On the Heun equation

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A new approach to the theory of finite-gap integration for the Heun equation is constructed. As an application, global monodromies of the Heun equation are calculated and expressed as hyperelliptic integrals. The relationship between the Heun equation and the spectral problem for the BC\text{1} Inozemtsev model is also discussed.

Keywords: Heun equation; elliptic function; finite-gap integration; monodromy; hyperelliptic integral

1. Introduction

The Heun equation is a differential equation given by

$$
\left( \frac{d}{dw} \right)^2 + \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t} \right) \frac{d}{dw} + \frac{\alpha \beta w - q}{w(w-1)(w-t)} \tilde{f}(w) = 0, \tag{1.1}
$$

with the condition

$$
\gamma + \delta + \epsilon = \alpha + \beta + 1. \tag{1.2}
$$

The Heun equation is the standard canonical form of a Fuchsian equation with four singularities. It is well known that the Fuchsian equation with three singularities is the hypergeometric differential equation.

In this paper, we calculate global monodromies of the Heun equation in terms of the hyperelliptic integral for the case $\gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} + (1/2)$. We also explain the relationship of the Heun equation to finite-gap theory, which plays an important role in finding solutions to the Heun equation.

Ince (1940) showed that the Lamé potential $l(l+1) \varphi(x + \tau/2)$ for $l \in \mathbb{Z}$ is a finite-gap potential. Here, $\varphi(x)$ is the Weierstrass $\varphi$-function with periods $(1, \tau)$. Other examples of the finite-gap potential were found by Treibich and Verdier. Around 1987–1992, they found that the potential $\sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i) (\omega_0 = 0, \omega_1 = 1/2, \omega_2 = -(\tau + 1)/2$ and $\omega_3 = \tau/2)$, which is called the Treibich & Verdier (1992) potential, is an algebra-geometric finite-gap potential iff $l_i \in \mathbb{Z}$ for all $i \in \{0, 1, 2, 3\}$. They proved it by using a theory of elliptic soliton, which is closely related to the geometry of Riemann surfaces. Subsequently, several others (Gesztesy & Weikard 1995; Weikard 1998; Smirnov 2002 and so forth) have

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produced more precise statements and concerned results on this subject. For more
details, see references therein. On the other hand, it is known that the spectral
problem for the Treibich–Verdier potential is equivalent to solving the Heun
equation (§4).

Aside from the above-mentioned activities, at first the author took an interest
in the Heun equation from an integrable systems viewpoint, especially the
Calogero–Moser–Sutherland model and the Inozemtsev model. The $BC_N$
Inozemtsev (1989) model is a system of quantum mechanics with $N$-particles
whose Hamiltonian is given by

\[
H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2l(l+1) \sum_{1 \leq j < k \leq N} (\varphi(x_j - x_k) + \varphi(x_j + x_k))
+ \sum_{j=1}^{N} \sum_{i=0}^{3} l_i(l_i+1) \varphi(x_j + \omega_i),
\]

(1.3)

where $l$ and $l_i$ ($i=0, 1, 2, 3$) are coupling constants.

It is known that the $BC_N$ Inozemtsev model is quantum completely integrable.
More precisely, there exist operators of the form $H_k = \sum_{j=1}^{N} (\partial/\partial x_j)^{2k} + \text{lower terms}$ ($k=2, \ldots, N$) such that $[H, H_k]=0$ and $[H_{k_1}, H_{k_2}]=0$ ($k, k_1, k_2=2, \ldots, N$). Note that the $BC_N$ Inozemtsev model is a universal completely integrable model
of quantum mechanics with $B_N$ symmetry, which follows from the classification
due to Ochiai et al. (1994), and the $BC_N$ Calogero–Moser–Sutherland system is a special case of the $BC_N$ Inozemtsev system.

For the case $N=1$, the potential coincides with the Treibich–Verdier potential and the spectral problem for the $BC_1$ Inozemtsev model is equivalent to solving
the Heun equation.

To analyse the $BC_1$ Inozemtsev model, the Bethe Ansatz method for the Heun
equation is proposed in the work of Takemura (2003), and it is rediscovered that
the potential of the $BC_1$ Inozemtsev model is a finite-gap potential (Takemura
2004b). In this paper, we will explain how to obtain the finite-gap property with
examples. It will be simpler than an original proof of Treibich–Verdier because
we do not need any advanced knowledge of algebraic geometry of Riemann
surfaces. We will also illustrate how to calculate functions related to the finite-
gap integration with examples. In particular, we will calculate global
monodromies of the Heun equation.

First, a method for proving that the function $\sum_{i=0}^{3} l_i(l_i+1) \varphi(x + \omega_i)$ ($l_0, l_1, l_2, l_3 \in \mathbb{Z}$) is an algebro-geometric finite-gap potential is presented in §§2 and 3. The
strategy is as follows: in §2a, a product of two certain solutions to $Hf(x) = Ef(x)$
($H$, Hamiltonian; $E$, eigenvalue) is shown to be doubly periodic for all $E$. As a
by-product, a differential operator of odd degree that commutes with the
Hamiltonian $H$ is constructed in §2b, which ensures the algebro-geometric finite-
gap property. In §2c, formulae for global monodromies of eigenfunctions of the $BC_1$
Inozemtsev model are obtained. These formulae are expressed as hyperelliptic
integrals. In §3a, finite-dimensional spaces of elliptic functions are introduced. Note that these spaces are related to the concept of quasi-exact solvability (Turbiner
1988; Gonzalez-Lopez et al. 1994; Takemura 2002, 2004a) and play important roles
in the presentation and investigation of a spectral curve, i.e. a curve determined by
the equality between the Hamiltonian $H$ and the commuting operator of odd degree.

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As an application, we obtain a determinant formula for the commuting operator in §3c. In §4, a relationship between the Heun equation and the Inozemtsev model is reviewed, and a translation between the two formalisms is detailed. From this, formulae for the monodromies of the Heun equation on cycles around two regular singular points are derived and expressed by hyperelliptic integrals. In §5, we give comments. In appendices A and B, definitions and formulae for the elliptic functions and the proof of theorem 3.1 are provided.

Throughout this paper, it is assumed that \( l_i \in \mathbb{Z} \) \((i=0, 1, 2, 3)\) and \((l_0, l_1, l_2, l_3) \neq (0, 0, 0, 0)\).

2. Finite-gap property and monodromy

(a) Integral formula

The \( BC_1 \) Inozemtsev model is a one-particle quantum mechanics model whose Hamiltonian is given as

\[
H = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i), \tag{2.1}
\]

where \( \varphi(x) \) is the Weierstrass \( \varphi \)-function with periods \( (1, \tau) \); \( \omega_0 = 0, \omega_1 = 1/2, \omega_2 = -(\tau + 1)/2 \) and \( \omega_3 = \tau/2 \) are half-periods; and \( l_i (i=0, 1, 2, 3) \) are coupling constants. Note that the function \( \sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i) \) is also called the Treibich–Verdier potential.

Fix the eigenvalue \( E \) of the Hamiltonian \( H \) (see (2.1)) and consider the second-order differential equation

\[
(H - E)f(x) = \left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i) - E\right)f(x) = 0. \tag{2.2}
\]

Let \( h(x) \) be the product of any pair of solutions to equation (2.2). Then, the function \( h(x) \) satisfies the following third-order differential equation:

\[
\left(\frac{d^3}{dx^3} - 4\left(\sum_{i=0}^{3} l_i (l_i + 1) \varphi(x + \omega_i) - E\right)\frac{d}{dx} - 2\left(\sum_{i=0}^{3} l_i (l_i + 1) \varphi'(x + \omega_i)\right)\right)h(x) = 0. \tag{2.3}
\]

It is known that equation (2.3) has a non-zero doubly periodic solution for all \( E \) if \( l_i \in \mathbb{Z}_{\geq 0} \) \((i=0, 1, 2, 3)\).

Proposition 2.1. \((\text{Takemura 2003, proposition 3.5})\) If \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \), then equation (2.3) has a non-zero doubly periodic solution \( \Xi(x, E) \), which has the expansion

\[
\Xi(x, E) = c_0(E) + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_j^{(i)}(E) \varphi(x + \omega_i)^{l_i-j}, \tag{2.4}
\]

where the coefficients \( c_0(E) \) and \( b_j^{(i)}(E) \) are polynomials in \( E \) and do not have common divisors, and the polynomial \( c_0(E) \) is monic. Moreover, the function \( \Xi(x, E) \) is determined uniquely. Set \( g = \deg_E c_0(E) \), then the coefficients satisfy \( \deg_E b_j^{(i)}(E) < g \) for all \( i \) and \( j \).
Now, we calculate the function $\Xi(x, E)$ explicitly for fixed coupling constants $l_0, l_1, l_2$ and $l_3$.

**Example 2.1.** (i) (The case $l_0=1, l_1=l_2=l_3=0$) By considering poles of solutions to (2.3), the function $\Xi(x, E)$ should be expressed as

$$\Xi(x, E) = c_0(E) + b_0^{(0)}(E) \varphi(x).$$

Substituting this into (2.3), the coefficients $c_0(E)$ and $b_0^{(0)}(E)$ are determined. As a result, we have

$$\Xi(x, E) = \varphi(x) + E. \quad (2.5)$$

(ii) (The case $l_0=2, l_1=1, l_2=0, l_3=1$) The function $\Xi(x, E)$ should be expressed as

$$\Xi(x, E) = c_0(E) + b_0^{(0)}(E) \varphi(x)^2 + b_1^{(0)}(E) \varphi(x)^2 + b_0^{(1)}(E) \varphi(x + \frac{1}{2}) + b_0^{(3)}(E) \varphi(x + \frac{3}{2}).$$

Substituting this into (2.3), the coefficients are determined as follows:

$$\Xi(x, E) = 9 \varphi(x)^2 + 3(E - 2e_1 - 2e_3) \varphi(x) + (E + e_1 - 8e_3) \varphi(x + \frac{1}{2})$$

$$+ (E + e_3 - 8e_1) \varphi(x + \frac{3}{2}) + (E^2 - 5(e_1 + e_3)E - 15e_1^2 + 33e_1 e_3 - 15e_3^2), \quad (2.6)$$

where $e_1 = \varphi(1/2)$ and $e_3 = \varphi(\tau/2)$.

Let us consider an integral formula for the solution to (2.2) in terms of the doubly periodic function $\Xi(x, E)$. Set

$$Q(E) = \Xi(x, E)^2 \left( E - \sum_{i=0}^{3} l_i(l_i + 1) \varphi(x + \omega_i) \right) + \frac{1}{2} \Xi(x, E) \frac{d^2 \Xi(x, E)}{dx^2}$$

$$- \frac{1}{4} \left( \frac{d\Xi(x, E)}{dx} \right)^2. \quad (2.7)$$

Since $\Xi(x, E)$ satisfies (2.3), $Q(E)$ is independent of $x$. From the expression for $\Xi(x, E)$ given in (2.4), it follows that $Q(E)$ is a monic polynomial in $E$ of degree $2g+1$. Then, we have the following integral representation of a solution to (2.2).

**Proposition 2.2.** (Takemura 2003, proposition 3.7) Let $\Xi(x, E)$ be the doubly periodic function defined in proposition 2.1 and $Q(E)$ be the monic polynomial defined in (2.7). Then, the function

$$A(x, E) = \sqrt{\overline{\Xi(x, E)}} \exp \int \frac{\sqrt{-Q(E)}dx}{\Xi(x, E)}, \quad (2.8)$$

is a solution to the differential equation (2.2).

**Example 2.2.** (i) (The case $l_0=1, l_1=l_2=l_3=0$) The polynomial $Q(E)$ is determined as

$$Q(E) = (E + e_1)(E + e_2)(E + e_3), \quad (2.9)$$

where $e_1 = \varphi(1/2)$, $e_2 = (\varphi(1+\tau)/2)$ and $e_3 = \varphi(\tau/2)$. Therefore, the solution $A(x, E)$ to (2.2) is expressed as

$$A(x, E) = \sqrt{\varphi(x) + E} \exp \int \frac{\sqrt{-(E + e_1)(E + e_2)(E + e_3)}dx}{\varphi(x) + E}. \quad (2.10)$$
(ii) (The case \( l_0=2, l_1=1, l_2=0, l_3=1 \)) The polynomial \( Q(E) \) is determined as
\[
Q(E) = (E + e_1 - 8e_3)(E + e_3 - 8e_1)\left\{E^3 - 3(e_1 + e_3)E^2 + (-29e_1^2 + 22e_1e_3 - 29e_3^2)E + (-97e_1^3 + 77e_1^2e_3 + 77e_1e_3^2 - 97e_3^3)\right\}.
\]

Therefore, the solution \( A(x, E) \) to (2.2) is expressed as
\[
A(x, E) = \sqrt{\Xi(x, E)} \exp \int \frac{\sqrt{-Q(E)} \, dx}{\Xi(x, E)},
\]
where \( \Xi(x, E) \) is defined in (2.6).

(b) Construction of commuting operator

Let us consider the Schrödinger operator \( H = -(d^2/dx^2) + q(x) \) with conditions that \( q(x) \) is smooth on \( \mathbb{R} \) and \( q(x+1) = q(x) \). Let \( \sigma_b(H) \) be the set defined by
\[
E \in \sigma_b(H) \iff \text{Every solution to } (H-E)f(x) = 0 \text{ is bounded on } x \in \mathbb{R}.
\]

If the closure of the set \( \sigma_b(H) \) can be written as
\[
\overline{\sigma_b(H)} = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, \infty],
\]
where \( E_0 < E_1 < \cdots < E_{2g} \), then \( q(x) \) is called the finite-gap potential.

If there exists an odd-order differential operator \( A = (\frac{d}{dx})^{2g+1} + \sum_{j=0}^{2g-1} b_j(x) (\frac{d}{dx})^{2g-1-j} \) such that \([A, -(d^2/dx^2) + q(x)] = 0\), then \( q(x) \) is called the algebro-geometric finite-gap potential.

It is known that if \( q(x) \) is a real-holomorphic function on \( \mathbb{R} \) satisfying \( q(x+1) = q(x) \), then \( q(x) \) is a finite-gap potential if and only if \( q(x) \) is an algebro-geometric finite-gap potential (for details see Novikov (1974), Dubrovin (1975), Its & Matveev (1975) and Weikard (1998)). The equation \([A, -(d^2/dx^2) + q(x)] = 0\) is equivalent to the function \( q(x) \) being a solution to a certain stationary higher-order KdV equation.

Since the potential \( \sum_{i=0}^{3} l_i(l_i + 1) \varphi(x + \omega_i) \) has poles on \( \mathbb{R} \), it would contain some obstacles if investigated from a spectral viewpoint. Instead, it is shown that \( \sum_{i=0}^{3} l_i(l_i + 1) \varphi(x + \omega_i) \) (\( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \)) is an algebro-geometric finite-gap potential. Now we describe the odd-order commuting differential operator \( A \).

Write
\[
\Xi(x, E) = \sum_{i=0}^{g} a_{g-i}(x)E^i.
\]
(see (2.4)). From proposition 2.1, we have \( a_0(x) = 1 \). The following statements are shown by standard technique of finite-gap theory.

**Theorem 2.1.** (cf. Takemura 2004b, theorem 3.1) Set \( u(x) = \sum_{i=0}^{3} l_i(l_i + 1) \varphi(x + \omega_i) \) and define the \((2g+1)\)st-order differential operator \( A \) by
\[
A = \sum_{j=0}^{g} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_j(x) \right) \right\} \left( \frac{d^2}{dx^2} + u(x) \right)^{g-j},
\]

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where $a_j(x)$ are defined in (2.15). Then, the operator $A$ commutes with the Hamiltonian $H=-(d^2/dx^2)+u(x)$. In other words, the function $u(x)=\sum_{i=0}^{3}l_i(l_i+1)\varphi(x+\omega_i)$ is an algebro-geometric finite-gap potential.

**Proposition 2.3.** (cf. Takemura 2004b, proposition 3.2) Let $Q(E)$ be the polynomial defined in (2.7). Then,

$$A^2 = -Q(H).$$  \hspace{1cm} (2.17)

**Example 2.3.** (i) (The case $l_0=1, l_1=l_2=l_3=0$) Write $a_0(x)E+a_1(x)=E+\varphi(x)(=\Xi(x,E))$. Set

$$A = \sum_{j=0}^{1} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_j(x) \right) \right\} \left( -\frac{d^2}{dx^2} + u(x) \right)^{1-j}$$

$$= \left( \frac{d}{dx} \right)^3 - 3\varphi(x) \frac{d}{dx} - \frac{3}{2} \varphi'(x).$$  \hspace{1cm} (2.18)

From theorem 2.1 and proposition 2.3, it follows that $[A,H]=0$ and $A^2=-(H+e_1)(H+e_2)(H+e_3)$. Note that $H=-(d^2/dx^2)+2\varphi(x)$ for this case.

(ii) (The case $l_0=2, l_1=1, l_2=0, l_3=1$) We define $a_j(x)$ ($j=0, 1, 2$) as follows:

$$a_0(x)E^2+a_1(x)E+a_1(x)=E^2+\left(3\varphi(x)+\varphi(x+\frac{1}{2})+\varphi(x+\frac{\tau}{2})-5e_1-5e_3\right)E$$

$$+9\varphi(x)^2-6(e_1+e_3)\varphi(x)+(e_1-8e_3)\varphi(x+\frac{1}{2})$$

$$+(e_3-8e_1)\varphi(x+\frac{\tau}{2})-15e_1^2+33e_1e_3-15e_3^2(=\Xi(x,E)).$$

Set

$$A = \sum_{j=0}^{2} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_j(x) \right) \right\} \left( -\frac{d^2}{dx^2} + u(x) \right)^{2-j}$$

$$= \left( \frac{d}{dx} \right)^5 - 5 \left( 3\varphi(x)+\varphi(x+\frac{1}{2})+\varphi(x+\frac{\tau}{2})-e_1-e_3 \right) \left( \frac{d}{dx} \right)^3$$

$$- \frac{15}{2} \left( 3\varphi'(x)+\varphi'(x+\frac{1}{2})+\varphi'(x+\frac{\tau}{2}) \right) \left( \frac{d}{dx} \right)^2$$

$$- 15 \left( 3\varphi(x)^2 + \left( 2\varphi(x+\frac{1}{2})-e_1+2e_3 \right) \varphi(x+\frac{1}{2}) \right)$$

$$+ \left( 2\varphi(x+\frac{\tau}{2})-e_3+2e_1 \right) \varphi(x+\frac{\tau}{2}) - \frac{19}{5} (e_1^2+e_1e_3+e_3^2) \frac{d}{dx}$$

$$- \frac{15}{2} \left( 2\varphi(x+\frac{1}{2})-e_1+2e_3 \right) \varphi'(x+\frac{1}{2})$$

$$+ \left( 2\varphi(x+\frac{\tau}{2})-e_3+2e_1 \right) \varphi'(x+\frac{\tau}{2}) \right).$$  \hspace{1cm} (2.19)

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From theorem 2.1 and proposition 2.3, it follows that \([A,H]=0\) and \(A^2 = -Q(H)\), where the polynomial \(Q(E)\) was defined in (2.11). Note that \(H = -(d/dx)^2 + 6 \varphi(x) + 2 \varphi(x + (1/2)) + 2 \varphi(x + (\tau/2))\) for this case.

(c) Monodromy

Since the potential of the operator \(H\) (see (2.2)) is doubly periodic, it follows immediately that if \(A(x, E)\) is an eigenfunction of the operator \(H\) with an eigenvalue \(E\), then \(A(x+1, E)\) and \(A(x+\tau, E)\) are also eigenfunctions with the eigenvalue \(E\).

In this section, we will calculate the monodromies, especially the constants \(B_1(E)\) and \(B_2(E)\) which satisfy \(A(x+1, E) = B_1(E)A(x, E)\) and \(A(x+\tau, E) = B_2(E)A(x, E)\), respectively, where the function \(A(x, E)\) is the solution to the equation \((H-E)A(x, E)=0\) of the form (2.8). The monodromies play an important role in investigating eigenvalues and eigenstates for systems with physical boundary conditions. They are expressed by hyperelliptic integrals of the second kind, as we see in the following.

**Theorem 2.2.** (Takemura 2004b, theorem 3.7) Let \(l_1 \in \mathbb{Z}_{\geq 0}\) and \(E_0\) be eigenvalues of the Hamiltonian \(H\) (see (2.2)) such that \(A(x+1, E_0) = \pm A(x, E_0)\). Then,

\[
A(x + 1, E_0) = \pm A(x, E_0) \exp \left( -\frac{1}{2} \int_{E_0}^{E} \frac{Q_1(E)}{\sqrt{-Q(E)}} \, dE \right),
\]

where the polynomial \(Q_1(E)\) in \(E\) of degree \(g\) is given by

\[
Q_1(E) = \int_{0+\epsilon}^{1+\epsilon} \Xi(x, E) \, dx,
\]

with \(\epsilon\) a constant so determined as to avoid passing through the poles while integrating.

If \(E_0'\) is an eigenvalue of \(H\) such that \(A(x + \tau, E_0') = \pm A(x, E_0')\), then

\[
A(x + \tau, E_0) = \pm A(x, E_0) \exp \left( -\frac{1}{2} \int_{E_0'}^{E} \frac{Q_2(E)}{\sqrt{-Q(E)}} \, dE \right),
\]

where \(Q_2(E) = \int_{0+\epsilon}^{\tau+\epsilon} \Xi(x, E) \, dx\).

Theoretically, the integral \(\int_{0+\epsilon}^{1+\epsilon} \Xi(x, E) \, dx\) can be calculated explicitly. Let us explain this with some examples.

**Example 2.4.** (i) (The case \(l_0=1, l_1=l_2=l_3=0\)) Now we calculate the polynomials \(Q_1(E)\) and \(Q_2(E)\). We have

\[
Q_1(E) = \int_{0+\epsilon}^{1+\epsilon} \Xi(x, E) \, dx = \int_{0+\epsilon}^{1+\epsilon} (E + \varphi(x)) \, dx = E - \zeta(1+\epsilon) + \zeta(\epsilon) = E - 2\eta_1,
\]

\[
Q_2(E) = \int_{0+\epsilon}^{\tau+\epsilon} \Xi(x, E) \, dx = E\tau - 2\eta_3,
\]
where \( \eta_1 = \zeta(1/2) \), \( \eta_3 = \zeta(\tau/2) \) and \( \zeta(x) \) is the Weierstrass \( \zeta \)-functions. Note that we used formulae \( \zeta'(x) = -\varphi(x) \), \( \zeta(x+1) = \zeta(x) = 2\eta_1 \) and \( \zeta(x+\tau) = \zeta(x) = 2\eta_3 \). It will be shown in \( \S 3a \) that \( E = -e_1 \) is a solution to \( Q(E) = 0 \) such that \( A(x, E) = A(x+1, E) = -A(x+\tau, E) \). Hence,

\[
A(x+1, E_e) = A(x, E_e) \exp \left( -\frac{1}{2} \int_{-e_1}^{E_e} \frac{E - 2\eta_1}{\sqrt{-(E + e_1)(E + e_2)(E + e_3)}} dE \right),
\]

\[
A(x + \tau, E_e) = -A(x, E_e) \exp \left( -\frac{1}{2} \int_{-e_1}^{E_e} \frac{E\tau - 2\eta_3}{\sqrt{-(E + e_1)(E + e_2)(E + e_3)}} dE \right),
\]

which are elliptic integrals of the second kind.

(ii) (The case \( l_0 = 2 \), \( l_1 = 1 \), \( l_2 = 0 \), \( l_3 = 1 \)) From the formulae \( \varphi(x)^2 = (1/6) \varphi''(x) - (1/3)(e_1 e_2 + e_2 e_3 + e_1 e_3) \), \( \zeta'(x) = -\varphi(x) \), \( \zeta(x+1) = \zeta(x) = 2\eta_1 \) and \( \zeta(x+\tau) - \zeta(x) = 2\eta_3 \), the periodicity of the function \( \varphi'(x) \), we have

\[
Q_1(E) = \int_{0+e_2}^{1+e_2} Z(x, E) dx = E^2 - 5(e_1 + e_3)E - 12(e_1^2 - 3e_1 e_3 + e_3^2)
\]

\[\quad - 2\eta_1(5E - 13(e_1 + e_3)),\]

\[
Q_\tau(E) = (E^2 - 5(e_1 + e_3)E - 12(e_1^2 - 3e_1 e_3 + e_3^2))\tau - 2\eta_3(5E - 13(e_1 + e_3)).
\]

It will be shown in \( \S 3a \) that \( E = 8e_1 - e_3 \) is a solution to \( Q(E) = 0 \) such that \( A(x, E) = -A(x+1, E) = A(x+\tau, E) \). Hence,

\[
A(x+1, E_e) = -A(x, E_e) \exp \left( -\frac{1}{2} \int_{8e_1-e_3}^{E_e} \frac{Q_1(E)}{\sqrt{-Q(E)}} dE \right),
\]

\[
A(x + \tau, E_e) = A(x, E_e) \exp \left( -\frac{1}{2} \int_{8e_1-e_3}^{E_e} \frac{Q_\tau(E)}{\sqrt{-Q(E)}} dE \right),
\]

which are hyperelliptic integrals related to the genus 2 curve.

3. Invariant subspace and determinant formula

(a) Invariant subspace of elliptic functions

It is shown in the papers of Gonzalez-Lopez et al. (1994) and Takemura (2002, 2003, 2004a) that there exist finite-dimensional spaces of elliptic functions on which the action of the Hamiltonian \( H \) (see (2.1)) is well defined. In this section, we introduce them and discuss their relationship with the polynomial \( Q(E) \) defined in (2.7).
Let $\beta_0$, $\beta_1$, $\beta_2$ and $\beta_3$ be integers such that $-\sum_{i=0}^{3} \beta_i/2 \in \mathbb{Z}_{\geq 0}$ and $V_{\beta_0, \beta_1, \beta_2, \beta_3}$ be the $(d+1)$-dimensional vector space spanned by $\{ \varphi_1(x)^{\beta_1}, p_2(x)^{\beta_2}, \varphi_3(x)^{\beta_3} \varphi(x)^n \}_{n=0,\ldots,d}$, where $d = -\sum_{i=0}^{3} \beta_i/2$ and $\varphi_j(x) (i=1, 2, 3)$ are co-$\varphi$ functions (see appendix A). Let $\alpha_0$, $\alpha_1$, $\alpha_2$ and $\alpha_3$ be integers such that $\alpha_i \in \{-l_i, l_i+1\} (i=0, 1, 2, 3)$ and $\sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}$. Set

$$U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \begin{cases} V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\leq 0}; \\ \{0\} & \sum_{i=0}^{3} \alpha_i/2 = 1; \\ V_{1-\alpha_0, 1-\alpha_1, 1-\alpha_2, 1-\alpha_3} & \sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\geq 2}. \end{cases} \tag{3.1}$$

Then, it is shown that the space $U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ is invariant under the action of the Hamiltonian $H$, i.e. $H \cdot U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \subset U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ (Takemura 2003, 2004).

We define a vector space $V$ by

$$V = \left\{ \begin{array}{ll} U_{-l_0, -l_1, -l_2, -l_3} \oplus U_{-l_0, -l_1, l_2+1, l_3+1} \oplus U_{-l_0, l_1+1, -l_2, l_3+1} \oplus U_{-l_0, l_1+1, l_2+1, -l_3} & (l_0 + l_1 + l_2 + l_3 : \text{even}); \\ U_{-l_0, -l_1, -l_2, l_3+1} \oplus U_{-l_0, -l_1, l_2+1, -l_3} \oplus U_{-l_0, l_1+1, -l_2, -l_3} \oplus U_{l_0+1, -l_1, -l_2, -l_3} & (l_0 + l_1 + l_2 + l_3 : \text{odd}). \end{array} \right\} \tag{3.2}$$

Then, $H \cdot V \subset V$. Let $P(E)$ be the monic characteristic polynomial of the Hamiltonian $H$ (see (2.1)) on the space $V$, i.e. $P(E) = \det_V (E \cdot 1 - H)$. By extending the theorem in the work of Takemura (2003, theorem 3.2), we have theorem 3.1.

**Theorem 3.1.** The roots of the equation $P(E)=0$ are distinct for generic $\tau$.

We will prove this in appendix B. Note that Gesztesy & Weikard (1995, theorem 3.5 (iii)) have already stated it.

Related to the polynomials $P(E)$ and, we have the following statements.

**Proposition 3.1.** (Takemura 2003, theorem 3.8) The set of zeros of $Q(E)$ coincides with the set of zeros of $P(E)$.

**Conjecture 3.1.** (Takemura 2003, 2004b) Let $l_i \in \mathbb{Z}_{\geq 0} (i=0, 1, 2, 3)$. Then, $P(E)=Q(E)$.

**Proposition 3.2.** (Takemura 2003, propositions 3.9 and 3.10) If two of $l_i (i=0, 1, 2, 3)$ are zero or $l_0 + l_1 + l_2 + l_3 \leq 8$, then conjecture 3.1 is true.

The curve $\Gamma : y^2 = -Q(x)$ is called the spectral curve of the operators $A$ and $H$. This curve plays important roles in equations (2.8) and (2.17). By proposition 3.2, edges of the hyperelliptic curve $\Gamma$ are eigenvalues on the invariant space $V$. The genus of the curve $\Gamma$ is $g$, where $g$ is defined in paragraph 2.1. It is determined as follows.

**Proposition 3.3.** (Takemura 2004b, proposition 3.3; cf. Gesztesy & Weikard 1995, theorem 3.5) Let $k_0 = \max(l_0, l_1, l_2, l_3)$ and $k_3 = \min(l_0, l_1, l_2, l_3)$. If conjecture
3.1 is true, then the arithmetic genus of the curve $\Gamma$ is given by

$$k_0, \quad l_0 + l_1 + l_2 + l_3 \text{ is even and } k_0 + k_3 \geq \frac{1}{2}(l_0 + l_1 + l_2 + l_3);$$

$$\frac{1}{2}(l_0 + l_1 + l_2 + l_3 - 2k_3), \quad l_0 + l_1 + l_2 + l_3 \text{ is even and } k_0 + k_3 < \frac{1}{2}(l_0 + l_1 + l_2 + l_3);$$

$$k_0, \quad l_0 + l_1 + l_2 + l_3 \text{ is odd and } k_0 \geq \frac{1}{2}(l_0 + l_1 + l_2 + l_3 + 1);$$

$$\frac{1}{2}(l_0 + l_1 + l_2 + l_3 + 1), \quad l_0 + l_1 + l_2 + l_3 \text{ is odd and } k_0 < \frac{1}{2}(l_0 + l_1 + l_2 + l_3 + 1).$$

(3.3)

**Corollary 3.1.** (Takemura 2004b, corollary 3.4) If two of $l_i$ ($i=0,1,2,3$) are zero, then the arithmetic genus of the curve $\Gamma$ is $\max(l_0, l_1, l_2, l_3)$.

**Remark 3.1.** A referee pointed out that conjecture 3.1 and proposition 3.3 become theorems, although the author could not comprehend it as in writing. The referee explained as follows: the corresponding spectral curve is an exceptional hyperelliptic tangential cover of type $(\mu_0, \mu_1, \mu_2, \mu_3)$ given in terms of $(l_0, l_1, l_2, l_3)$ and arithmetic genus is equal to $g = (1 + \mu_0 + \mu_1 + \mu_2 + \mu_3)/2$ (Treibich & Verdier 1992). In particular, the genus $g$ is expressed as (3.3) which was found independently and written down for the first time in the work of Gesztesy & Weikard (1995, theorem 3.5), and it is established by them that for generic $\tau$ the spectral curve is smooth. It follows that $\deg P(E) = 2g + 1 = \deg Q(E)$ and $P(E) = Q(E)$.

**Example 3.1.** (i) (The case $l_0=1, l_1=1, l_2=1, l_3=0$) The space $V$ defined in (3.2) is decomposed as

$$V = U_{-1,1,0,0} \oplus U_{-1,0,1,0} \oplus U_{-1,0,0,1};$$

(4.4)

where

$$U_{-1,1,0,0} = C \varphi_1(x), \quad U_{-1,0,1,0} = C \varphi_2(x), \quad U_{-1,0,0,1} = C \varphi_3(x).$$

(3.5)

The spaces $V, U_{-1,1,0,0}, U_{-1,0,1,0}$ and $U_{-1,0,0,1}$ are invariant under the action of the Hamiltonian $H$ and $\dim V = 3$. The characteristic polynomial of the Hamiltonian $H$ on the space $V$ is

$$P(E) = (E + e_1)(E + e_2)(E + e_3).$$

(3.6)

In this case, we have $Q(E) = P(E)$, which reproduces proposition 3.2.

An eigenvalue of the operator $H$ on the space $U_{-1,1,0,0}$ is $-e_1$ and $A(x, -e_1) = \varphi_1(x)$. It follows that $A(x + 1, -e_1) = A(x, -e_1)$ and $A(x + \tau, -e_1) = -A(x, -e_1)$ (appendix B).

(ii) (The case $l_0=2, l_1=1, l_2=0, l_3=1$) The space $V$ is decomposed as

$$V = U_{-2,-1,0,-1} \oplus U_{-2,2,1,-1} \oplus U_{-2,-1,1,2};$$

(3.7)
on the Heun equation

\[ U_{-2,-1,0,-1} = \mathbb{C} \frac{1}{\varphi_1(x) \varphi_3(x)} \Theta \mathbb{C} \frac{\varphi(x)}{\varphi_1(x) \varphi_3(x)} \Theta \mathbb{C} \frac{\varphi(x)^2}{\varphi_1(x) \varphi_3(x)} \]

\[ U_{-2,2,1,-1} = \mathbb{C} \frac{\varphi_1(x)^2 \varphi_2(x)}{\varphi_3(x)} \], \quad U_{-2,-1,1,2} = \mathbb{C} \frac{\varphi_3(x)^2 \varphi_2(x)}{\varphi_1(x)} . \]

(3.8)

We have \( \dim \mathbf{V} = 5 \). The characteristic polynomial of the Hamiltonian \( H \) on the space \( \mathbf{V} \) is

\[ P(E) = (E + e_1 - 8e_3)(E + e_3 - 8e_1)\{E^3 - 3(e_1 + e_3)E^2 \]

\[ + (-29e_1^2 + 22e_1e_3 - 29e_3^2)E + (-97e_1^3 + 77e_1^2e_3 + 77e_1e_3^2 - 97e_3^3)\} = Q(E). \]

(3.9)

An eigenvalue of the operator \( H \) on the space \( U_{-2,-1,1,2} \) is \( 8e_1 - e_3 \) and \( A(x, 8e_1 - e_3) = 3(\varphi_3(x)^2 \varphi_2(x)/\varphi_1(x)) \). It follows that \( A(x + 1, 8e_1 - e_3) = -A(x, 8e_1 - e_3) \) and \( A(x + \tau, 8e_1 - e_3) = A(x, 8e_1 - e_3) \).

(b) Monodromy matrix

Let us calculate the monodromy matrix. If \( Q(E_0) \neq 0 \), then it is shown that the functions \( A_1(x, E_0)(=A(x, E_0)) \) and \( A_2(x, E_0)(=A(-x, E_0)) \) are linearly independent. From theorem 2.2, monodromy matrices on the basis \( (A_1(x, E_0), A_2(x, E_0)) \) are obtained as follows:

\[ (A_1(x + 1, E_0), A_2(x + 1, E_0)) \]

\[ = (A_1(x, E_0), A_2(x, E_0)) \begin{pmatrix} \exp \left(-\frac{1}{2} \int_{E_0}^{E} \frac{Q_1(E)}{\sqrt{-Q(E)}} dE \right) & 0 \\ 0 & \exp \left(\frac{1}{2} \int_{E_0}^{E} \frac{Q_1(E)}{\sqrt{-Q(E)}} dE \right) \end{pmatrix} \]

(3.10)

\[ (A_1(x + \tau, E_0), A_2(x + \tau, E_0)) \]

\[ = (A_1(x, E_0), A_2(x, E_0)) \begin{pmatrix} \exp \left(-\frac{1}{2} \int_{E_0}^{E} \frac{Q_1(E)}{\sqrt{-Q(E)}} dE \right) & 0 \\ 0 & \exp \left(\frac{1}{2} \int_{E_0}^{E} \frac{Q_1(E)}{\sqrt{-Q(E)}} dE \right) \end{pmatrix} \]

(3.11)

Let us consider the case \( Q(E_0) = 0 \). Let \( E_0 \) be a solution to \( Q(E) = 0 \). Then, we have \( P(E_0) = 0 \), and the functions \( A(x, E_0) \) and \( A(-x, E_0) \) are linearly dependent. In this case, we have \( A(x, E_0) = \sqrt{\Xi(x, E_0)} \), and another solution to (2.2) can be
derived as $\sqrt{\Xi(x, E_0)} \int dx/\Xi(x, E_0)$. Set $A_1(x, E_0) = \sqrt{\Xi(x, E_0)}$ and $A_2(x, E_0) = \sqrt{\Xi(x, E_0)} \int dx/\Xi(x, E_0)$, then we have $A_1(x + 1, E_0) = A_1(x, E_0)$ or $A_1(x + 1, E_0) = -A_1(x, E_0)$ (respectively, $A_1(x + \tau, E_0) = A_1(x, E_0)$ or $A_1(x + \tau, E_0) = -A_1(x, E_0)$), so that we write $A_1(x + 1, E_0) = (-1)^n A_1(x, E_0)$ (respectively $A_1(x + \tau, E_0) = (-1)^n A_1(x, E_0)$). By the way, it is shown by Takemura (2004b, §3.3) that

$$
\int_{0+\varepsilon}^{P+\varepsilon} \frac{\Xi(x, E)}{\sqrt{-Q(E)}} dx + 2 \int_{0+\varepsilon}^{P+\varepsilon} \frac{d}{dE} \left( \frac{\sqrt{-Q(E)}}{\Xi(x, E)} \right) dx = 0, \tag{3.12}
$$

for $P \in \mathbb{Z} \oplus \mathbb{Z} \tau$. By the limit $E \to E_0$, we have $\int_0^1 dx/\Xi(x, E_0) = Q_1(E)/(d/dE)Q(E)|_{E \to E_0}$ and $\int_0^1 dx/\Xi(x, E_0) = Q_1(E)/(d/dE)Q(E)|_{E \to E_0}$. Hence, monodromy matrices on the basis $(A_1(x, E_0), A_2(x, E_0))$ are obtained as follows:

$$(A_1(x + 1, E_0), A_2(x + 1, E_0))$$

$$= (-1)^n (A_1(x, E_0), A_2(x, E_0)) \begin{pmatrix} 1 & \frac{Q_1(E)}{dE} \bigg|_{E \to E_0} \\ 0 & 1 \end{pmatrix}, \tag{3.13}$$

$$(A_1(x + \tau, E_0), A_2(x + \tau, E_0))$$

$$= (-1)^n (A_1(x, E_0), A_2(x, E_0)) \begin{pmatrix} 1 & \frac{Q_1(E)}{dE} \bigg|_{E \to E_0} \\ 0 & 1 \end{pmatrix}. \tag{3.14}$$

**Example 3.2.** (i) (The case $l_0 = 1, l_1 = l_2 = l_3 = 0$) Set $E_0 = -e_1$. Then, $E = E_0$ is a solution to $Q(E) = 0$ and the function $A(x, -e_1)$ is equal to $\varphi_1(x)$. By the relation $\varphi_1(x)^2 = \varphi(x) - e_1$, it follows that the function $\varphi_1(x) \int dx/(\varphi(x) - e_1)$ which is equal to $-\varphi_1(x)\zeta(x + 1/2) + e_1x/((e_2 - e_1)(e_3 - e_1))$ is also a solution to the differential equation $(H - (-e_1))f(x) = 0$. Set $(A_1(x), A_2(x)) = (\varphi_1(x), \varphi_1(x) \int dx/(\varphi(x) - e_1))$. From the periodicity of $\varphi_1(x)$ (appendix B), we have $A_1(x + 1) = A_1(x)$ and $A_1(x + \tau) = -A_1(x)$. Since $Q_1(E) = E - 2a_1$ and $Q_1(E) = E\tau - 2\eta_3$, we have

$$\left. \frac{Q_1(E)}{dE} \right|_{E \to -e_1} = \frac{-e_1 - 2\eta_1}{(e_2 - e_1)(e_3 - e_1)}, \left. \frac{Q_1(E)}{dE} \right|_{E \to -e_1} = \frac{-e_1 \tau - 2\eta_3}{(e_2 - e_1)(e_3 - e_1)}. \tag{3.15}$$

Hence,

$$(A_1(x + 1), A_2(x + 1)) = (A_1(x), A_2(x)) \begin{pmatrix} 1 & \frac{-e_1 - 2\eta_1}{(e_2 - e_1)(e_3 - e_1)} \\ 0 & 1 \end{pmatrix}. \tag{3.16}$$

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(iii) (The case $l_0=2, l_1=1, l_2=0, l_3=1$) Set $E_0 = 8e_1 - e_3$. Then, $E = E_0$ is a solution to $Q(E) = 0$ and the function $A(x, 8e_1 - e_3)$ is equal to $3 \varphi_2(x)(\varphi(x) - e_3)/\varphi_1(x)$. Since $\Xi(x, 8e_1 - e_3) = A(x, 8e_1 - e_3)^2$, the function $A(x, 8e_1 - e_3) \int (dx/\Xi(x, 8e_1 - e_3)$ which is equal to $\{ \varphi_2(x)(\varphi(x) - e_3)/(3 \varphi_1(x)) \} \int dx(\varphi(x) - e_3)/((\varphi(x) - e_3)^2$ $(\varphi(x) - e_2)$ is also a solution to the differential equation $(H - (8e_1 - e_3))f(x) = 0$. Set $(A_1(x), A_2(x)) = (3 \varphi_2(x)(\varphi(x) - e_3)/\varphi_1(x)$, $\{ \varphi_2(x)(\varphi(x) - e_3)/(3 \varphi_1(x)) \} \int dx(\varphi(x) - e_3)/((\varphi(x) - e_3)^2 (\varphi(x) - e_2))$. Then,

\[
\frac{Q_1(E)}{\frac{d}{dE} Q(E)} \bigg|_{E=8e_1-e_3} = \frac{-4e_1^2 + 5e_1e_3 + 2e_3^2 + 6\eta_1(3e_1 - 2e_3)}{27(e_1 - e_3)(e_1 + 2e_3)^3},
\]

\[
\frac{Q_\tau(E)}{\frac{d}{dE} Q(E)} \bigg|_{E=8e_1-e_3} = \frac{-4e_1^2 + 5e_1e_3 + 2e_3^2 \tau + 6\eta_3(3e_1 - 2e_3)}{27(e_1 - e_3)(e_1 + 2e_3)^3}.
\]

Hence,

\[
(A_1(x + 1), A_2(x + 1)) = (A_1(x), A_2(x)) \begin{pmatrix} -1 & -\frac{-4e_1^2 + 5e_1e_3 + 2e_3^2 + 6\eta_1(3e_1 - 2e_3)}{27(e_1 - e_3)(e_1 + 2e_3)^3} \\ 0 & -1 \end{pmatrix},
\]

\[
(A_1(x + \tau), A_2(x + \tau)) = (A_1(x), A_2(x)) \begin{pmatrix} 1 & \frac{-4e_1^2 + 5e_1e_3 + 2e_3^2 \tau + 6\eta_3(3e_1 - 2e_3)}{27(e_1 - e_3)(e_1 + 2e_3)^3} \\ 0 & 1 \end{pmatrix}.
\]

(c) **Determinant formula for commuting operator**

In this section, a determinant formula for the commuting operator $A$ is introduced. We present a conjectural determinant formula which is proved for some cases.

**Conjecture 3.2.** (Takemura 2004b) Let $l_i\ (i=0, 1, 2, 3)$ be non-negative integers, $V$ be the invariant subspace defined in §3a and $f_1(x), ..., f_m(x)$ be a
basis for the space $V$. Then,

$$
A = A_0 \begin{pmatrix}
  f_1(x) & \ldots & f_m(x) & 1 \\
  \frac{d}{dx} f_1(x) & \ldots & \frac{d}{dx} f_m(x) & \frac{d}{dx} \\
  (\frac{d}{dx})^2 f_1(x) & \ldots & (\frac{d}{dx})^2 f_m(x) & (\frac{d}{dx})^2 \\
  \vdots & \vdots & \vdots & \vdots \\
  (\frac{d}{dx})^m f_1(x) & \ldots & (\frac{d}{dx})^m f_m(x) & (\frac{d}{dx})^m
\end{pmatrix},
$$

(3.22)

for some non-zero constant $A_0$. Here, the determinant for the matrix

$$
\begin{pmatrix}
  a_{1,1} & \ldots & a_{1,n} \\
  \vdots & \ddots & \vdots \\
  a_{n,1} & \ldots & a_{n,n}
\end{pmatrix},
$$

is defined by $\sum_{\sigma \in S_n} \text{sgn}(\sigma)a_{\sigma(1),1}\ldots a_{\sigma(n),n}$.

**Proposition 3.4.** (Takemura 2004b, proposition 3.6) If two of $l_i$ ($i=0, 1, 2, 3$) are zero, conjecture 3.2 is true.

**Example 3.3.** (i) (The case $l_0=1$, $l_1=l_2=l_3=0$) The commuting operator $A$ is written as (2.18). From proposition 3.4, the operator $A$ admits a determinant formula. More precisely, we have

$$(e_1 - e_2)(e_2 - e_3)(e_3 - e_1)A =
\begin{pmatrix}
  \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & 1 \\
  \varphi'_1(x) & \varphi'_2(x) & \varphi'_3(x) & \frac{d}{dx} \\
  \varphi''_1(x) & \varphi''_2(x) & \varphi''_3(x) & (\frac{d}{dx})^2 \\
  \varphi'''_1(x) & \varphi'''_2(x) & \varphi'''_3(x) & (\frac{d}{dx})^3
\end{pmatrix}.
$$

(3.23)

(ii) (The case $l_0=2$, $l_1=1$, $l_2=0$, $l_3=1$) The commuting operator $A$ is written as (2.19). Let $f_1(x) = \varphi_1(x)^2 \varphi_2(x)/\varphi_3(x)$, $f_2(x) = \varphi_3(x)^2 \varphi_2(x)/\varphi_1(x)$, $f_3(x) = 1/(\varphi_1(x)\varphi_3(x))$, $f_4(x) = \varphi(x)/(\varphi_1(x)\varphi_3(x))$ and $f_5(x) = \varphi(x)^2/(\varphi_1(x)\varphi_3(x))$. 

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By lengthy calculation, it is seen that
\[ 432(e_1 - e_2)^3(e_2 - e_3)^3(e_3 - e_1)A \]
\[ \begin{pmatrix}
  f_1(x) & \ldots & f_5(x) & 1 \\
  \frac{d}{dx}f_1(x) & \ldots & \frac{d}{dx}f_5(x) & \frac{d}{dx} \\
  \left(\frac{d}{dx}\right)^2f_1(x) & \ldots & \left(\frac{d}{dx}\right)^2f_5(x) & \left(\frac{d}{dx}\right)^2 \\
  \vdots & \vdots & \vdots & \vdots \\
  \left(\frac{d}{dx}\right)^5f_1(x) & \ldots & \left(\frac{d}{dx}\right)^5f_5(x) & \left(\frac{d}{dx}\right)^5
\end{pmatrix}.
\]
(3.24)

Therefore, conjecture 3.2 is true for this case.

4. Heun equation and Inozemtsev model

In this section, the relationship between the Heun equation and the Inozemtsev model is reviewed. Results for the BC1 Inozemtsev model are translated to ones for the Heun equation.

Let \( f(x) \) be an eigenfunction of the Hamiltonian \( H \) with an eigenvalue \( E \), i.e.
\[ -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1) \phi(x + \omega_i) - E f(x) = 0. \]
\[ (4.1) \]
Set
\[ t = \frac{e_1 - e_3}{e_2 - e_3}, \tilde{\phi}(w) = w^{l_0+1/2}(w-1)^{l_1+1/2}(w-t)^{l_2+1/2}. \]
\[ (4.2) \]
Under the transformation
\[ w = \frac{e_1 - e_3}{\phi(x) - e_3}, \]
\[ (4.3) \]
equation (4.1) is equivalent to
\[ \left( \left( \frac{d}{dw}\right)^2 + \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t} \right) \frac{d}{dw} + \frac{\alpha \beta w - q}{w(w-1)(w-t)} \right) \tilde{f}(w) = 0, \]
\[ (4.4) \]
where \( \tilde{f}((e_1 - e_3)/\phi(x) - e_3))\tilde{\phi}((e_1 - e_3)/\phi(x) - e_3)) = f(x), \alpha = (l_0 + l_1 + l_2 + l_3 + 4)/2, \beta = (l_0 + l_1 + l_2 - l_3 + 3)/2, \gamma = l_0 + (3/2), \delta = l_1 + (3/2),
\[ \epsilon = l_2 + (3/2), q = -(t/4)(E/(e_1 - e_3) + c_0) \text{ and } c_0 = ((t+1)/(3t)) \sum_{i=0}^{3} l_i(l_i+1) 
- (1/t)(l_0 + l_2 + 2)^2 - (l_0 + l_1 + 2)^2. \] (For details see Takemura (2004a).) Note that the relation \( \gamma + \delta + \epsilon = \alpha + \beta + 1 \) is satisfied.

Equation (4.4) with the condition \( \gamma + \delta + \epsilon = \alpha + \beta + 1 \) is called the Heun equation. The Heun equation is the standard canonical form of a Fuchsian equation with four singularities. By the aforementioned correspondence between

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functions \( x \) and \( w \), the parameters \((l_0, l_1, l_2, l_3, E, \tau)\) correspond essentially one-to-one to the parameters \((\alpha, \beta, \gamma, \delta, \varepsilon, q, t)\) with the condition \(\gamma + \delta + \varepsilon = \alpha + \beta + 1\) (Ochiai et al. 1994; Rouveaux 1995; Takemura 2004b).

**Remark 4.1.** An expression of the Heun equation in terms of elliptic functions as in (4.1) was basically established by Darboux (1882) more than 100 years ago.

There are other possible choices for the function \(\Phi(w)\) of (4.2) and the relationship (4.3) between \(x\) and \(w\). For example, Smirnov (2002) chose the correspondence defined by \( t = (e_3 - e_1)/(e_2 - e_1) \), \( \Phi(w) = w^{-l_2/2}(w-1)^{-l_2/2}(w-t)^{-l_3/2} \) and \( w = (\varphi(x) - e_1)/(e_2 - e_1) \).

The transformation \( w = (e_1 - e_3)/(\varphi(x) - e_3) \) produces the following correspondence:

\[
\begin{align*}
x & \quad 0 \quad \frac{1}{2} \quad \frac{\tau + 1}{2} \quad \frac{\tau}{2}, \\
w & \quad 0 \quad 1 \quad t \quad \infty.
\end{align*}
\]

Other mathematical objects and concepts discussed in this paper are transformed under equation (4.3) as follows.

— In §§2 and 3, we considered the finite-gap property for the Inozemtsev model for the case \(l_0, l_1, l_2, l_3 \in \mathbb{Z}\). The condition \(l_0, l_1, l_2, l_3 \in \mathbb{Z}\) is transformed into the condition \(\gamma, \delta, \varepsilon, \alpha - \beta \in (1/2) + \mathbb{Z}\).

— Some statements in this paper have assumptions such as \(l_i = 0\). If the assumption \(l_i = 0\) is changed to \(l_i = -1\), all the applicable statements in this paper remain valid. The condition \(l_0 = 0\) or \(-1\) (respectively \(l_1 = 0\) or \(-1\), \(l_2 = 0\) or \(-1\), \(l_3 = 0\) or \(-1\)) is transformed under (4.3) into the condition \(\gamma = 1/2\) or \(3/2\) (respectively \(\delta = 1/2\) or \(3/2\), \(\varepsilon = 1/2\) or \(3/2\), \(\alpha - \beta = \pm (1/2)\)).

— By transforming the functions (e.g. \(A(x, E)\), \(\Xi(x, E)\)) and the formulae (e.g. (2.7) and (2.8)) in this paper into a form appropriate for the Heun equation (4.4), we recover the corresponding formulae of Smirnov (2002).

— Let \(C_1\) be the cycle enclosing two points \(0, 1\) in \(w\). The path from \(\varepsilon'\) to \(\varepsilon' + 1\) (\(|\varepsilon'|\) sufficiently small) in \(x\) corresponds to the cycle \(C_1\) in \(w = (e_1 - e_3)/(\varphi(x) - e_3)\) (figure 1).

— Let \(C_\tau\) be the cycle enclosing two points \(1, t\) in \(w\). The path from \(\varepsilon'\) to \(\varepsilon' + \tau\) in \(x\) corresponds to the cycle \(C_\tau\) in \(w\).

Assume \(\gamma, \delta, \varepsilon, \alpha - \beta \in (1/2) + \mathbb{Z}\) and \(\gamma + \delta + \varepsilon = \alpha + \beta + 1\). The monodromy on the cycle \(C_1\) (respectively \(C_\tau\)) is written by a hyperelliptic integral in \(q\) (\(q\) is the parameter given in (4.4)). Now we explain it concretely. Let \(\Lambda_1(w, q)\) and \(\Lambda_2(w, q)\) be functions defined by relations \(\Lambda_1((e_1 - e_3)/\varphi(x) - e_3), q) = A(x, (e_3 - e_2)((4q/t) + c_0))/\Phi((e_1 - e_3)/\varphi(x) - e_3)\) and \(\Lambda_2((e_1 - e_3)/\varphi(x) - e_3), q) = A(-x, (e_3 - e_2)/(4q/t) + c_0))/\Phi((e_1 - e_3)/\varphi(x) - e_3)\)
where the initial point and (2.22), it follows that \( F \) where the signs in (4.8) depend on the initial point. Phil. Trans. R. Soc. A (2008)

\[
((4q/t + c_0)) \Phi((e_1 - c_3)/(e_2 - e_3)) \text{, where } t = (e_1 - e_3)/(e_2 - e_3) \text{ and } c_0 = ((t + 1)/(3t)) \sum_{l=0}^{3} l_1(l_1 + 1) - (1/t)(l_0 + l_2 + 2)^2 - (l_0 + l_1 + 2)^2. \text{ Then, the functions } \tilde{A}_1(w, q) \text{ and } \tilde{A}_2(w, q) \text{ satisfy (4.4). Let } \tilde{Q}(q), \tilde{Q}_1(q) \text{ and } \tilde{Q}_r(q) \text{ be monic polynomials defined by } \tilde{Q}(q) = (4(e_3 - e_2))^{-2(2s+1)} \tilde{Q}_1((e_3 - e_2)((4q/t + c_0))) \text{ and } \tilde{Q}_r(q) = (4(e_3 - e_2))^{-2} \tilde{Q}((e_3 - e_2)((4q/t + c_0))) \text{ and } \tilde{Q}_r(E) = (4(e_3 - e_2))^{-2} \tilde{Q}((e_3 - e_2)((4q/t + c_0)) / \tau.

Fix \( q_* \) and assume \( \tilde{Q}(q_*) \neq 0. \) Let \( \tilde{A}_1^{C_i}(w, q_*) \) and \( \tilde{A}_2^{C_i}(w, q_*) \) (respectively, \( \tilde{A}_1^{C_r}(w, q_*) \) and \( \tilde{A}_2^{C_r}(w, q_*) \)) be functions obtained by analytic continuation in \( w \) along the cycle \( C_1 \) (respectively, \( C_r \)) from the functions \( \tilde{A}_1(w, q_*) \) and \( \tilde{A}_2(w, q_*) \), and define the monodromy matrix \( M^{C_1} \) (respectively, \( M^{C_r} \)) by \( (\tilde{A}_1^{C_1}(w, q_*), \tilde{A}_2^{C_1}(w, q_*)) = (\tilde{A}_1(w, q_*), \tilde{A}_2(w, q_*)) \) \( M^{C_1} \) (respectively, \( (\tilde{A}_1^{C_r}(w, q_*), \tilde{A}_2^{C_r}(w, q_*)) = (\tilde{A}_1(w, q_*), \tilde{A}_2(w, q_*)) \) \( M^{C_r} \)). Set

\[
M_1 = \exp \left( \sqrt{(e_3 - e_2)} \int_{q_0}^{q_*} \frac{\tilde{Q}_1(q)}{\sqrt{\tilde{Q}(q)}} \, dq \right), \tag{4.6}
\]

\[
M_r = \exp \left( \sqrt{(e_3 - e_2)} \int_{q_0}^{q_*} \frac{\tilde{Q}_r(q)}{\sqrt{\tilde{Q}(q)}} \, dq \right), \tag{4.7}
\]

where the initial point \( q_0 \) is the value which satisfies \( \tilde{Q}(q_0) = 0. \) From formulae (2.20) and (2.22), it follows that

\[
M^{C_1} = \pm \left( \begin{array}{cc}
1/M_1 & 0 \\
0 & M_1 \\
\end{array} \right), \quad M^{C_r} = \pm \left( \begin{array}{cc}
1/M_r & 0 \\
0 & M_r \\
\end{array} \right), \tag{4.8}
\]

where the signs in (4.8) depend on the initial point \( q_0 \) and the branching of the function \( \tilde{\Phi}(w) \). Hence, the monodromies along the cycles \( C_1 \) and \( C_r \) are expressed by hyperelliptic integrals as in (4.6)–(4.8). Now we calculate the monodromies explicitly by the following examples.

**Example 4.1.** (i) (The case \( l_0 = 1, l_1 = l_2 = l_3 = 0 \)) The functions \( \tilde{A}_1(w, q), \tilde{A}_2(w, q), \tilde{Q}(q), \tilde{Q}_1(q) \) and \( \tilde{Q}_r(q) \) are calculated as follows:

\[
\tilde{Q}(q) = \left( q - 2 - \frac{9t}{4} \right) \left( q - \frac{9}{4} - 2t \right)(q - 2 - 2t),
\]

\[
\tilde{Q}_1(q) = q - \frac{25}{12} (1 + t) - \frac{\eta_1}{2(e_3 - e_2)},
\]

\[
\tilde{Q}_r(q) = q - \frac{25}{12} (1 + t) - \frac{\eta_3}{2(e_3 - e_2)},
\]

\[
\tilde{A}_1(w, q) = 4(e_3 - e_2)\phi(w)\xi(w, q)\exp \left( \int \frac{-\sqrt{Q(q)} \, dw}{\xi(w, q)\sqrt{w(w-1)(w-t)}} \right), \tag{4.9}
\]

\[
\tilde{A}_2(w, q) = 4(e_3 - e_2)\phi(w)\xi(w, q)\exp \left( \int \frac{-\sqrt{Q(q)} \, dw}{\xi(w, q)\sqrt{w(w-1)(w-t)}} \right),
\]

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where \( \phi(w) = w(w-1)^{1/2}(w-t)^{1/2} \), \( \xi(w,q) = q-t/(4w)-2(1+t) \) and \( t = (e_1 - e_3)/(e_2 - e_3) \).

Set \( q_0 = 2 + (9/4)t \), which corresponds to \( E_0 = -e_1 \). Then, the monodromy matrices are determined as

\[
M^{c_1} = -\begin{pmatrix} 1 & 0 \\ 0 & M_1 \end{pmatrix}, \quad M^{c_r} = -\begin{pmatrix} 1 & 0 \\ 0 & M_r \end{pmatrix},
\]

where \( M_1 \) and \( M_r \) are determined by (4.6) and (4.7). Since \( \deg_q \tilde{Q}(q) = 3 \) and \( \deg_q \tilde{Q}_r(q) = 2 \), the monodromies are expressed by elliptic integrals of the second kind.

We can choose another transformation. Set \( l_0 = -2, l_1 = l_2 = l_3 = -1 \), then the operator (4.1) is not changed. On this case, the functions \( \tilde{A}_1(w,q), \tilde{A}_2(w,q), \tilde{Q}(q), \tilde{Q}_1(q) \) and \( \tilde{Q}_r(q) \) are

\[
\tilde{Q}(q) = q\left(q - \frac{1}{4}\right)\left(q - \frac{t}{4}\right), \quad \tilde{Q}_1(q) = q - \frac{1 + t}{12} - \frac{\eta_1}{2(e_3 - e_2)},
\]

\[
\tilde{Q}_r(q) = q - \frac{1 + t}{12} - \frac{\eta_3}{2\tau(e_3 - e_2)},
\]

\[
\tilde{A}_1(w,q) = 4(e_3 - e_2)\phi(w)\xi(w,q)\exp\left(\int \frac{-\sqrt{-\tilde{Q}(q)}dw}{\xi(w,q)sqrt{(w-1)(w-t)}}\right),
\]

\[
\tilde{A}_2(w,q) = 4(e_3 - e_2)\phi(w)\xi(w,q)\exp\left(\int \frac{\sqrt{-\tilde{Q}(q)}dw}{\xi(w,q)sqrt{(w-1)(w-t)}}\right),
\]

where \( \phi(w) = 1/\sqrt{w} \) and \( \xi(w,q) = q - (t/(4w)) \).

Set \( q_0 = t/4 \). Then, the monodromy matrices are determined as

\[
M^{c_1} = -\begin{pmatrix} 1 & 0 \\ 0 & M_1 \end{pmatrix}, \quad M^{c_r} = -\begin{pmatrix} 1 & 0 \\ 0 & M_r \end{pmatrix},
\]

where \( M_1 \) and \( M_r \) are determined by (4.6) and (4.7).
(ii) (The case \( l_0=2, l_1=1, l_2=0, l_3=1 \)) The functions \( \tilde{A}_1(w, q), \tilde{A}_2(w, q), \tilde{Q}(q), \tilde{Q}_1(q) \) and \( \tilde{Q}_\tau(q) \) are calculated as follows:

\[
\begin{align*}
\tilde{Q}(q) &= \left( q - \frac{15}{4} - \frac{25}{4} t \right) \left( q - \frac{15}{4} - 4t \right) \left( q^3 - (10 + 16t)q^2 + (11 + 21t)(3 + 4t)q - \left( 36 + \frac{33887}{192} t + \frac{54433}{192} t^2 + \frac{27551}{192} t^3 \right) \right), \\
\tilde{Q}_1(q) &= q^2 - \left( \frac{43}{6} + \frac{125}{12} t + \frac{5\eta_1}{2(e_3 - e_2)} \right) q + \left( \frac{51}{4} + \frac{449}{12} t + \frac{157}{6} t^2 + \frac{(9 + 13t)\eta_1}{e_3 - e_2} \right), \\
\tilde{Q}_\tau(q) &= q^2 - \left( \frac{43}{6} + \frac{125}{12} t + \frac{5\eta_3}{2(e_3 - e_2)} \right) q + \left( \frac{51}{4} + \frac{449}{12} t + \frac{157}{6} t^2 + \frac{(9 + 13t)\eta_3}{\tau(e_3 - e_2)} \right), \\
\tilde{A}_1(w, q) &= 16(e_3 - e_2)^2 \phi(w) \xi(w, q) \exp \left( \int \frac{-\tilde{Q}(q) \mathrm{d}w}{\xi(w, q) \sqrt{w(w-1)(w-t)}} \right), \\
\tilde{A}_2(w, q) &= 16(e_3 - e_2)^2 \phi(w) \xi(w, q) \exp \left( \int \frac{\sqrt{-\tilde{Q}(q) \mathrm{d}w}}{\xi(w, q) \sqrt{w(w-1)(w-t)}} \right),
\end{align*}
\]

(4.13)

where \( \phi(w) = w^{3/2}(w-1)(w-t)^{1/2} \) and \( \xi(w, q) = q^2 - \{ (w^2 + 27w - 27)/(w-1) + (40w^2 - 38w - 3)t/(w(w-1)) \} q/4 + (1/16) \{ 15(w^2 + 12w - 12)/(w-1) + 2t(8w^3 + 276w^2 - 261w - 18)/(w(w-1)) + (384w^3 - 352w^2 - 48w - 9)t^2/(w^2(w-1)) \} \).

Set \( q_0 = (15/4) + 4t \), which corresponds to \( E_0 = 8e_1 - e_3 \). Then, the monodromy matrices are determined as

\[
M^{C_1} = \begin{pmatrix} 1 & 0 \\ \frac{1}{M_1} & 0 \end{pmatrix}, \quad M^{C_\tau} = \begin{pmatrix} 1 & 0 \\ 0 & M_\tau \end{pmatrix},
\]

(4.14)

where \( M_1 \) and \( M_\tau \) are determined by (4.6) and (4.7). Hence, the monodromies along \( C_1 \) and \( C_\tau \) are expressed by hyperelliptic integrals of genus 2.

For the case \( \tilde{Q}(q_\tau) = 0 \), we can also obtain monodromies along \( C_1 \) and \( C_\tau \) similarly by transforming equations (3.16) and (3.17) with variables \( w \) and \( q \).

5. Concluding remarks

By continuing arguments in §2, we can obtain results related with Bethe ansatz (for details see Takemura (2000, 2003)).

From a physics viewpoint, obtaining square-integrable eigenfunctions of the Hamiltonian and their corresponding eigenvalues is often desired. In §§2 and 3, eigenfunctions of the Hamiltonian \( H \) without physical boundary conditions are constructed, whereas physical boundary conditions are included in Takemura (2004a,b). It is shown in Takemura (2004a) that an eigenvalue satisfying a physical boundary condition is holomorphic in \( \tau \) if \( p = \exp(\pi \sqrt{-1}\tau) \in \mathbb{R} \) and

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where yields the relations

|p|<1. In the work of Takemura (2004b), a sufficient condition for holomorphy of the physical eigenvalues in $p \in \mathbb{C}$ and |p|<1 is provided. This condition will involve the radius of convergence of the eigenvalue as a power series in p, which can be calculated by the method of perturbation.

Note added in 2006. The Heun equation was also investigated from other viewpoints, Hermite–Krichever ansatz (Takemura 2005) and Darboux transformation (Takemura 2006b). Consequently, conjectures 3.1 and 3.2 were proved. Further results on §5 were obtained in the work of Takemura (2006a).

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Appendix A. Elliptic functions

This section presents the definitions and formulae for elliptic functions.

Let $\omega_1$ and $\omega_3$ be complex numbers such that the value $\omega_3/\omega_1$ is an element of the upper half-plane. In this paper, we set $\omega_1=1/2$ and $\omega_3=\tau/2$.

The Weierstrass $\wp$-function, the Weierstrass ($\sigma$-function and the Weierstrass $\zeta$-function with periods $2\omega_1$, $2\omega_3$ are defined as follows:

$$
\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \frac{1}{(z-2m\omega_1-2n\omega_3)^2} - \frac{1}{(2m\omega_1+2n\omega_3)^2},
$$

$$
\sigma(z) = z \prod_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left(1 - \frac{z^2}{2m\omega_1 + 2n\omega_3}\right) \exp\left(\frac{z^2}{2(2m\omega_1 + 2n\omega_3)^2}\right),
$$

$$
\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.
$$

(A1)

Setting $\omega_2 = -\omega_1 - \omega_3$ and

$$
e_i = \wp(\omega_i), \eta_i = \zeta(\omega_i) \quad (i = 1, 2, 3),
$$

(A2)

yields the relations

$$
e_1 + e_2 + e_3 = \eta_1 + \eta_2 + \eta_3 = 0,
$$

$$
\wp(z) = -\zeta'(x), (\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),
$$

$$
\wp(z + 2\omega_j) = \wp(z), \zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j \quad (j = 1, 2, 3),
$$

$$
\wp(z + \omega_i) = e_i + \frac{(e_i - e_i')(e_i - e_i'')}{\wp(z) - e_i} \quad (i = 1, 2, 3),
$$

(A3)

where $i', i'' \in \{1, 2, 3\}$ with $i' < i''$, $i \neq i'$ and $i \neq i''$. The co-$\sigma$ functions $\sigma_i(z) (i=1, 2, 3)$ and co-$\wp$ functions $\wp_i(z) (i=1, 2, 3)$ are defined by

$$
\sigma_i(z) = \exp(-\eta_i z) \frac{\sigma(z + \omega_i)}{\sigma(\omega_i)}, \wp_i(z) = \frac{\sigma_i(z)}{\sigma(z)}.
$$

(A4)
and satisfy
\[ \varphi_i(z)^2 = \varphi(z) - e_i \quad (i = 1, 2, 3), \]
\[ \varphi_1(z) = \varphi_1(z + 2\omega_1) = -\varphi_1(z + 2\omega_3), \quad \varphi_1'(z) = -2\varphi_2(z)\varphi_3(z), \]
\[ \varphi_2(z) = -\varphi_2(z + 2\omega_1) = -\varphi_2(z + 2\omega_3), \quad \varphi_2'(z) = -2\varphi_1(z)\varphi_3(z), \]
\[ \varphi_3(z) = -\varphi_3(z + 2\omega_1) = \varphi_3(z + 2\omega_3), \quad \varphi_3'(z) = -2\varphi_1(z)\varphi_2(z). \] (A 5)

\section*{Appendix B. Proof of theorem 3.1}

\textbf{Theorem 3.1.} The roots of the equation \( P(E) = 0 \) are distinct for generic \( \tau \).

\textit{Proof.} The polynomial \( P(E) \) is a characteristic polynomial of the Hamiltonian \( H \) on the space \( \mathbf{V} \), which is decomposed as (3.2). In the proof of Takemura (2003, theorem 3.2), it is shown that any two characteristic polynomials of distinct subspaces which appear on the r.h.s. of (3.2) are relatively prime. Hence, it is sufficient to show that zeros of the characteristic polynomial of \( H \) on the space \( U_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} (\alpha_i \in \{-l_i, l_i + 1\} (i = 0, 1, 2, 3) \) and \( d = -\sum_{i=0}^{3} \alpha_i/2 \in \mathbb{Z}_{\geq 0} \) are simple for generic \( \tau \).

Set \( v_n = \varphi_1(x)^{\alpha_1} \varphi_2(x)^{\alpha_2} \varphi_3(x)^{\alpha_3} \varphi(x)^n \). Then, the action of the operator \( H \) is written as
\[
Hv_r = -4 \left( (r + \gamma_1)(r + \gamma_2)v_{r+1} + r\left( r + \alpha_2 - \frac{1}{2} \right)(e_2 - e_3)(e_2 - e_1)v_{r-1} \right. \\
\left. + \left( (e_2 - e_3)(r + \alpha_2 + \alpha_1)r + (e_2 - e_1)(r + \alpha_2 + \alpha_3)r + q' \right)v_r \right),
\] (B 1)

where \( \gamma_1 = \frac{\alpha_1 + \alpha_2 + \alpha_3 - l_0}{2} \) and \( \gamma_2 = \frac{\alpha_1 + \alpha_2 + \alpha_3 + l_0 + 1}{2} \).

Set \( p = \exp(\pi\sqrt{-1} \tau) \). Then, \( e_1, e_2 \) and \( e_3 \) are expressed as power series in \( p \) and we have \( e_2 - e_1 = \pi^2(-1 - 8p + O(p^2)) \) and \( e_2 - e_3 = \pi^2(-16p + O(p^2)) \). If \( p \to 0 \), then the operator \( H \) acts triangularly and the eigenvalues of \( H \) are \( (r + \alpha_2 + \alpha_3)r + C \), where \( r = 0, \ldots, d \) and \( C \) is a constant. Since eigenvalues are quadratic in \( r \), the multiplicity of the eigenvalues is one or two. Hence, it is sufficient to show that the eigenvalue with multiplicity two in the case \( p = 0 \) separates when \( p \) varies.

If the multiplicity of the eigenvalue \( E_0 \) is two, the eigenvalues are expressed as
\[
\begin{cases} 
E_0 + c_{1/2} \sqrt{p} + c_1 p + \ldots \quad (r = (\alpha_2 - \alpha_3 \pm 1)/2), \\
E_0 + c_1 p + \ldots \quad \text{(otherwise)}.
\end{cases} \quad \text{(B 2)}
\]

First we deal with the case \( r \neq (\alpha_2 - \alpha_3 \pm 1)/2 \). By evaluating (B2) in (B1) we obtain a quadratic equation for \( c_1 \). It is seen that if \( \alpha_1 \neq \alpha_2 \), then the quadratic equation does not have multiple roots. Hence, the eigenvalue around \( E_0 \) separates when \( p \neq 0 \) and \( |p| \) is sufficiently small. For the case \( r = (\alpha_2 - \alpha_3 \pm 1)/2 \), it is shown by determining \( c_{1/2} \) and \( c_1 \) that if \( \alpha_1 \neq \alpha_2 \), \( p \neq 0 \) and \( |p| \) is sufficiently small, then the eigenvalue around \( E_0 \) separates.
Hence, if $\alpha_1 \neq \alpha_2$, then the characteristic equation of $H$ on the space $U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ does not have multiple roots for generic $p$.

If $\alpha_i = \alpha_j$ for some $i, j$ ($i \neq j$), then it is shown similarly that the characteristic equation of $H$ on the space $U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ does not have multiple root for generic $p$.

Therefore, theorem 3.1 is proved unless $l_0 = l_1 = l_2 = l_3$.

For the case $l_0 = l_1 = l_2 = l_3$, the Hamiltonian is expressed as

$$H = -\frac{d^2}{dx^2} + 4l_0(l_0 + 1)\varphi(2x),$$

and the finite-dimensional space $V(=U_{-l_0, -l_0, -l_0, -l_0})$ (see (3.2)) for the case $l_0 = l_1 = l_2 = l_3 = 0$ by replacing basic periods $(1/2, \tau/2) \leftrightarrow (1, au)$. For the case $l_0 \neq 0$, $l_1 = l_2 = l_3 = 0$, the corresponding theorem is proved by Whittaker & Watson (1962, §23.4) or Takemura (2003, theorem 3.2).

Thus, theorem 3.1 is proved. 

References


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