsymbol{u}}_{\mathrm{T}} = -\lambda\boldsymbol{\sigma}_{\mathrm{T}}, \quad \lambda \geqslant 0, \end{aligned}\right\} \tag{3.6}$$

where μ is the friction coefficient and a superposed dot denotes time derivative.

Relations (3.1)–(3.6) constitute the quasi-static frictional contact problem. The problem is time dependent via the time derivative in Coulomb's friction law. Therefore the external forces $\boldsymbol{f}$ and $\boldsymbol{t}$ should be taken as time dependent.

The Signorini–Fichera conditions (3.5) are approximations to the behaviour of contacting surfaces. Although these conditions are very useful in many situations, an alternative which somewhat reflects the physical nature of contacting surfaces has been suggested and analysed in the literature. This is the so-called normal compliance law:

$$-\sigma_{\mathrm{N}} = c_{\mathrm{N}}(u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{N}}}, \tag{3.7}$$

where c_{N} and m_{N} are positive parameters representing the physical characteristics of the interface and $(z)_{+} = \max(0, z)$. Since (3.7) is a local relation between σ_{N} and u_{N}, we can eliminate the former from the friction law (3.6). However, one can also generalize the friction law somewhat in the following way

$$\left.\begin{aligned} |\boldsymbol{\sigma}_{\mathrm{T}}| &\leqslant c_{\mathrm{T}}(u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{T}}}, \\ |\boldsymbol{\sigma}_{\mathrm{T}}| &< c_{\mathrm{T}}(u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{T}}} \Rightarrow \dot{\boldsymbol{u}}_{\mathrm{T}} = 0, \\ 0 < |\boldsymbol{\sigma}_{\mathrm{T}}| &= c_{\mathrm{T}}(u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{T}}} \Rightarrow \dot{\boldsymbol{u}}_{\mathrm{T}} = -\lambda\boldsymbol{\sigma}_{\mathrm{T}}, \quad \lambda \geqslant 0, \end{aligned}\right\} \tag{3.8}$$

where c_{T} and m_{T} are new physical parameters. One retrieves (3.6) from (3.7) and (3.8) by putting $c_{\mathrm{T}} = \mu c_{\mathrm{N}}$ and $m_{\mathrm{T}} = m_{\mathrm{N}}$.

Relations (3.1)–(3.4), (3.7) and (3.8) constitute the quasi-static frictional contact problem with normal compliance.

4. The static problem

We now formulate a static version of the time-dependent problem with the Signorini–Fichera conditions (3.5) of the previous section. Therefore we consider time-independent force and traction fields satisfying equations (3.1)–(3.4). The time-dependent friction condition (3.6) is then replaced by

$$\left.\begin{aligned} &|\boldsymbol{\sigma}_{\mathrm{T}}| \leqslant -\mu\sigma_{\mathrm{N}}, \\ |\boldsymbol{\sigma}_{\mathrm{T}}| < -\mu\sigma_{\mathrm{N}} \Rightarrow \boldsymbol{u}_{\mathrm{T}} = 0, \quad &0 < |\boldsymbol{\sigma}_{\mathrm{T}}| = -\mu\sigma_{\mathrm{N}} \Rightarrow \boldsymbol{u}_{\mathrm{T}} = -\lambda\boldsymbol{\sigma}_{\mathrm{T}}, \quad \lambda \geqslant 0. \end{aligned}\right\} \tag{4.1}$$

Here (4.1) may be considered as a backward difference version of the time derivative condition (3.6).

Using Green's formula, it follows in a straightforward way that equations (3.1)–(3.4) and (4.1) are equivalent to the following preliminary variational problem, provided that the displacement field $\boldsymbol{u}$ and the fields $\boldsymbol{f}$ and $\boldsymbol{t}$ are sufficiently regular.

VPstat′. Find $\boldsymbol{u}$ with $\boldsymbol{u}|_{S_u} = \bar{\boldsymbol{u}}$ and $u_{\mathrm{N}} \leqslant g$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \int_{S_{\mathrm{c}}} \mu\sigma_{\mathrm{N}}(|\boldsymbol{v}_{\mathrm{T}}| - |\boldsymbol{u}_{\mathrm{T}}|)\,\mathrm{d}S \geqslant \int_{\Omega} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}x + \int_{S_t} \boldsymbol{t} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}S \quad (4.2)$$

for all $\boldsymbol{v}$ with $\boldsymbol{v}|_{S_u} = \bar{\boldsymbol{u}}$ and $v_{\mathrm{N}} \leqslant g$. Here $a(\boldsymbol{u}, \boldsymbol{v})$ denotes the bilinear energy functional.

For the problem **VPstat** to make mathematical sense, we must specify appropriate function spaces. We therefore require that $\boldsymbol{u}, \boldsymbol{v} \in K \subset V$ and, for the force field, that $\boldsymbol{f} \in (L^2(\Omega))^3$ and, for the traction field, that $\boldsymbol{t} \in (H^{-1/2}(\Omega))^3$. Here

$$V = \{\boldsymbol{v} \in (H^1(\Omega))^3 : \boldsymbol{v}|_{S_{\mathrm{c}}} = \bar{\boldsymbol{u}}\} \quad \text{and} \quad K = \{\boldsymbol{v} \in V : v_{\mathrm{N}}|_{S_{\mathrm{c}}} \leqslant g\}.$$

Under this assumption the second and fourth terms in (4.2) are not *a priori* properly defined, since σ_{N} is in $H^{-1/2}(\partial\Omega)$ and not necessarily in $L^2(\partial\Omega)$. Therefore we should take, for example,

$$\int_{S_{\mathrm{c}}} \mu\sigma_{\mathrm{N}}|\boldsymbol{v}_{\mathrm{T}}|\,\mathrm{d}S := \langle \psi_{\mathrm{c}}\mu\sigma_{\mathrm{N}}, |\boldsymbol{v}_{\mathrm{T}}|\rangle_{-1/2,1/2,\partial\Omega}$$

and

$$\int_{S_t} \boldsymbol{t} \cdot \boldsymbol{v}\,\mathrm{d}S := \langle \boldsymbol{t}, \boldsymbol{v}\rangle_{(H^{-1/2}(\partial\Omega))^3,(H^{1/2}(\partial\Omega))^3} = \langle \boldsymbol{t}, \boldsymbol{v}\rangle_{-1/2,1/2,\partial\Omega},$$

where $\psi_{\mathrm{c}} \in C_0^{\infty}(\mathbb{R}^3)$ is some cut-off function such that $\psi_{\mathrm{c}} = 1$ is in a neighbourhood of $\bar{S}_{\mathrm{c}}$ and $\psi_{\mathrm{c}} = 0$ is in a neighbourhood of $\bar{S}_t \cup \bar{S}_u$. By the variational inequality (4.2) it follows that $\sigma_{\mathrm{N}} = 0$ outside $\bar{S}_t \cup \bar{S}_u \cup \bar{S}_{\mathrm{c}}$ and, therefore, the value of $\langle \psi_{\mathrm{c}}\mu\sigma_{\mathrm{N}}, |\boldsymbol{v}_{\mathrm{T}}|\rangle$ will be independent of the particular choice of ψ, i.e. any solution is independent of ψ_{c}.

With this notation we have the following formulation of the static problem.

VPstat. Find $\boldsymbol{u} \in K$ such that, for all $\boldsymbol{v} \in K$,

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle \psi_{\mathrm{c}}\mu\sigma_{\mathrm{N}}(\boldsymbol{u}), |\boldsymbol{v}_{\mathrm{T}}| - |\boldsymbol{u}_{\mathrm{T}}|\rangle_{-1/2,1/2,\partial\Omega} \\ \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{2,\Omega} + \langle \boldsymbol{t}, \boldsymbol{v} - \boldsymbol{u}\rangle_{-1/2,1/2,\partial\Omega}. \end{aligned} \quad (4.3)$$

The contact problem without friction, i.e. **VPstat** with $\mu = 0$, can be formulated as a problem of minimizing a convex (quadratic) functional over a convex domain of definition, and was treated by Fichera (1964, 1972*b*).

Contact problems with friction were first considered by Duvaut & Lions (1971) and formulated somewhat later in Duvaut & Lions (1972) with the English translation appearing in Duvaut & Lions (1976). They gave, however, a simplified version of the variational problem **VPstat**, where the normal pressure σ_{N} was prescribed on the contact surface and where no non-penetration condition of the form $u_{\mathrm{N}} \leqslant g$ was present. Under these simpler conditions a proof of existence and uniqueness was given.

An early existence and uniqueness result for a modified **VPstat** problem was given by Duvaut (1980). He introduced a non-local frictional law by replacing σ_{N} in (4.2) by a mollified function $\sigma_{\mathrm{N}}^* \in L^2(\partial\Omega)$, where the mapping

$$H^{-1/2}(\partial\Omega) \ni \sigma_{\mathrm{N}} \mapsto \sigma_{\mathrm{N}}^* \in L^2(\partial\Omega)$$

was assumed to be bounded and linear and mapping positive functions on positive functions. Using a fixed-point argument, Duvaut showed existence of a solution and, if the coefficient of friction was taken small enough, its uniqueness.

Similar results on existence and uniqueness were given by Cocu (1984) and Demkowicz & Oden (1982) using different methods. Cocu made use of an abstract theory of inequalities in Banach spaces, due to Fan (1972) and Brézis *et al*. (1972).

The first result for the problem **VPstat**, without any regularization of the friction term, was given in Nečas *et al*. (1980), where the existence of a solution was proved, under the condition that the coefficient was *small enough*. The proof was rather complicated and based on a shifting technique for estimating Sobolev norms, introduced by Fichera (1972*a*, *b*). Also, for technical reasons the treatment in Nečas *et al*. (1980) was restricted to the situation when the domain Ω was an infinite strip in the plane. In a later article, Jarušek (1983) considered the more general situation with a domain $\Omega \in \mathbb{R}^3$, where the part S_c was locally $C^{2,1}$ (meaning that for some local coordinate-system the boundary was represented as the graph of a function with bounded third derivatives). Moreover, it was assumed that $0 \leqslant \mu$, $\operatorname{supp} \mu \in S_c$ and that $\mu \in C^1(\Omega)$. For the elasticity tensor it was assumed that the functions a_{ijkl} are Lipschitz continuous and satisfy the usual positivity and symmetry conditions. Then existence was proved for small enough values of the sup–norm $\|\mu\|_\infty$. The proof which supplied explicit bounds for the sup–norm of μ was quite technical. In the articles by Nečas *et al*. (1980) and Jarušek (1983) it was assumed that a part S_u of the boundary had a prescribed motion (as in the formulation of **VPstat**). This means that, by Korn's inequality, the energy functional $a(\boldsymbol{u}, \boldsymbol{u})$ is coercive, in contrast to the case when the set S_c is empty where one has to take into account that rigid-body motions might be possible. This non-coercive case was treated in Jarušek (1984), where a similar existence result as in Jarušek (1983) was given.

A slight generalization of these results was given by Kato (1987), who proved existence for **VPstat**. In particular one should note that Kato did not impose the restriction that the coefficient of friction, μ, should have compact support in the open set S_c. Kato assumed that the gap function g is zero, that the boundary $\partial\Omega$ is smooth, and that $\mu \in C^1(\bar{S}_c)$, with sufficiently small sup–norm $\|\mu\|_\infty$. He also gave some results of regularity for the solution.

Later Eck and Jarušek (see Eck 1996; Eck & Jarušek 1998) gave an alternative and somewhat simpler proof of the results in Nečas *et al*. (1980) and Jarušek (1983), by using a penalty formulation of the normal compliance type and taking limits as the penalty parameter tends to infinity. They also considered the case when μ was dependent on the tangential displacement.

One may note that essentially the same shifting technique mentioned above is used in all of the works Nečas *et al*. (1980), Jarušek (1983, 1984), Kato (1987), Eck (1996), Eck & Jarušek (1998), as well as in Andersson (2001), cited below, in order to obtain necessary compactness arguments. One may also note that no uniqueness results seem to be known for **VPstat** in its original form without any kind of regularization of the friction term.

5. Steady sliding problem

A substantial simplification of the quasi-static problem appears if it can be argued that the direction of tangential sliding on S_c is known. This occurs if the rigid obstacle has prescribed motion or if the obstacle is fixed and a superimposed rigid-body motion with zero acceleration is assumed for the body. This latter situation may be viewed as a *steady sliding* problem.

In such a case Coulomb's law of friction may be replaced by

$$\boldsymbol{\sigma}_{\mathrm{T}} = \mu\sigma_{\mathrm{N}}\boldsymbol{e}, \tag{5.1}$$

where $\boldsymbol{e}$ is a unit vector defining the direction of sliding. The steady sliding problem is now defined by (3.1)–(3.5) and (5.1).

The variational formulation of this problem becomes similar to that of **VPstat**, but the non-differentiable second term on the left-hand side is replaced by a bilinear form as follows.

VPsteady–sliding. Find $\boldsymbol{u} \in K$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle \psi_{\mathrm{c}}\mu\sigma_{\mathrm{N}}, (\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{e}\rangle_{-1/2,1/2,\partial\Omega} \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{2,\partial\Omega} + \langle \boldsymbol{t}, \boldsymbol{v} - \boldsymbol{u}\rangle_{-1/2,1/2,\partial\Omega}$$

for all $\boldsymbol{v} \in K$.

For the coercive case, i.e. when S_{c} is large enough to prevent any rigid-body motions, modifications of this problem were treated by Pires & Trabucho (1990) and Rabier *et al*. (1986). In the first of these papers the non-local friction law of Duvaut was used and the existence of a unique solution for sufficiently small friction coefficient or 'sufficiently non-local' friction law was shown. In Rabier *et al*. (1986) the normal compliance regularization was used, which transforms **VPsteady–sliding** into a variational equality. The existence of a locally unique solution for small friction coefficient or small load was shown. The semi-coercive version of the problem, i.e. when S_{c} is empty, has been considered only for special geometries. Gastaldi & Martins (1988) considered the sliding on a flat surface. Andersson & Klarbring (1997) considered unidirectional sliding occurring, for instance, when uncorking a wine bottle. It may be noted that no regularization of the friction term was used in Gastaldi & Martins (1988) and Andersson & Klarbring (1997). The discrete version of the steady sliding problem has been studied in Klarbring (1997) and Klarbring & Pang (1999).

6. Existence results for quasi-static friction problems

We will now formulate a variational problem corresponding to the classical formulation (3.1)–(3.6) in § 3. Using Green's formula, as in § 4, we are led to the following time-dependent variational problem, consisting of two simultaneous inequalities (for notation see Appendix A).

VPQ. Given $\boldsymbol{f} \in W^1(0, T; (L^2(\Omega))^3)$ and $\boldsymbol{t} \in W^1(0, T; (H^1(\partial\Omega))^3)$, find a mapping $\boldsymbol{u} \in W^1(0, T; K)$ such that for almost all $t \in [0, T]$ we have

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) - \langle \psi_{\mathrm{c}}\sigma_{\mathrm{N}}(\boldsymbol{u}), v_{\mathrm{N}} - \dot{u}_{\mathrm{N}}\rangle - \langle \psi_{\mathrm{c}}\mu\sigma_{\mathrm{N}}(\boldsymbol{u}), |\boldsymbol{v}_{\mathrm{T}}| - |\dot{\boldsymbol{u}}_{\mathrm{T}}|\rangle \geqslant (\boldsymbol{f}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + \langle \boldsymbol{t}, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle \tag{6.1}$$

for all $\boldsymbol{v} \in V$ and such that

$$\langle \psi_{\mathrm{c}}\sigma_{\mathrm{N}}(\boldsymbol{u}), w_{\mathrm{N}} - u_{\mathrm{N}}\rangle \geqslant 0 \tag{6.2}$$

for all t and all $\boldsymbol{w} \in K$.

For the initial state $\boldsymbol{u}(0) = \boldsymbol{u}_0$ we have the following necessary compatibility condition

$$a(\boldsymbol{u}_0, \boldsymbol{v} - \boldsymbol{u}_0) - \langle \psi_c \mu \sigma_N(\boldsymbol{u}), |\boldsymbol{v}_T - \boldsymbol{u}_{0T}|\rangle \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_0) + \langle \boldsymbol{t}, \boldsymbol{v} - \boldsymbol{u}_0\rangle \tag{6.3}$$

for all $\boldsymbol{v} \in K$. Formally, the latter condition expresses that the initial state is in equilibrium, satisfies the contact law (3.5), and that $|\sigma_T(\boldsymbol{u}_0)| \leqslant -\mu\sigma_N(\boldsymbol{u}_0)$ on S_c.

Cocu *et al.* (1996) gave an existence result for a regularized version of this problem, when σ_N is replaced by σ_N^*, where $*$, as in § 4, denotes a bounded linear mapping,

$$H^{-1/2}(\partial\Omega) \ni \sigma_N \mapsto \sigma_N^* \in L^2(\Omega).$$

It was also assumed that the coefficient of friction is chosen sufficiently small. We may also note that, in their work, μ is allowed to be time dependent. Their proof is based on an incremental formulation of the problem **VPQ** and employs a technique for passing to the limit which is similar to that in Andersson (1991).

Another way of regularizing **VPQ** is by normal compliance. The appropriate variational formulation is then as follows.

VPQNC. Find $\boldsymbol{u} \in W^1(0, T; K)$ such that for almost all t we have

$$a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + \varphi_{nc}(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + j_{nc}(\boldsymbol{u}, \boldsymbol{v}) - j_{nc}(\boldsymbol{u}, \dot{\boldsymbol{u}}) \geqslant \langle F, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle \tag{6.4}$$

for all $\boldsymbol{v} \in V$.

Here we have introduced the notation

$$\varphi_{nc}(\boldsymbol{u}, \boldsymbol{v}) = \int_{S_c} c_N (u_N - g)_+^{m_N} v_N \,\mathrm{d}S, \qquad j_{nc}(\boldsymbol{u}, \boldsymbol{v}) = \int_{S_c} c_T (u_N - g)_+^{m_T} |\boldsymbol{v}_T| \,\mathrm{d}S$$

and

$$\langle F, \boldsymbol{u}\rangle = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \int_{S_t} \boldsymbol{t} \cdot \boldsymbol{v} \,\mathrm{d}S.$$

The problem **VPQNC** was formulated by Klarbring *et al.* (1988). They gave a proof of existence for the so-called rate problem, i.e. the problem of calculating the derivative $\dot{\boldsymbol{u}}$ at a fixed time with a given field $\boldsymbol{u}$, by the aid of (6.4). They also formulated a related incremental version of **VPQNC**, i.e. a time-discretized problem. They observed that the incremental problem was essentially a sequence of static problems, similar to **VPstat**, but with a normal compliance frictional term, and proved a theorem of existence. In this proof they made the assumptions $1 \leqslant m_N, m_T < 3$ (in three space dimensions). There were, however, no restrictions on the size of the coefficients c_N and c_T, and consequently not on the coefficient of friction μ, for the case that $m_N = m_T$ and $c_T = \mu c_N$.

In Klarbring *et al.* (1989) the same authors gave a local proof of uniqueness, under certain additional restrictions on the size of c_N and c_T (with bounds depending on $\boldsymbol{f}$, $\boldsymbol{t}$ and g). Klarbring *et al.* (1991) investigated a modified version of the time-dependent problem **VPQNC**, where the terms $c_N(u_N - g)_+^{m_N}$ and $c_T(u_T - g)_+^{m_T}$ were replaced by mollifications backwards in time.

In Andersson (1991) the problem **VPQNC** was treated without any additional regularization, and a proof of existence was given under similar restrictions on m_N,

m_{T}, on the size of the loads and the size of coefficients c_{N} and c_{T} as above. An estimate of the form $\|\dot{\boldsymbol{u}}(t)\| \leqslant C(\|\dot{\boldsymbol{f}}(t)\| + \|\dot{\boldsymbol{t}}(t)\|)$ for the solution, valid almost everywhere in t, was proved.

In a later paper (Andersson 1995) a similar existence result was obtained without any restrictions on the size of the loads or the size of coefficients c_{N} and c_{T}. For this solution $\boldsymbol{u}(t)$ the conclusion regarding time-regularity was that $\boldsymbol{u} \in BV(0, t_0; V)$, i.e. that it was of bounded variation in time. It is therefore not necessarily differentiable everywhere and might even have countably many jump discontinuities.

In Andersson (1999*a*) the problem **VPQNC** was treated with $m_{\mathrm{T}} = m_{\mathrm{N}}$ and

$$\varphi_{\mathrm{nc}}(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_{S_{\mathrm{c}}} (u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{N}}} \boldsymbol{v}_{\mathrm{N}} \, \mathrm{d}S, \qquad j_{\mathrm{nc}}(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_{S_{\mathrm{c}}} \mu (u_{\mathrm{N}} - g)_{+}^{m_{\mathrm{N}}} |\boldsymbol{v}_{\mathrm{T}}| \, \mathrm{d}S,$$

where λ is a, possibly large, penalization parameter. It was shown that if

$$\|\mu\|_{\infty} < \frac{c_0}{C_0} \frac{1}{\|\operatorname{tr}_0\| \|\mathcal{E}_0\|} \quad \text{and} \quad \|\mu\| < \frac{c_0}{C_0} \frac{1}{\|\operatorname{tr}\| \|\mathcal{E}\|}, \tag{6.5}$$

then there exists a solution $\boldsymbol{u}$ of **VPQNC** (for notation see Appendix A). Further it was shown that

$$\|\dot{\boldsymbol{u}}(t)\| \leqslant C(\|\dot{\boldsymbol{f}}(t)\| + \|\dot{\boldsymbol{t}}(t)\|) \tag{6.6}$$

for almost all t, with the constant C independent of the penalty parameter λ. Furthermore, we note that the upper bounds on the norms of μ given in (6.5) are also independent of λ.

Using the uniform (in λ) estimates (6.6) and the shifting technique of Fichera, the following result was proved in Andersson (2001).

Theorem 6.1. *Assume regularity, etc., for Ω, S_{c}, S_u, a_{ijkl} and g given in Appendix A is valid and that $0 < \alpha < \beta \leqslant 1$, $\alpha \leqslant \frac{1}{2}$. For the initial state $\boldsymbol{u}_0$ we require that $\sigma_{\mathrm{N}}(\boldsymbol{u}_0) = 0$ on S_{c}. Also assume that μ satisfies (6.5). Then **VPQ** has a solution $\boldsymbol{u}$ satisfying an inequality (6.6) almost everywhere, with the constant C depending only on $\|\mu\|$ and $\|\mu\|_{\infty}$. Further, on S_{c} we have the regularity*

$$\sigma_{\mathrm{N}}(\boldsymbol{u}) \in H_{\mathrm{loc}}^{-(1/2)+\alpha}(S_{\mathrm{c}}).$$

Rocca (2000) later gave a slightly different version of the same theorem, valid for the more general initial condition (6.3). Rocca used a time-incremental version of **VPQ**, and with the aid of similar *a priori* estimates as in Andersson (2001), he proved existence by taking limits as the time-step tends to zero.

7. A quasi-static problem with finitely many spatial degrees of freedom

In this section we will consider the question of existence and uniqueness for the quasi-static problem with finitely many spatial degrees of freedom. The counterexample presented in § 2 shows that for large coefficients of friction we cannot expect any general existence or uniqueness results for this problem. Furthermore, Ballard (1999) has given a remarkable counterexample to *uniqueness* for a 3DOF system with a single contact node confined to a half-space. His counterexample is valid for an *arbitrarily small* coefficient of friction. The applied force in his example is

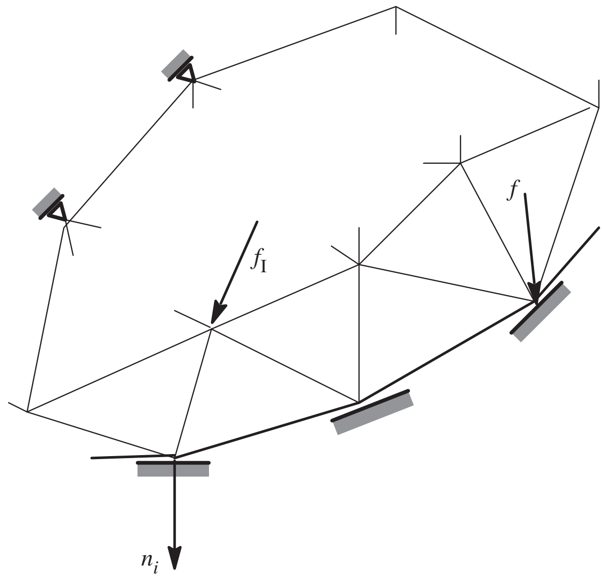

Figure 2. Triangulation of a plane elastic body, resulting in the discrete quasi-static friction problem.

in $W^{1,\infty}(0,T;\mathbb{R}^3)$ but has an oscillating direction close to the bifurcation point. Therefore the question remains of whether one can obtain some uniqueness results if additional time-regularity is required of the applied force field.

In the following subsections we will review the recent work of Andersson (1999*b*), also reported in Andersson & Klarbring (2000). An upper bound $\tilde{\mu}$ for the coefficients of friction which guarantees the existence of a solution is provided, and also a rather general uniqueness result for forces which are right piecewise real analytic in time is given. Only the coercive case is treated. Results for the discrete static non-coercive problem are reported in Klarbring & Pang (1998).

(*a*) *Formulation of the problem*

We consider a particular type of discrete structure which is composed in such a way that its displacement state can be represented by three-dimensional (or two-dimensional) geometric displacement vectors associated with non-coinciding points in the physical Euclidean three-dimensional space. These points are called (displacement) nodes. There are two classes of nodes: the contact nodes, which upon deformation may come into frictional contact with rigid obstacles, and the rest, which are called interior nodes. There are l contact nodes. From the displacement vectors $\boldsymbol{u}_i$ and the reaction forces $\boldsymbol{r}_i$, $1 \leqslant i \leqslant l$, associated with these nodes, we form a global contact displacement vector and a global reaction force vector:

$$u = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_l \end{bmatrix} \in \mathbb{R}^{3l}, \qquad r = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_l \end{bmatrix} \in \mathbb{R}^{3l}.$$

Similarly, the displacement of the interior nodes are collected in a vector u_{I}. External prescribed forces are denoted by

$$F = \begin{bmatrix} f_{\mathrm{I}} \\ f \end{bmatrix},$$

with f acting on contact nodes and f_{I} on interior nodes, respectively.

For a linear elastic structure under a small displacement assumption and ignoring inertia we have the following stiffness relation:

$$\begin{bmatrix} f_{\mathrm{I}} \\ f + r \end{bmatrix} = \begin{bmatrix} K_{\mathrm{II}} & K_{\mathrm{I}} \\ (K_{\mathrm{I}})^{\mathrm{T}} & K \end{bmatrix} \begin{bmatrix} u_{\mathrm{I}} \\ u \end{bmatrix} \tag{7.1}$$

where

$$\begin{bmatrix} K_{\mathrm{II}} & K_{\mathrm{I}} \\ (K_{\mathrm{I}})^{\mathrm{T}} & K \end{bmatrix}$$

is a symmetric stiffness matrix which, if the structure is sufficiently anchored, is positive definite and a superscript 'T' denotes the transpose of a matrix. Solving equation (7.1) for u we obtain the structural equation in flexibility form:

$$u = Lr + MF, \tag{7.2}$$

where L is a symmetric positive definite flexibility matrix and M a matrix. In the following we will be interested in time-dependent problems. The external forces $F(t)$ are then prescribed functions of time.

Collecting the structural equation, the non-penetration contact conditions and Coulomb's friction law we then have the following space-discrete quasi-static evolution problem. Here $\dot{u}^+$ denotes the time derivative to the right, and displacements and contact forces are decomposed into tangential and normal vectors: $\boldsymbol{u}_i = \boldsymbol{u}_{i\mathrm{T}} + u_{i\mathrm{N}}\boldsymbol{n}_i$, $\boldsymbol{u}_{i\mathrm{T}} \cdot \boldsymbol{n}_i = 0$, $\boldsymbol{r}_i = \boldsymbol{r}_{i\mathrm{T}} + r_{i\mathrm{N}}\boldsymbol{n}_i$, $\boldsymbol{r}_{i\mathrm{T}} \cdot \boldsymbol{n}_i = 0$, where a centred dot indicates the standard scalar product of geometric vectors and $\boldsymbol{n}_i$ is a normal vector of obstacle i.

DQP. For a given exterior force vector $F = F(t)$, find displacements $u = u(t)$ and reaction forces $r = r(t)$ satisfying (7.2), Signorini contact conditions

$$u_{i\mathrm{N}} \leqslant 0, \quad r_{i\mathrm{N}} \leqslant 0, \quad u_{i\mathrm{N}} r_{i\mathrm{N}} = 0, \tag{7.3}$$

and Coulomb's friction law,

$$\begin{gathered} \boldsymbol{u}_{i\mathrm{N}} = \mathbf{0} \Rightarrow |\boldsymbol{r}_{i\mathrm{T}}| \leqslant -\mu_i r_{i\mathrm{N}}, \\ 0 < |\boldsymbol{r}_{i\mathrm{T}}| = -\mu_i r_{i\mathrm{N}} \Rightarrow \dot{\boldsymbol{u}}_{i\mathrm{T}}^+ = -\lambda_i(t) r_{i\mathrm{T}}, \quad \lambda_i(t) \geqslant 0, \\ |\boldsymbol{r}_{i\mathrm{T}}| < -\mu_i r_{i\mathrm{N}} \Rightarrow \dot{\boldsymbol{u}}_{i\mathrm{T}}^+ = \mathbf{0}. \end{gathered}$$

By taking the right-hand time derivative of the inequalities and equations of **DQP**, it is possible to derive a so-called rate problem which, given an initial state, concerns finding right-hand time derivatives of unknown variables. This rate problem turns out to be a nonlinear complementarity problem, which we denote **NLCP**. In this way it is possible to conclude that the full problem **DQP** can be divided into two subproblems as follows.

Rate problem. Given r and $\dot{F}$, is there a unique solution $\dot{r}^+$ to the **NLCP**, so that we may write

$$\dot{r}^+ = \mathcal{F}(r, \dot{F})?$$

Integration problem. If so, can we integrate this integral equation for a (unique?) solution $r(t)$ with

$$\dot{r}^+(t) = \mathcal{F}(r(t), \dot{F}(t))?$$

(*b*) *A fundamental frictional parameter and existence and uniqueness results*

In the following we give, in the case of a small friction coefficient, an existence and uniqueness result for the rate problem as well as an existence result for **DQP**. The smallness of the friction coefficient is expressed by a fundamental parameter which we first define as follows.

Definition 7.1.

$$\tilde{\varphi}(L, n) = \max_{\substack{u \neq 0 \\ u_i \perp n_i}} \{ \min_{1 \leqslant i \leqslant l} \varphi_i(L, n; u) \},$$

$$\tilde{\mu} = \tilde{\mu}(L, n) = \cot \tilde{\varphi}(L, n),$$

where

$$\varphi_i(L, n; u) = \begin{cases} \angle(\boldsymbol{u}_i, P_i \boldsymbol{r}_i), & \text{if } |\boldsymbol{u}_i|\,|P_i \boldsymbol{r}_i| \neq 0, \\ \pi, & \text{else,} \end{cases}$$

and P_i is the orthogonal projection on $\operatorname{span}\{\boldsymbol{u}_i, \boldsymbol{n}_i\}$.

The parameter $\tilde{\varphi}$ has the following simple mechanical interpretation: if the system is acted upon by an arbitrary force vector r on the contact nodes ($F = 0$) such that all displacements are tangential, then, for at least one node, we have

$$\varphi_i \leqslant \tilde{\varphi}.$$

In Andersson (1999*b*) the following result was obtained.

Theorem 7.2. *If*

$$\mu_i < \tilde{\mu}(L, n)$$

for all contact nodes, then the rate problem has a unique solution $\dot{r}^+$*, provided that* r *and* u *satisfy the compatibility conditions:*

$$u_{i\mathrm{N}} \leqslant 0, \quad r_{i\mathrm{N}} \leqslant 0, \quad u_{i\mathrm{N}} r_{i\mathrm{N}} = 0, \tag{7.4}$$

$$u_{i\mathrm{N}} = 0 \Rightarrow |\boldsymbol{r}_{i\mathrm{T}}| \leqslant -\mu_i r_{i\mathrm{N}}. \tag{7.5}$$

In the same report the following theorem of existence for the discrete quasi-static problem **DQP** is given.

Theorem 7.3. *If in addition* $r(0)$, $u(0)$ *satisfy the conditions (7.4) and (7.5),* $F(t)$ *is absolutely continuous and if* $\dot{F} \in L^\infty(0, T)$*, then there exists a solution* r *to the quasi-static problem so that, almost everywhere,*

$$\dot{r}^+(t) = \mathcal{F}(r(t), \dot{F}(t)).$$

Moreover, there exists a constant $C = C(\mu)$ *so that, almost everywhere,*

$$\|\dot{r}(t)\| \leqslant C \|\dot{F}(t)\|.$$

The main steps in the proof of this existence theorem are the following.

(1) First prove unique solvability of the 'rate problem'.

(2) Then exploit the theory of differential inclusions and differential equations with set-valued mappings, as given, for example, in Filippov (1988).

Alternatively,

(1) make a time-discretization and show that the sequence of incremental problems is uniquely solvable;

(2) obtain an estimate of the form $\|\Delta u_k\| \leqslant C\|\Delta F_k\|$;

(3) pass to the limit to obtain a solution of the time-continuous problem.

For a system with *one node and two DOFs* it is possible to show that the solution given in the previous existence theorem is *unique* (see, for example, Andersson 1999*b*).

(*c*) *Piecewise real analytic forces*

For piecewise real analytic forces it is possible to obtain a partial uniqueness result for the full problem **DQP**.

Definition 7.4. The mapping

$$[0, T) \ni t \mapsto F(t) \in \mathbb{R}^{3l}$$

is *right piecewise real analytic* (RPRA) if, for every $t_0 \in [0, T)$, there exists $\epsilon > 0$ such that

$$F(t) = \sum_{k=0}^{\infty} F^k (t - t_0)^k, \quad \text{for } t \in [t_0, t_0 + \epsilon).$$

Further, F is said to be *absolutely continuous* (AC) if

$$\int_0^T \|\dot{F}(t)\| \, \mathrm{d}t < \infty.$$

Right piecewise real analyticity implies continuity on $[0, T)$, and that $v(t) = M\dot{F}(t)$ has at most countably many discontinuities.

We first formulate an existence and uniqueness result for the case that we have two DOFs for each node.

Theorem 7.5. *Assume that* $\mu_i < \tilde{\mu}$, *that* $F : [0, T) \to \mathbb{R}^{2l}$ *is RPRA and AC and that the initial values* $r(0)$ *and* $u(0)$ *satisfy the compatibility conditions (7.4) and (7.5). Then there exists* a unique *RPRA mapping*

$$[0, T) \ni t \mapsto r(t) \in \mathbb{R}^{2l},$$

that solves our quasi-static friction problem (for all t*). Further there exists a constant* $C = C(\mu)$ *such that*

$$\|\dot{r}^+(t)\| \leqslant C\|\dot{F}(t)\|$$

for all t, *i.e.* $r(t)$ *is AC.*

Sketch of proof.

(1) For a given t_0 construct a *formal* power series solution $r(t) = \sum_{k=0}^{\infty} r^k (t - t_0)^k$. The coefficients r^k are uniquely determined by a sequence (tree) of $\mathbf{NLCP}(\boldsymbol{k})$, for $k \geqslant 1$. These NLCPs are similar to that for the 'rate problem', although a little more complicated.

(2) Prove that the power series converges in some interval $[t_0, t_0 + \epsilon)$, giving a local solution.

(3) Repeat with new initial point.

∎

The proof does not go through completely for the case with three DOFs at each node. However, we can still construct unique formal power series solutions

$$\boldsymbol{r}_i(t) = \sum_{k=s_i}^{\infty} \boldsymbol{r}_i^k (t - t_0)^k, \qquad \boldsymbol{r}_i^{s_i} \neq 0,$$

$$\boldsymbol{u}_i(t) = \boldsymbol{u}_i^0 + \sum_{k=q_i}^{\infty} \boldsymbol{u}_i^k (t - t_0)^k, \quad \boldsymbol{u}_i^{q_i} \neq 0,$$

and convergence is obtained if, for all nodes in contact, we have

$$s_i \leqslant q_i.$$

The exceptional case, $s_i > q_i$, should be interpreted as a case with *grazing tangential contact*, meaning that (formally) if the obstacle were removed, then the velocity vector would be perpendicular to the normal vector. We may therefore formulate the following corollary.

Corollary 7.6. *We have uniqueness for the case with three DOFs at each node, provided that* non-grazing contact *or* grazing non-tangential contact *does not occur.*

Summarizing we have the following:

existence, if $\mu_i < \tilde{\mu}$ for all contact nodes and with 'relatively arbitrary' external forces: $\dot{F} \in L^{\infty}$;

existence and uniqueness, if, in addition,

(i) 'arbitrary forces', one single node, two DOFs;

(ii) 'regular forces', F in RPRA and AC:

  (a) many-particle systems, two DOFs at each node, and

  (b) many-particle systems, three DOFs at each node, provided that *grazing non-tangential contact* does not occur!

8. Conclusion

Contact problems with friction are among the most basic problems of mechanics. Yet fundamental questions of existence and uniqueness of solutions are only partly understood. In the present paper we have attempted to give the state of the art of a class of frictional contact problems: that of static and quasi-static small-displacement linear elasticity. Even for such a limited class of problems we cannot report a full understanding, although substantial progress has recently been made. As a basic open question we mention that of uniqueness of solutions.

First we note that for quasi-static evolution problems, as described in § 6, without any viscosity terms, there do not seem to be any uniqueness results at all available. This is also true if we introduce various regularizations of the friction terms, such as normal compliance or mollification of the normal pressure.

For static problems, treated in § 4, there are, as mentioned above, uniqueness results given for the case of a small coefficient of friction, μ, and with some kind of mollification (see Duvaut 1980; Cocu 1984; Demkowicz & Oden 1982). For the more general problem with Signorini–Fichera conditions and without regularization, treated by Nečas *et al*. (1980), no uniqueness result is known. Here one might add that for μ large, even existence has not been established.

As appropriate areas of future research in contact problems with friction, we would like to identify the following.

For continuous elastic systems with Signorini–Fichera contact conditions (and without regularization of the frictional terms and without viscosity), questions of uniqueness in the static case should be settled.

Questions of uniqueness for elastic quasi-static evolution problems are of great importance, as well as the question of whether the (unique?) quasi-static solutions appear as limits of dynamic solutions or of solutions with viscosity present, as the mass or the viscosity tends to zero, respectively. For the simplest case with two DOFs, Martins *et al*. (1994, 1995) have some results of this kind.

Most of the problems treated here have been for linearly elastic systems. It is natural to try to extend them to cases with large deformations, which seems more fundamental than expanding to, for instance, plastic material behaviour.

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Appendix A.

We let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, denote an open bounded and connected set with Lipschitz boundary $\partial\Omega$. The set Ω is understood to be the reference configuration of an elastic body which may come into contact with a rigid obstacle along some part S_c of its boundary. The body is subjected to time-dependent volume and traction forces, $\boldsymbol{f}(t)$ and $\boldsymbol{t}(t)$, the latter acting on some part S_t of $\partial\Omega$. Some part $S_c \subset \partial\Omega$ is kept fixed (or has a prescribed motion).

For the subsets S_c, S_a and $S_u \subset \partial\Omega$ we further assume that they are relatively open with mutually disjoint closures.

The (Hilbert space) Sobolev norm of order p over Ω or its boundary is denoted by $\|\cdot\|_{p,\Omega}$ or $\|\cdot\|_{p,\partial\Omega}$, respectively. The same notation is also used when the functions are vector valued. So we write, for example,

$$\|\cdot\|_{L^2(\Omega)^n} = \|\cdot\|_{0,\Omega} \quad \text{and} \quad \|\cdot\|_{H^1(\Omega)^n} = \|\cdot\|_{1,\Omega}.$$

Similarly, the dual pairing, for example, between the spaces

$$H^{-1/2}(\partial\Omega) \quad \text{and} \quad H^{1/2}(\partial\Omega)$$

is written $\langle\cdot,\cdot\rangle_{-1/2,1/2(\partial\Omega)}$. When no confusion is likely to appear, the sub-indices denoting the particular spaces are omitted.

To denote the dual pairing between a Hilbert space H and its dual H' we use $\langle\cdot,\cdot\rangle_{H',H}$. The inner product and the norm in H will be denoted by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$.

We also introduce a gap function $g \in H^{1/2}\partial\Omega$ defining the initial gap between the elastic body in the reference configuration. Note that we do not assume that $g \geqslant 0$, which means that the body may be preloaded by the obstacle in the absence of external forces $\boldsymbol{f}$ and $\boldsymbol{t}$.

The trace operator is denoted by

$$\mathrm{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega). \tag{A 1}$$

Further we use the same notation for the trace operator,

$$\mathrm{tr} : (H^1(\Omega))^n \to (H^{1/2}(\partial\Omega))^n,$$

and denote its norm by $\|\,\mathrm{tr}\,\|$. We also use that there exists a linear bounded extension operator

$$\mathcal{E} : (H^{1/2}(\partial\Omega))^n \to (H^1(\Omega))^n$$

with norm denoted by $\|\mathcal{E}\|$, such that $\mathrm{tr}\circ\mathcal{E} = \mathrm{id}_{(H^{1/2}(\partial\Omega))^n}$. It is clear that $\|\mathcal{E}\|\|\,\mathrm{tr}\,\| \geqslant 1$ and that an optimal value of this product depends only on the geometry of Ω. For the case that $\Omega = \mathbb{R}^n_+ = \{(x, x_n) : x_n > 0\}$ these mappings are denoted by tr_0 and $\mathcal{E}_0$.

Next, let us introduce the affine subspace

$$V = \{\boldsymbol{u} \in (H^1(\Omega))^n : \boldsymbol{u}|_{S_u} = \bar{\boldsymbol{u}}\}$$

of the Sobolev space $(H^1(\Omega))^n$ and its closed convex subset

$$K = \{\boldsymbol{u} \in V : \boldsymbol{u}|_{S_u} = 0 \text{ and } u_{\mathrm{N}}|_{S_{\mathrm{c}}} \leqslant g\}.$$

Here, and subsequently, restrictions of functions are interpreted in the trace sense.

The bilinear elastic energy form is given by

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega a_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x,$$

where the stress tensor is

$$\sigma_{ij} = \sigma_{ij}(\boldsymbol{u}) = a_{ijkl}\frac{\partial u_k}{\partial x_l}.$$

For the coefficients a_{ijkl} of the elasticity tensor we require, besides the usual symmetry conditions, that $a_{ijkl} \in L^\infty(\Omega)$. Further we assume that there exists a constant $\alpha > 0$ such that

$$a_{ijkl}\xi_{ij}\xi_{kl} \geqslant \alpha|\xi|^2 = \alpha\xi_{ij}\xi_{ij}$$

for all ξ_{ij} such that $\xi_{ij} = \xi_{ji}$. Here we have used the summation convention. The mapping

$$(\boldsymbol{u}, \boldsymbol{v}) \mapsto a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} a_{ijkl}(x) \frac{\partial u_k}{\partial x_l} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x$$

then defines a bounded, symmetric bilinear form on $V \times V$. By Korn's inequality and the boundedness of a_{ijkl} there exist constants $c_0 = c_0(\alpha, \Omega, S_u)$ and C_0 such that for all $u \in V$

$$c_0 \|\boldsymbol{u}\|^2_{1,\Omega} < a(\boldsymbol{u}, \boldsymbol{u}) = a(\boldsymbol{u}) < C_0 \|\boldsymbol{u}\|^2_{1,\Omega}. \tag{A 2}$$

For convenience we may let the non-negative coefficient of friction μ be defined on the whole of the boundary and we assume that $\mu \in L^{\infty}$ with sup–norm $\|\mu\|_{\infty}$.

(*a*) *Assumptions made in* § 6

For the coefficients a_{ijkl} of the elasticity tensor we require in addition that they are locally in $C^{0,\beta}$ in a neighbourhood of S_{c}, i.e. that each point $x_0 \in S_{\mathrm{c}}$ has a neighbourhood $U_0 \ni x_0$ such that for some constant L_0 and all $x, y \in U_0$ we have the inequality

$$|a_{ijkl}(x) - a_{ijkl}(y)| < L_0 |x - y|^{\beta}. \tag{A 3}$$

For the coefficient of friction μ we assume that it is a multiplier on $H^{1/2}(\partial\Omega)$, i.e. that the mapping

$$H^{1/2}(\partial\Omega) \ni v \mapsto \mu v \in H^{1/2}(\partial\Omega)$$

is bounded with norm $\|\mu\|$, so that $\|\mu v\|_{1/2,\partial\Omega} \leqslant \|\mu\| \, \|v\|_{1/2,\partial\Omega}$. By duality, it follows that μ is a multiplier on $H^{1/2}(\partial\Omega)$ if and only if it is a multiplier on $H^{-1/2}(\partial\Omega)$, and that the respective norms are equal. We note that μ is such a multiplier if μ is a Lipschitz function on $\partial\Omega$. For more information on multipliers and for more general sufficient conditions that μ is a multiplier, we refer to Maz'ya & Shaposhnikova (1985). We also assume that $\mu \in L^{\infty}(\partial\Omega)$ with the norm $\|\mu\|_{\infty}$.

For a Hilbert space H, $L^2(0, T; H)$ is the class of mappings $[0, T] \ni t \mapsto f(t) \in H$, where f is weakly measurable and

$$\|f\|^2_{L^2(0,T;H)} = \int_0^T \|f(t)\|^2_H \, \mathrm{d}t < \infty.$$

One can prove (see, for example, Lions & Magenes 1968) that $L^2(0, T; H)$ is a Hilbert space. Next, $W^1(0, T; H)$ denotes the class of all functions $f \in L^2(0, T; H)$ which are strongly absolutely continuous and such that their weak derivative $\dot{f} \in L^2(0, T; H)$. One can prove that the derivative exists also in the strong sense, i.e. that

$$\lim_{h \to 0} \|\dot{f}(t) - [f(t+h) - f(t)]/h\|_H = 0 \quad \text{for almost all } t,$$

and that $W^1(0, T; H)$ is a Hilbert space with norm given by

$$\|f\|^2_{W^1(0,T;H)} = \int_0^T \|f(\tau)\|^2_H \, \mathrm{d}\tau + \int_0^T \|\dot{f}(\tau)\|^2_H \, \mathrm{d}\tau.$$

For the set S_{c} we assume that it is locally in $C^{1,\beta}$ for some $\beta \in (0, 1]$. This means that each point of S_{c} has a neighbourhood where it can be represented by an equation

$$x_n = \Psi(x'), \quad x = (x', x_n), \quad x' \in \mathbb{R}^{n-1}$$

with $\Psi \in C^{1,\beta}$, i.e. such that

$$|\nabla\Psi(x') - \nabla\Psi(y')| \leqslant c|x' - y'|^{\beta}$$

for some constant c.

Further, for the sets S_u and S_{c} we make the assumption that their relative boundaries ∂S_u and ∂S_{c} have $H^{1/2}(\partial\Omega)$-capacity equal to zero:

$$\mathrm{cap}(\partial S_u, H^{1/2}(\partial\Omega)) = \mathrm{cap}(\partial S_{\mathrm{c}}, H^{1/2}(\partial\Omega)) = 0.$$

Here the capacity of a compact subset $e \subset \partial\Omega$ is defined as

$$\mathrm{cap}(e, H^{1/2}(\partial\Omega)) = \inf\{\|\psi\|_{1/2,\partial\Omega} : \psi \in C_0^{\infty}(\mathbb{R}^n),\ \psi \geqslant 1 \text{ on } e\}.$$

A sufficient condition for this is, for example, that ∂S_u and ∂S_{c} are Lipschitz curves. The condition that ∂S_u and ∂S_{c} have capacity zero implies that the distributions in $H^{1/2}(\partial\Omega)$ with support contained in $\partial\Omega \setminus \partial S_u$ are dense in $H^{1/2}(\partial\Omega)$ and similarly for ∂S_{c}. For C^{∞}-domains with C^{∞}-boundaries, this is proved in Lions & Magenes (1968), and follows for Lipschitz domains with Lipschitz boundaries by the fact that H^1- and $H^{1/2}$-norms are invariant with respect to Lipschitz transformations (see also Maz'ya 1985).

We finally assume that $g \in H_{\mathrm{loc}}^{(1/2)+\alpha}(S_{\mathrm{c}}) \cap H^{1/2}(\partial\Omega)$. Here $g \in H_{\mathrm{loc}}^{(1/2)+\alpha}(S_{\mathrm{c}})$ means that for each $x_0 \in S_{\mathrm{c}}$ there exists some function $\rho \in C_0^{\infty}(\mathbb{R}^n)$ with $x_0 \in \mathrm{supp}\,\rho$, $\mathrm{supp}\,\rho \cap \partial\Omega \subset S_{\mathrm{c}}$ and such that $\rho g \in H^{(1/2)+\alpha}(\partial\Omega)$.

The parameters α and β should satisfy the inequalities, $0 < \alpha < \beta \leqslant 1$ and $\alpha \leqslant \frac{1}{2}$.

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