Electronic properties and the quantum Hall effect in bilayer graphene

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In this paper, I review the quantum Hall effect (QHE) and far-infra red (FIR) absorption properties of bilayer graphene in a strong magnetic field. This includes a derivation of the effective low-energy Hamiltonian for this system and the consequences of this Hamiltonian for the sequencing of the Landau levels in the material: the form of this effective Hamiltonian gives rise to the presence of a level with doubled degeneracy at zero energy. The effect of a potential difference between the layer of a bilayer is also investigated. It is found that there is a density-dependent gap near the K points in the band structure. The consequences of this gap on the QHE are then described. Also, the magneto-absorption spectrum is investigated and an experiment proposed to distinguish between model groundstates of the bilayer QHE system based on the different absorption characteristics of right- and left-handed polarization of FIR light. Finally, the effects of trigonal warping are taken into account in the absorption picture.

Keywords: graphene; quantum Hall effect; electronic properties

1. Introduction

The experimental investigation of monolayer graphene in a strong perpendicular magnetic field has shown several interesting features. The quantum Hall effect (QHE) shows an unusual phase of Shubnikov–de Haas oscillations and the QHE plateau sequencing is markedly different from what one sees in other semiconductors (Novoselov et al. 2005; Zhang et al. 2005). These effects have been explained as manifestations of the chiral nature of the Dirac-like quasiparticles near the K points of graphene which contain a Landau level (LL) at zero energy (McClure 1956).

The low-energy Hamiltonian of bilayer graphene describes the quadratic band structure of chiral quasiparticles which carry a Berry’s phase of $2\pi$ (in contrast to the monolayer where the Berry’s phase is $\pi$). The difference in the band structure and chiral properties makes the bilayer material an interesting topic of study on its own. In this work we will describe the properties of the bilayer material in a strong magnetic field and contrast its properties with those of the monolayer. In this section we briefly review the form of the Hamiltonians in the monolayer and bilayer and set out our notations. Then in §2 we show how to derive the effective

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One contribution of 11 to a Discussion Meeting Issue ‘Carbon-based electronics: fundamentals and device applications’.
low-energy Hamiltonian in the bilayer, and in §3 we use this Hamiltonian to
discuss the formation of a doubly degenerate LL at zero energy and the effect of
this arrangement on the sequencing of the QHE plateaux. In §4, we describe the
absorption properties of graphene and propose a spectroscopic method of
determining the form of the groundstate of the bilayer when this zero-energy LL
is exactly half-filled. Then in §5 we discuss the effect of a potential difference
between the two layers on the low-energy band structure of the bilayer and the
influence of the resulting gap at zero excess density on observations of the QHE.
Finally, in §6 we discuss the effect of trigonal warping for intermediate values of
the magnetic field.

Bilayer graphene consists of two coupled monolayers in the Bernal stacking
arrangement where the A sites in the top layer sit above the B sites in the lower layer
so that the dimer bond is formed between the A2 and B1 atoms, where 1 and 2 refer
to the two layers. The coupling between layers at these points is represented by the
energy of the dimer bond $\gamma_1$. The bases in which we write the Hamiltonians are
$(\psi_{A,K}, \psi_{B,K})$ in valley $K$ and $(\psi_{B,K'}, \psi_{A,K'})$ in valley $K'$ for the monolayer and
$(\psi_{A1,K}, \psi_{B2,K}, \psi_{A2,K}, \psi_{B1,K})$ in valley $K$ and $(\psi_{B2,K'}, \psi_{A1,K'}, \psi_{B1,K'}, \psi_{A2,K'})$ in
valley $K'$ for the bilayer. Using this nomenclature, the Hamiltonians $\hat{H}_1$ and $\hat{H}_2$
of the monolayer and bilayer are, respectively

$$\hat{H}_1 = \xi v \sigma \cdot p \quad \text{and} \quad \hat{H}_2 = \begin{pmatrix} \xi v_3 \sigma^i \cdot p & \xi v \sigma \cdot p \\ \xi v \sigma \cdot p & \gamma_1 \sigma_x \end{pmatrix}. \tag{1.1}$$

The velocity $v = (\sqrt{3}/2) a \gamma_0 / \hbar$ is determined by the intra-layer $A \equiv B$
hopping (characterized by energy $\gamma_0$ and the lattice constant $a$), $\xi$ takes the value +1 in
valley $K$ and −1 in valley $K'$, and $\sigma = (\sigma_x, \sigma_y)$ is a vector of Pauli matrices which act
in the space of electronic amplitudes on the two inequivalent lattice points (which
we call the isospin space). In the bilayer Hamiltonian, the velocity $v_3 \ll v$ represents
the weak direct hop between $A1$ and $B2$ sites. (We use $\gamma_1 = 0.39$ eV, $c_0 = 3.35$ Å,
$\varepsilon_r = 1$ and $v = 8.0 \times 10^5$ m s$^{-1}$; Dresselhaus & Dresselhaus 2002.)

2. Formation of approximate low-energy Hamiltonian for the bilayer

In a magnetic field we write the electron momentum as $p = -i \hbar \nabla - eA$ and
introduce the operators $\pi^\dagger = p_x - ip_y$ and $\pi = p_x + ip_y$ such that $[\pi, \pi^\dagger] = 2 \hbar eB$.
Following the notation of equation (1.1), the Hamiltonians can be rewritten
(disregarding the inter-layer asymmetry for now) as

$$\hat{H}_1 = \xi v \begin{pmatrix} 0 & \pi^\dagger \\ \pi & 0 \end{pmatrix} \quad \text{and} \quad \hat{H}_2 = \begin{pmatrix} 0 & \xi v_3 \pi & \xi v \pi^\dagger \\ \xi v_3 \pi^\dagger & 0 & \xi v \pi \\ 0 & \xi v \pi^\dagger & \gamma_1 \end{pmatrix}. \tag{2.1}$$

In the case of a magnetic field oriented perpendicularly to the plane of
the graphene we use the Landau gauge $A = (0, Bx)$ in which operators $\pi^\dagger$ and $\pi$
coincide with the raising and lowering operators in the basis of Landau functions

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\[ e^{i \psi \phi_n(x)} \], such that
\[
\pi^\dagger \phi_n = \frac{\sqrt{2i \hbar}}{\lambda_B} \sqrt{n + 1} \phi_{n+1}, \quad \pi \phi_n = -\frac{\sqrt{2i \hbar}}{\lambda_B} \sqrt{n} \phi_{n-1}, \quad \pi \phi_0 = 0, \quad \lambda_B = \sqrt{\frac{\hbar}{eB}}.
\]

(2.2)

The transport properties are dominated by low-energy quasiparticles, so the high-energy split band is irrelevant. We separate \( \mathcal{H}_2 \) into \( 2 \times 2 \) blocks where
\[
\mathcal{H}_{11} = \xi v_3 (\sigma_z p_x - \sigma_y p_y), \quad \mathcal{H}_{12} = \gamma_1 \sigma_z, \quad \mathcal{H}_{12} = \mathcal{H}_{21} = v_3 (\sigma_z p_x + \sigma_y p_y).
\]

(2.3)

We now take the 4x4 Green's function determined by \( \mathcal{H}_2 \) and compute the block \( G_{11} \) relating to the lower-band states. The effective low-energy bilayer Hamiltonian \( \hat{H} \) can be evaluated using this Green's function. If \( G_{11}^{(0)} = (\mathcal{H}_{11} - \epsilon)^{-1} \) we write
\[
G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (G_{11}^{(0)})^{-1} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & (G_{22}^{(0)})^{-1} \end{pmatrix}^{-1} = (\hat{H}_2 - \epsilon)^{-1}.
\]

(2.4)

By calculating the matrix inversion, we find that \( G_{11} = (1 - \mathcal{H}_{11}^0 \mathcal{H}_{12} \mathcal{H}_{22} \mathcal{H}_{21})^{-1} G_{11}^{(0)} \), so that \( G_{11}^{-1} + \epsilon = \mathcal{H}_{11} - \mathcal{H}_{12} G_{22}^{(0)} \mathcal{H}_{21} \). Since \( |\epsilon| \ll \gamma_1 \), we expand \( G_{22}^{(0)} = (\mathcal{H}_{22} - \epsilon)^{-1} \) in \( \gamma_1^{-1} \), keeping only terms up to quadratic in \( p \) (and therefore in \( \pi^\dagger, \pi \)), and arrive at the following expression for the low-energy bilayer Hamiltonian:
\[
\hat{H}_2 = -\frac{1}{2m} \begin{pmatrix} 0 & (\pi^\dagger)^2 \\ \pi^2 & 0 \end{pmatrix} + \xi v_3 \begin{pmatrix} 0 & \pi \\ \pi^\dagger & 0 \end{pmatrix}.
\]

(2.5)

The basis of this Hamiltonian is \( (\psi_{A1}, \psi_{B2}) \) the valley \( K \) and \( (\psi_{B2}, \psi_{A1}) \) in the valley \( K' \). For low quasiparticle energies \( |\epsilon| \ll \gamma_1 \), the spectrum determined by \( \mathcal{H}_2 \) agrees with that found using the 4x4 Hamiltonian \( \hat{H}_2 \). An explanation of the effect of the second term in this Hamiltonian is found in §6, but for now we describe the properties of the first term only.

The spectrum of LLs derived from the first term of (2.5) is an almost equidistant staircase of levels characterized in the following way (McCann & Fal’ko 2006):

\[
\begin{align*}
\varepsilon_0 &= 0; \quad \chi_{0,K} = (\varphi_0, 0), \quad \chi_{0,K'} = (\varphi_0, 0); \\
\varepsilon_1 &= 0; \quad \chi_{1,K} = (\varphi_1, 0), \quad \chi_{1,K'} = (\varphi_1, 0); \\
\varepsilon_{n\pm} &= \pm \hbar \omega_c \sqrt{n(n-1)}; \\
\chi_{n\pm,K} &= \frac{1}{\sqrt{2}} (\varphi_n, \pm \varphi_{n-2}), \quad \chi_{n\pm,K'} = \frac{1}{\sqrt{2}} (\varphi_n, \pm \varphi_{n-2});
\end{align*}
\]

(2.6)

where \( \chi_{n\pm,K} \) represents the wave function of the electron in valley \( K \) in the valence (\( - \)) or conduction (\( + \)) band. According to the different definitions of the two-component wave function in each valley, the \( n=0 \) and \( 1 \) states in the valley \( K \) are formed by orbitals predominantly on the A sites from the bottom layer, whereas the corresponding states in the valley \( K' \) are located on the B sites in the top layer. Therefore, if there is no splitting between the two layers they form an eightfold degenerate LL at \( \varepsilon = 0 \) (taking spin degeneracy into account). The levels for \( |n| \geq 2 \) are all fourfold degenerate.
For comparison, the monolayer spectrum and wave functions consist of (McClure 1956) a fourfold degenerate zero-energy level $\epsilon_0 = 0$ with $\psi_{0,K} = (\varphi_0, 0)$, $\psi_{0,K'} = (\varphi_0, 0)$; and a ladder of fourfold degenerate levels $\epsilon_{n\pm} = \pm \sqrt{2n}\hbar v/\lambda_B$, with $\psi_{n\pm,K} = 1/\sqrt{2}(\varphi_n, \pm i\varphi_{n-1})$, $\psi_{n\pm,K'} = 1/\sqrt{2}(\varphi_n, \mp i\varphi_{n-1})$.

3. Quantum Hall effect

We can define $J$ as the degree of chirality of the low-energy Hamiltonians of the monolayer ($J=1$) and bilayer ($J=2$). Then $\hat{H}_J = \bar{\xi}^J f(|\rho|)\mathbf{\sigma} \cdot \mathbf{n}$—with $n = l_x \cos(J\varphi) + l_y \sin(J\varphi)$ and where $\varphi$ is the angle that the momentum makes with the $x$-axis—form a family of Hamiltonians describing particles which are chiral in the sublattice space. The group of eight states at $|\epsilon|=0$ (four for electrons and four for holes) embedded into the ladder of fourfold degenerate LLs with $n \geq 2$ is specific to the magneto-spectrum of $J=2$ chiral quasiparticles. It would be reflected by the Hall conductivity dependence on carrier density, $\sigma_{xy}(N)$ as shown in figure 1. The solid line sketches the form of the Hall conductivity $\sigma_{xy}^{(2)}(N)$ in a bilayer which exhibits plateaux at integer values of $4e^2/h$ and has a ‘double’ $8e^2/h$ step between the hole and electron gases across $N=0$ that would be accompanied by a maximum in $\sigma_{xx}$. Figure 1 is sketched assuming that temperature and the LL broadening are large enough to overcome any small valley and spin splittings as well as the splitting between $n=0$ and 1 electron/hole LLs in equations (2.6), so that the percolating states from these levels would not be resolved. To compare, a monolayer has a spectrum containing fourfold (spin and valley) degenerate LLs $\epsilon_0 = 0$ and $\epsilon_{n\pm}^{\pm} = \pm \sqrt{2n}\hbar v/\lambda_B$ which corresponds to Hall conductivity $\sigma_{xy}^{(1)}(N)$ exhibiting plateaux at $(4n + 2)e^2/h$.
(dashed lines), as discussed in earlier publications (McClure 1956). Note that two parallel equivalent monolayers would display Hall conductivity $2\sigma_{xy}^{(1)}(N)$, thus missing every second plateau, as compared to $\sigma_{xy}^{(2)}(N)$. The absence of a $\sigma_{xy}=0$ plateau in the QHE accompanied by the maximum in $\sigma_{xx}$ in the vicinity of zero density is the result of the existence of the zero-energy LL, which is the fingerprint of a chiral nature of two-dimensional quasiparticles. This contrasts with a gradual freeze-out of both Hall and dissipative conductivities in semiconductor structures upon their depletion. Having compared various types of density-dependent Hall conductivity, we suggest that two kinds of chiral (Berry phase $J\pi$) quasiparticles specific to monolayer ($J=1$) and bilayer ($J=2$) systems can be distinguished on the basis of QHE measurements. It is interesting to note that the Hall effect study of ultra-thin films by Novoselov et al. (2006) featured both types of $\sigma_{xy}(N)$ dependence shown in figure 1.

It is also worth mentioning that the eightfold degeneracy of the group of $\epsilon=0$ LLs in a bilayer is quite unusual in two-dimensional systems. It suggests that electron–electron interaction in a bilayer may give rise to a variety of strongly correlated QHE states. For structures studied by Novoselov et al. (2006), with electron/hole densities $N \sim 10^{12}$ cm$^{-2}$, such a regime may be realized in fields $B \sim 10$ T.

4. Optical properties: far-infrared absorption

We now describe the magneto-absorption properties of the bilayer and propose an experimental method of determining between two model groundstates of the strongly correlated QHE system. The absorption of an electromagnetic field by a two-dimensional electron gas with conductivity $\sigma(\omega) \ll c/2\pi$ can be represented by the ratio of the Joule heating and the incident energy flux

$$ g = \frac{E_i E_f \sigma(\omega)}{S} = \frac{8e^2}{c \omega} R \frac{F \, d\epsilon}{N} \text{Tr} \left\{ \hat{v}_j \hat{\ell}_i \hat{G}^R(\epsilon) \hat{v}_j \hat{\ell}_i \hat{G}^A(\epsilon + \omega) \right\}, \quad (4.1) $$

where $\hat{v} = \partial p / \partial H$ is the velocity operator; $\text{Tr}$ includes the summation both over the sublattice indices ‘tr’ and over single-particle orbital states; $N$ is the normalization area of the sample; and $F = n_e(\epsilon) - n_h(\epsilon + \omega)$ takes into account the occupancy of the initial and final states. Here we have included spin and valley degeneracy. The energy flux $S = cE \times H / 4\pi = -Sl_z$ is transported by the EM field $E_\omega = \ell E e^{-i\omega t}$ arriving along the direction antiparallel to a magnetic field and carrying polarization $\ell$. We denote the two orientations of circularly polarized light as $\ell_\Theta = [l_x - il_y] / \sqrt{2}$ for right-handed radiation and $\ell_\Theta = [l_x + il_y] / \sqrt{2}$ for left-handed radiation.

Figure 2 illustrates the selection rules for the single-particle inter-LL transitions in bilayer and monolayer graphene derived by calculating matrix elements of the perturbation $\hat{v} \cdot (e/c) \hat{A}$. In both cases, photons with in-plane polarization $\ell_\Theta$ are absorbed via transitions where the LL index changes from $n$ to $n-1$ ($n \geq 1$), whereas absorption of $\ell_\Theta$ photons happens via transitions from the LL $n$ to $n+1$ ($n \geq 0$). Assuming the same broadening $\hbar \tau^{-1}$ of all LLs, we arrive at the following magneto-absorption spectra for monolayer (Sadowski et al. 2006; Abergel & Fal’ko 2007) and bilayer graphene (for $\epsilon_L < \hbar \omega < 1/4 \gamma_1$):

$$ g_j^{\oplus}(B, \omega) = \frac{2\pi e^2}{\hbar c} f_j^{\oplus}(B, \omega), \quad (4.2) $$
where

\[
 f_1^\Theta = \sum_{n \geq 1} \frac{w_1^2}{\pi \hbar^2} \frac{2h\omega}{a\sqrt{n-1-a'}} \left( \nu_{n,a'} - \nu_{n-1,a} \right),
\]

\[
 f_1^\Omega = \sum_{n \geq 0} \frac{w_1^2}{\pi \hbar^2} \frac{2h\omega}{a\sqrt{n+1-a'} \sqrt{n}} \left( \nu_{n,a'} - \nu_{n+1,a} \right),
\]

\[
 f_2^\Theta = \sum_{n \geq 2} \frac{4e_2^2}{\pi \omega_c \tau} \frac{\nu_{n,a'} - \nu_{n-1,a}}{\alpha\sqrt{n^2 - n - \alpha'\sqrt{(n-1)(n-2)}}} \times \frac{1}{\left[ \frac{\omega}{\omega_c} - \alpha\sqrt{n^2 - n} + \alpha'\sqrt{(n-1)(n-2)} \right]^2 + \frac{\tau^2}{\omega_c^2}},
\]

\[
 f_2^\Omega = \sum_{n \geq 1} \frac{4e_2^2}{\pi \omega_c \tau} \frac{\nu_{n,a'} - \nu_{n+1,a}}{\alpha\sqrt{n^2 + n - \alpha'\sqrt{n^2 - n}}} \times \frac{1}{\left[ \frac{\omega}{\omega_c} - \alpha\sqrt{n^2 + n} + \alpha'\sqrt{n^2 - n} \right]^2 + \frac{\tau^2}{\omega_c^2}},
\]

where \( w_1 = \sqrt{2h\nu\lambda_B^{-1}} \) and \( \hbar\omega_c = \hbar^2/m_2\lambda_B^2 \) are the characteristic energy scales for the LL spectra in monolayer (\( J=1; \epsilon_{n\alpha} \)) and bilayer (\( J=2; \epsilon_{n\alpha} \)) graphene, respectively, and \( \alpha, \alpha' = \pm \) determine whether the corresponding state belongs to
the conduction (+) or valence (−) band. Also, $v_{n,\alpha}$ are the filling factors of the corresponding LLs, and $b_0=1$, $b_{n\geq 1} = 1/\sqrt{2}$ for the monolayer and $c_{0,1} = 1$, $c_{n\geq 2} = 1/\sqrt{2}$ for the bilayer.

In contrast to monolayer graphene, a weak absorption of light polarized perpendicular to the bilayer is possible. A perturbation $\sigma_z eE_x d/2$ distinguishes between the on-site energies in the top and bottom layers separated by spacing $d$, which leads to weak absorption $g_2^z = (2\pi e^2/\hbar c)f_2^z$,

$$
f_2^z = a_z^2 \frac{1}{\Omega + 1} + \frac{\theta(\Omega - 2)}{\Omega - 1}, \quad \Omega \equiv \hbar \omega / \gamma_1,
$$

$$
f_2^z(B, \omega) = a_z^2 \frac{\tau^2}{\pi} \sum_{n\geq 2} \frac{\omega}{\omega_c} \frac{\tau \omega}{\omega_c - 2\sqrt{n^2 - n} + 1},
$$

where the constant $a_z = \gamma_1 d/2\hbar v \sim 10^{-1}$ and the magneto-absorption spectrum at $\hbar \omega < (1/4)\gamma_1$ involves $\varepsilon_{n-} \rightarrow \varepsilon_{n+}$ inter-LL transitions.

One can test the selection rules shown in figure 2 and the actual polarization of these transitions using gated graphene structures (Zhang et al. 2005). By filling the monolayer sheet with electrons up to $n=2$ (a completely filled $n=0$ LL), one would suppress the intensity of the lowest absorption peak in $\ell_\parallel$ polarization and increase the size of the $\ell_\perp$ peak. By depleting the monolayer to $n=-2$ state (emptying the $n=0$ LL), one would achieve the opposite effect. Similarly, in a bilayer with a completely filled pair of $n=0, 1$ LLs which takes place at $n=4$, light with $\omega = \sqrt{2}\omega_c$ can be absorbed only in the $\ell_\perp$ polarization. By depleting the bilayer down to $n=-4$, one could suppress the absorption in this line in $\ell_\parallel$ polarization while retaining $\ell_\perp$ absorption.

It is probable that, at a high magnetic field, electrons in an ideally clean bilayer with filling factor $-4<\nu<4$ would form a correlated groundstate in which the occupancy of $n=0$ and 1 LLs would be determined by the electron–electron interaction. This may be due to either the inter-layer asymmetry (Koshino & Ando 2006; Latil & Henrard 2006; Lu et al. 2006; McCann & Fal’ko 2006; Nilsson et al. 2006; Partoens & Peeters 2006) or caused by the AA/BB intra-layer hopping in the last term of $H_2$. Below we refer to such a QHE state as being antiferromagnetic, stressing that this state is not spin polarized.

An alternative form of the groundstate can arise when the splitting between the $n=0$ and 1 states is negligible. In this situation, the electron–electron repulsion may lead to the ferromagnetic alignment of electron spins and the
formation of a ferromagnetic QHE state with one spin component of each LL being completely full and the other completely empty.

Since transitions from/to the \( n = 0 \) LL with \( \varepsilon_0 \approx 0 \) to/from the states \( \varepsilon_{n \pm} \) with \( n \geq 2 \) are forbidden, the intensity and polarization of the peaks at \( \omega = \sqrt{2} \omega_c \) are determined by transitions from/to \( n = 1 \) LL (also with \( \varepsilon_1 \approx 0 \)) and directly reflect the occupancy of this state. If the bilayer groundstate is ferromagnetic, with a half-filled \( n = 1 \) LL, then the absorption peak at \( \omega = \sqrt{2} \omega_c \) will have the same intensity in both \( \ell_{\phi} \) and \( \ell_{\Theta} \) polarizations. Also, note that a weak transition between \( n = 0 \) and 1 LLs with a low frequency \( \beta \omega_c \) (microwave range) is possible, to the measure of a small \( \beta \) and depending on the LL filling. In contrast, absorption by a \( n = 0 \) bilayer with antiferromagnetic groundstate would contain the line at \( \omega = \sqrt{2} \omega_c \) only in one polarization: in \( \ell_{\phi} \) if \( n = 1 \) LL is empty (fully occupied \( n = 0 \) LL corresponding to the transition \( 2 \rightarrow -1 \)) and in \( \ell_{\Theta} \) if \( n = 1 \) LL is full (transition \( 1 \rightarrow 2 + \)).

These two cases are shown in figure 3. The increased intensity of the lowest absorption peak in the antiferromagnetic line is due to the increased size of the difference \( \nu_n - \nu_{n-1} = \pm 1 \) for full or zero occupancy of the \( n = 1 \) LL, relative to the ferromagnetic \( \nu_n - \nu_{n-1} = \pm 1/2 \) value. For comparison, the lowest peak in the spectrum of a monolayer with \( \nu = 0 \) appears in both polarizations, since both transitions to \( (\varepsilon_1 \rightarrow \varepsilon_0) \) and from \( (\varepsilon_0 \rightarrow \varepsilon_1 +) \) half-filled \( n = 0 \) monolayer LL are possible, both in the monolayer and bilayer material. All higher-energy absorption peaks which involve transitions between filled valence band states \( (\alpha = -) \) and empty states in the conduction band \( (\alpha = +) \) have equal strengths in both polarizations, which reflect effectively the inter-band nature of these transitions.

Figure 3. Far-infra red (FIR) absorption spectra for (a) monolayer and (b) bilayer graphene in \( \ell_{\phi} \) and \( \ell_{\Theta} \) polarizations for filling factor \( \nu = 0 \). Solid and dashed lines describe absorption by ferromagnetic and antiferromagnetic QHE states of the \( \nu = 0 \) bilayer, respectively. In both cases, \( B = 10 \) T, \( \omega_1 = (\sqrt{2} \hbar \nu)/\lambda_B \) and \( \hbar \omega_c = 20 \) meV.

\[ g_1^\Theta \]

\[ g_1^\phi \]

\[ g_2^\Theta \]

\[ g_2^\phi \]

\[ h\omega/\omega_1 \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ \ell_{\phi} \]

\[ \ell_{\Theta} \]

\( \nu_n - \nu_{n-1} = \pm 1 \) for full or zero occupancy of the \( n = 1 \) LL, relative to the ferromagnetic \( \nu_n - \nu_{n-1} = \pm 1/2 \) value. For comparison, the lowest peak in the spectrum of a monolayer with \( \nu = 0 \) appears in both polarizations, since both transitions to \( (\varepsilon_1 \rightarrow \varepsilon_0) \) and from \( (\varepsilon_0 \rightarrow \varepsilon_1 +) \) half-filled \( n = 0 \) monolayer LL are possible, both in the monolayer and bilayer material. All higher-energy absorption peaks which involve transitions between filled valence band states \( (\alpha = -) \) and empty states in the conduction band \( (\alpha = +) \) have equal strengths in both polarizations, which reflect effectively the inter-band nature of these transitions.

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5. Inter-layer asymmetry gap in bilayers

We now turn our attention to the effects of the asymmetry gap. As given in the works by McCann (2006) and McCann & Fal’ko (2006), we consider a graphene bilayer, with inter-layer separation \( c_0 \), located a distance \( d \) from a gate. An external gate voltage \( V_g = \varepsilon_0 \varepsilon' \varepsilon_0 \) induces a total excess density \( n = n_1 + n_2 \) on the bilayer where we label by \( n_1 \) the excess density on the layer closest to the gate and by \( n_2 \) the excess density on the layer furthest from the gate. Here \( \varepsilon_0 \) is the permittivity of free space, \( \varepsilon' \) is the dielectric constant and \( e \) is the electronic charge. Imperfect screening of the effective charge density \( en \) from the gate leads to an excess density on the layer furthest from the gate, with a corresponding change in potential energy \( \Delta U_2 = e^2 n_2 c_0 / \varepsilon_0 \) that determines the layer asymmetry

\[
\Delta(n) = \varepsilon_2 - \varepsilon_1 \equiv \Delta_0 + e^2 n_2 L^2 / C_b,
\]

where \( C_b = \varepsilon_r \varepsilon_0 L^2 / c_0 \) is the capacitance of a bilayer of area \( L^2 \) and \( \varepsilon_r \) is the bilayer dielectric constant. We introduce the bare asymmetry parameter \( \Delta_0 \) due to an additional transverse electric field producing finite asymmetry \( \Delta(0) \) at zero density.

The tight-binding Hamiltonian relevant in this situation operates in the same space of wave functions as equation (2.1) and takes into account the asymmetry \( \Delta = \varepsilon_2 - \varepsilon_1 \) between on-site energies in the two layers, \( \varepsilon_2 = (1/2) \Delta, \varepsilon_1 = -(1/2) \Delta \). Neglecting the trigonal warping effects, it is

\[
\hat{\mathcal{H}}_\Delta = \xi \left( \begin{array}{cccc}
-\frac{1}{2} \Delta & 0 & 0 & v \pi^i \\
0 & \frac{1}{2} \Delta & v \pi & 0 \\
0 & v \pi^i & \frac{1}{2} \Delta & \xi \gamma_1 \\
v \pi & 0 & \xi \gamma_1 & -\frac{1}{2} \Delta 
\end{array} \right).
\]

We evaluate the semiclassical expression for the cyclotron mass \( m_c \) that may be confirmed experimentally through the Shubnikov–de Haas oscillations at a finite magnetic field (Novoselov et al. 2004, 2006). For a circular Fermi surface, the semiclassical expression for cyclotron mass \( m_c = p/(\partial \varepsilon / \partial p) \) gives

\[
m_c = \frac{\varepsilon^{(a)}}{v^2} \left[ 1 + (-1)^\alpha \frac{\gamma_1^2 + \Delta^2}{2 \sqrt{v^2 p^2 [\gamma_1^2 + \Delta^2] + \gamma_1^4 / 4}} \right]^{-1}.
\]

In the limit \( \Delta = 0 \), \( m_c = (\gamma_1 / 2 v^2) \sqrt{1 + 4 \pi \hbar^2 v^2 |n| / \gamma_1^2} \) for the low-energy bands \( \alpha = 1 \). Figure 4 shows a self-consistent calculation of the cyclotron mass \( m_c \) as a function of total density \( n \). For \( \Delta_0 = 0 \) (solid line), the mass is symmetric and finite \( m_c = \gamma_1 / 2 v^2 \) at zero density. The other curves, for \( \Delta_0 > 0 \), show that the cyclotron mass for positive and negative densities is asymmetric with divergent behaviour at low density resulting from the ‘Mexican hat’ structure of

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the low-energy bands. Note that we neglect the role of additional weak couplings, such as $A_1-B_2$ coupling $g_3$ that results in trigonal warping and is relevant at low density $n \sim 1 \times 10^{11} \text{cm}^{-2}$ (McCann & Fal’ko 2006).

Another manifestation of finite $\Delta_0$ is in the sequencing of QHE plateaux at low density. The LL spectrum of bilayer graphene has been discussed in the absence of this gap, so we restate the low-energy Hamiltonian with this feature added:

$$
\hat{H} = -\frac{1}{2m} \begin{pmatrix} 0 & (\pi^\dagger)^2 \\ \pi^2 & 0 \end{pmatrix} - \xi \Delta \left[ \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{v^2}{\gamma_1^2} \begin{pmatrix} \pi \pi^\dagger & 0 \\ 0 & -\pi \pi^\dagger \end{pmatrix} \right] \\
- \frac{\alpha}{2m} \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} + \frac{\beta \hbar eB}{2m} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. 
$$

(5.4)

For completeness, we include the final two terms, with dimensionless parameters $|\alpha|, |\beta| \ll 1$, describing weak asymmetry between the conduction and valence bands. Neglected previously, they arise due to additional couplings such as next-nearest-neighbour in-plane coupling between $A1-A1$ and $B2-B2$ orbitals or inter-layer coupling between $A1-A2$ and $B1-B2$ orbitals. The LL spectrum is calculated from this Hamiltonian as before. The spectrum has almost equidistant energy levels for $N \geq 2$ with spacing $\hbar \omega_c$, $\omega_c = eB/m$, which are weakly split in valleys $K$ and $K'$ ($\xi = \pm 1$; McCann & Fal’ko 2006),

$$
\epsilon_N^{\pm} \approx \frac{\xi \Delta \hbar \omega_c}{2\gamma_1} - \alpha \hbar \omega_c \left( N - \frac{1}{2} \right) \pm \hbar \omega_c \sqrt{N(N-1)},
$$

(5.5)

where $\epsilon_N^+$ and $\epsilon_N^-$ are assigned to electron and hole states, respectively, and we assume $|\Delta|/\gamma_1 \ll 1, |\Delta|/\hbar \omega_c \leq 1$. 

**Figure 4.** The cyclotron mass $m_c$ in units of the bare electronic mass $m_e$, for different values of $\Delta_0$. Solid line, $\Delta_0 = 0$; dashed line, $\Delta_0 = 0.1 \gamma_1 = 39 \text{meV}$; dotted line, $\Delta_0 = 0.2 \gamma_1 = 78 \text{meV}$. Typical parameter values (Dresselhaus & Dresselhaus 2002) have been used.
The LL spectrum in each valley also contains two levels identified using the fact that \( \phi_2^2 \phi_1^2 = 0 \),

\[
\begin{align*}
\epsilon_0 &= -\frac{1}{2} \xi \Delta - \frac{1}{2} (\alpha - \beta) \hbar \omega_c; & \chi_{0\xi} &\equiv (\phi_0, 0); \\
\epsilon_1 &= -\frac{1}{2} \xi \Delta + \frac{\xi \Delta \hbar \omega_c}{\gamma_1} - \frac{1}{2} (3\alpha - \beta) \hbar \omega_c; & \chi_{1\xi} &\equiv (\phi_1, 0).
\end{align*}
\] (5.6)

The spectrum is shown in figure 5c assuming \( |\Delta|/\gamma_1 \ll 1, |\Delta|/\hbar \omega_c \leq 1 \). In a symmetric bilayer (\( \Delta = 0 \)) levels \( \epsilon_0 \) and \( \epsilon_1 \) are degenerate and have the same energy in valleys \( K \) and \( K' \), thus forming an eightfold degenerate LL at \( \epsilon = 0 \) (here, spin is taken into account) as shown in figure 5a.

The LL spectrum is reflected in the Hall conductivity dependence on carrier density, \( \sigma_{xy}(n) \), although in general finite temperature and LL broadening tend to mask small LL splitting. When the asymmetry at zero density \( \Delta(0) \) is large enough that the percolating states from the lowest levels in the two valleys can be resolved (figure 5c), then these levels, embedded into the ladder of fourfold degenerate LLs with \( n \geq 2 \), equation (5.5), result in the form of \( \sigma_{xy}(n) \) as sketched in figure 5b (solid line) which exhibits plateaus at all integer values of \( 4e^2/h \) including a plateau at zero density. The temperature dependence of the zero density plateau will differ from that of the other plateaux owing to the different activation energies, related to \( \Delta(0) \) and \( \hbar \omega_c \), respectively. When it is not possible to resolve the asymmetry splitting of the group of eight states at \( |\epsilon| = 0 \) (for \( \Delta(0) \approx \Delta_0 \approx 0 \)) then \( \sigma_{xy}(n) \) (dashed lines in figure 5) has plateaus at integer values of \( 4e^2/h \) and a double \( 8e^2/h \) step across \( n = 0 \) (McCann & Fal'ko 2006; Novoselov et al. 2006). The form of \( \sigma_{xy}(n) \) with plateaux at all integer values \( 4e^2/h \) has recently been observed in measurements of heavily doped bilayer graphene (Novoselov et al. 2006).

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6. The effect of trigonal warping on the LL spectrum of bilayer graphene

We now take into account the effect of the direct $A1 \equiv B2$ coupling characterized by the energy $\gamma_3 \ll \gamma_1$ which determines the velocity $v_3 = (\sqrt{3}/2) a \gamma_3 / \hbar$. This corresponds to the term which is linear in $\pi$, $\pi^2$ in the Hamiltonian in (2.5). This term is only relevant for very small electron momenta (i.e. in an electron gas with a small density at a very low magnetic field) whereas the energy spectrum in the interval $(1/2) \gamma_1 (v_3/v) < |\varepsilon| < \gamma_1/4$ is dominated by the quadratic term.

For low quasiparticle energies, $|\varepsilon| \ll \gamma_1$, the spectrum determined by $\hat{H}_2$ in equation (2.5) agrees with $\varepsilon_1(p)$ found using the $4 \times 4$ Hamiltonian $\hat{H}_2$. Similarly to bulk graphite (Dresselhaus 1974; Dresselhaus & Dresselhaus 2002), the effect of the second term in (2.5) consists of trigonal warping, which deforms the isoenergetic lines along the directions $\phi = \phi_0$. For the valley $K$, $\phi_0 = 0$, $2/3 \pi$ and $4/3 \pi$, whereas for $K'$, $\phi_0 = \pi$, $(1/3) \pi$ and $(5/3) \pi$. At the lowest energies $|\varepsilon| < (1/2) \gamma_1 (v_3/v)^2$, trigonal warping breaks the isoenergetic line into four pockets, which can be referred to as one ‘central’ and three ‘leg’ parts (Dresselhaus 1974). The central and leg parts have minimum $|\varepsilon| = (1/2) u$ at $p = 0$ and at $|p| = \gamma_1 v_3/v^2$, angle $\phi_0$, respectively. For $v_3 \sim 0.1 \mu$, we find (using the data in the review by Dresselhaus & Dresselhaus (2002)) that the separation of a two-dimensional Fermi line into four pockets would take place for very small carrier densities $N < N_c = 2 (v_3/v)^2 N^* \sim 1 \times 10^{11} \text{cm}^{-2}$. For $N < N_c$, the central part of the Fermi surface is approximately circular with area $A_c \approx \pi e^2/\hbar (v_3)^2$, and each leg part is elliptical with area $A_l \approx (1/3) A_c$. This determines the following sequencing of the first few LLs in a low magnetic field, $B \ll B_c \approx h N_c / 4 e \sim 1 \text{T}$. Every third LL from the central part has the same energy as levels from each of the leg pockets, resulting in groups of four degenerate states. These groups of four would be separated by two non-degenerate LLs arising from the central pocket.

The effect of the warping term in the first order of perturbation theory in the small parameter $v_3/v$ is to mix each state $\chi_{naxK}$ with states $\chi_{(n \pm 3)axK}$. This allows the perturbation $\mathbf{v} \cdot (e/c) \mathbf{A}$ to generate weak transitions (with the coefficient $\delta_{g2} \sim (v_3 m_2 \lambda_B^2 / \hbar)^2 g$) between states which are two and four levels apart (figure 6). If we denote the second term in (2.5) as $\hat{H}_w$ then in addition the perturbation $\hat{p} \cdot (e/c) \mathbf{A}$ allows direct transitions between states which are two LLs apart. Although such transitions are negligibly weak at high fields where the first term in $\hat{H}_w$ is dominant, they become relevant at weak fields, where $\lambda_B^{-1} \lesssim \gamma_1 v_3/v^2$, and the low-frequency absorption spectrum of a bilayer acquires an additional structure.

In figure 6 we compare the absorption spectra in a $\nu = 0$ bilayer (where our previous comments about the level-dependent broadening still apply) at $B = 1 \text{T}$ calculated numerically for $v_3 = 0$ and $0.2 v$. When the warping is included the energy of each LL is shifted (with the exception of the degenerate $\varepsilon = 0$ levels; figure 6) which shifts the positions of the peaks in the absorption spectrum. Also, the additional transitions reveal themselves as new peaks which were not present previously. For example, the peak at $\omega/\omega_c \approx 2.5$ corresponds to the transitions $3- \rightarrow 1$ and $1 \rightarrow 3+$.  

Phil. Trans. R. Soc. A (2008)
In this article we have described the optical properties and QHE in bilayer graphene in a strong magnetic field. The pairs of degenerate bands of electrons with quadratic dispersion near the $K$ points in bilayer graphene led to the formation of a doubly degenerate zero-energy LL, and this level changed the sequencing of the QHE plateaux relative to those seen in the monolayer material (Zhang et al. 2005; Novoselov et al. 2006). Based on the doubled degeneracy of the lowest energy LL and the specific form of the selection rules for transitions of electrons between LLs, a proposal is made for an FIR optical experiment with circularly polarized light which can enable one to determine the form of the QHE groundstate for bilayer graphene. While the selection rules for transitions of

7. Conclusions

In this article we have described the optical properties and QHE in bilayer graphene in a strong magnetic field. The pairs of degenerate bands of electrons with quadratic dispersion near the $K$ points in bilayer graphene led to the formation of a doubly degenerate zero-energy LL, and this level changed the sequencing of the QHE plateaux relative to those seen in the monolayer material (Zhang et al. 2005; Novoselov et al. 2006). Based on the doubled degeneracy of the lowest energy LL and the specific form of the selection rules for transitions of electrons between LLs, a proposal is made for an FIR optical experiment with circularly polarized light which can enable one to determine the form of the QHE groundstate for bilayer graphene. While the selection rules for transitions of
electrons between LLs are similar to those in the monolayer material, the increased density of the bilayer spectrum means that the density of peaks in its absorption spectrum is higher. There is also the possibility for a small peak at a very low frequency due to a transition of electrons between the degenerate $n=0$ and 1 levels which may be slightly split by a weak next-nearest-neighbour hopping.

I thank E. McCann and D. Abergel for their help in the preparation of this manuscript; K. Kechedzhi, V. Cheianov and B. Altshuler for their discussions; and the EPSRC-Lancaster Portfolio Partnership for financial support of this research.

References


A. K. Savchenko (University of Exeter, UK). How different is weak localization in bilayer graphene compared with that in single layer?

V. I. Fal’ko. One may consider graphene without lattice imperfections and deformation, where electron scattering would be generated exclusively by remote charges in the dielectric substrate, and ignore the warping of the electronic band structure. Then, one should expect antilocalization behaviour in monolayers (where the electrons are characterized by the Berry’s phase $\pi$) and weak localization (WL) effect in bilayers (where the electrons are characterized by the Berry phase $2\pi$). However, the effect of warping and AB-asymmetric disorder suppresses this difference, and the inter-valley scattering on sharp defects and edges leads to the usual WL effect both in monolayer and bilayer, with the WL magnetoresistance saturated at a relatively weak magnetic field determined by the inter-valley scattering rate (McCann et al. 2006; Kechedzhi et al. 2007). This seems to be consistent with the recent observations of weak localization effects in both monolayers (Morozov et al. 2006) and bilayers (Gorbachev et al. 2007)

Additional references


