Boundary integral equations in elastodynamics of interface cracks

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The paper concerns the validation of a method for solving elastodynamics problems for cracked solids. The proposed method is based on the application of boundary integral equations. The problem of an interface penny-shaped crack between two dissimilar elastic half-spaces under harmonic loading is considered as an example.

Keywords: elastodynamics; crack; interface; boundary integrals

1. Introduction and formulation of the problem

All existing structural materials contain various crack-like defects. Such defects appear in real-life materials during fabrication or in service (fatigue, consequences of an impact, etc). The presence of defects considerably decreases the strength and the lifetime of structures as well as essentially increases the cost of exploitation (Freund 1990; Zhang & Gross 1998). Therefore it is necessary to ensure that the residual strength of the cracked structure will not fall below an acceptable level over the required service life. It takes on special significance for the case of high-rate deformations.

Here we consider an unbounded elastic solid consisting of two dissimilar half-spaces, \( \Omega^{(1)} \) and \( \Omega^{(2)} \). They are assumed to be homogeneous and isotropic, with Young’s moduli \( E^{(1)} \) and \( E^{(2)} \), Poisson’s ratios \( \nu^{(1)} \) and \( \nu^{(2)} \), and mass densities \( \rho^{(1)} \) and \( \rho^{(2)} \), respectively. The planes \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) are the boundaries of domains \( \Omega^{(1)} \) and \( \Omega^{(2)} \) characterized by the outer normal vectors \( n^{(1)} \) and \( n^{(2)} \), respectively. The solid is under dynamic harmonic or non-harmonic loading.

In the absence of body forces, the stress–strain state of both domains is defined by the dynamic equations of the linear elasticity

\[
\left\{ \begin{array}{l}
(\lambda^{(m)} + \mu^{(m)}) \text{grad} \text{ div } u^{(m)}(x, t) + \mu^{(m)} \Delta u^{(m)}(x, t) = \rho^{(m)} \partial_t \partial_t^{(2)} u^{(m)}(x, t), \\
x \in \Omega^{(m)}, \ t \in T = [0, \infty),
\end{array} \right. 
\] (1.1)

where \( \Delta \) is the Laplace operator, and \( \lambda^{(m)} \) and \( \mu^{(m)} \) are the Lamé elastic constants. Henceforth, the vector of displacements is denoted as \( u^{(m)}(x, t) \), and the traction

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vector as \( p^{(m)}(\mathbf{x}, t) \). Surface \( \Gamma^{(m)} \) consists of the infinite part \( \Gamma^{(m)*} \) and the finite part \( \Gamma^{(m)\text{cr}} \). The crack surface \( \Gamma^{\text{cr}} \) is formed by two opposite faces, \( \Gamma^{(1)\text{cr}} \) and \( \Gamma^{(2)\text{cr}} \). The superscript ‘\((m)\)’ refers to the domain \( \Omega^{(m)} \), \( m=1, 2 \).

The body is strainless at the initial moment, which means there are no initial displacements of the points of the body. The conditions of continuity for displacements and stresses are satisfied on the interface \( \Gamma^* = \Gamma^{(1)} \cap \Gamma^{(2)} \). On the opposite faces of the crack the traction vectors \( g^{(1)}(\mathbf{x}, t) \) and \( g^{(2)}(\mathbf{x}, t) \) are prescribed.

In addition, the condition \( ||u(\mathbf{x}, t)|| \leq C/R \), where \( C \) is a constant, \( R \to \infty \) is the distance from the origin, is imposed at infinity on the displacement vector. This condition ensures a finite elastic energy of an infinite body. In the case of a wave equation, it was introduced by Sommerfeld and is called a radiation-type condition at infinity (Graff 1991).

### 2. Obtaining the system of integral equations

The detailed procedure of obtaining the system of boundary integral equations is given by Guz et al. (2006). Here, owing to the space restrictions, we outline only the important stages of the derivation.

Following Ugodchikov & Khutoryansky (1986) and Martin & Rizzo (1989), we represent the components of the displacement field in the upper and the lower half-spaces \( \Omega^{(m)} \) in terms of boundary displacements and tractions using the Somigliano dynamic identity

\[
U_j^{(m)}(\mathbf{x}, t) = \int_T \int_{\Gamma^{(m)}} \left[ p_i^{(m)}(\mathbf{y}, \tau) U_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) - u_i^{(m)}(\mathbf{y}, \tau) W_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \right] \, d\mathbf{y} \, d\tau,
\]

(2.1)

where \( U_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) is the Green’s fundamental displacement tensor, \( \mathbf{x} \) is the point of observation, \( \mathbf{y} \) is the point of loading, \( j=1, 2, 3 \), \( t \in T \).

The integral kernel \( W_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) can be obtained from \( U_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) by applying the differential operator

\[
P_{ik}[\mathbf{y}] = \lambda n_i(\mathbf{y}) \frac{\partial \delta[i]}{\partial y_k} + \mu \left[ \delta_{ik} \frac{\partial \delta[i]}{\partial n(\mathbf{y})} + n_k(\mathbf{y}) \frac{\partial \delta[i]}{\partial y_i} \right].
\]

(2.2)

Applying the differential operator (2.2) to the Somigliano identity (2.1), we obtain the components of the traction vector \( p_i^{(m)}(\mathbf{x}, t) \) in terms of boundary displacements and traction for both half-spaces \( \Omega^{(m)} \)

\[
p_j^{(m)}(\mathbf{x}, t) = \int_T \int_{\Gamma^{(m)}} \left[ p_i^{(m)}(\mathbf{y}, \tau) K_{ij}^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) - u_i^{(m)}(\mathbf{y}, \tau) F_{ij}^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \right] \, d\mathbf{y} \, d\tau,
\]

(2.3)

where kernels \( K_{ij}^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) and \( F_{ij}^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) are obtained in a manner similar to \( W_i^{(m)}(\mathbf{x}, \mathbf{y}, t-\tau) \) (see Zhang & Gross 1998; Men’shikov et al. 2007).

Since \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) are the regular surfaces, we assume that the distributions of boundary displacements and traction vectors are smooth enough. Then for the

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limiting case $x \to I^{(m)}$ after some non-trivial manipulations, we deduce from
the equations (2.1) and (2.3) the following boundary integral equations:
\[
\frac{1}{2} g_j^{(1)}(x, t) - \int_{T} \int_{I^{(1)cr}} g_{ij}^{(1)}(y, \tau) K_{ij}^{(1)}(x, y, t - \tau) dy d\tau
\]
\[
= - \int_{T} \int_{I^{(1)cr}} u_{ij}^{(1)}(y, \tau) F^{(1)}_{ij}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(1)cr}} p_i^*(y, \tau) K_{ij}^{(1)}(x, y, t - \tau) dy d\tau
+ \int_{T} \int_{I^{(1)cr}} u_i^*(y, \tau) F_{ij}^{(1)}(x, y, t - \tau) dy d\tau,
\qquad x \in I^{(1)cr},
\]
(2.4)
\[
\frac{1}{2} g_j^{(2)}(x, t) - \int_{T} \int_{I^{(2)cr}} g_{ij}^{(2)}(y, \tau) K_{ij}^{(2)}(x, y, t - \tau) dy d\tau
\]
\[
= - \int_{T} \int_{I^{(2)cr}} u_{ij}^{(2)}(y, \tau) F^{(2)}_{ij}(x, y, t - \tau) dy d\tau
+ \int_{T} \int_{I^{(2)cr}} p_i^*(x, \tau) K_{ij}^{(2)}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(2)cr}} u_i^*(x, \tau) F_{ij}^{(2)}(x, y, t - \tau) dy d\tau,
\qquad x \in I^{(2)cr},
\]
(2.5)
\[
\int_{T} \int_{I^{(1)cr}} g_i^{(1)}(y, \tau) K_{ij}^{(1)}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(2)cr}} g_i^{(2)}(y, \tau) K_{ij}^{(2)}(x, y, t - \tau) dy d\tau
\]
\[
= \int_{T} \int_{I^{(1)cr}} u_i^{(1)}(y, \tau) F^{(1)}_{ij}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(2)cr}} u_i^{(2)}(y, \tau) F^{(2)}_{ij}(x, y, t - \tau) dy d\tau
+ \int_{T} \int_{I^{(1)cr}} p_i^*(y, \tau) [K_{ij}^{(1)}(x, y, t - \tau) + K_{ij}^{(2)}(x, y, t - \tau)] dy d\tau
- \int_{T} \int_{I^{(2)cr}} u_i^*(y, \tau) [F^{(1)}_{ij}(x, y, t - \tau) + F^{(2)}_{ij}(x, y, t - \tau)] dy d\tau,
\qquad x \in I^*,
\]
(2.6)
\[
\int_{T} \int_{I^{(1)cr}} g_i^{(1)}(y, \tau) U_{ij}^{(1)}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(2)cr}} g_i^{(2)}(y, \tau) U_{ij}^{(2)}(x, y, t - \tau) dy d\tau
\]
\[
= \int_{T} \int_{I^{(1)cr}} u_i^{(1)}(y, \tau) W^{(1)}_{ij}(x, y, t - \tau) dy d\tau
- \int_{T} \int_{I^{(2)cr}} u_i^{(2)}(y, \tau) W^{(2)}_{ij}(x, y, t - \tau) dy d\tau
+ \int_{T} \int_{I^*} p_i^*(y, \tau) [U_{ij}^{(1)}(x, y, t - \tau) + U_{ij}^{(2)}(x, y, t - \tau)] dy d\tau
- \int_{T} \int_{I^*} u_i^*(y, \tau) [W^{(1)}_{ij}(x, y, t - \tau) + W^{(2)}_{ij}(x, y, t - \tau)] dy d\tau,
\qquad x \in I^*,
\]
(2.7)
where the integration is carried out over the infinite surface $I^*$ with the outer
normal vector $n_{I^*} = n_{I^{(2)cr}} = -n_{I^{(1)cr}}$, and the new notations are introduced
\[
u_i(x, t) = u_i^{(2)}(x, t), \quad p_i^*(x, t) = p_i^{(2)}(x, t), \quad x \in I^*, \quad t \in T.
\]
(2.8)
The boundary integral equations (2.4)–(2.7) contain four unknown quantities: the displacements on both crack faces, \( u^{(1)}(x, t) \) and \( u^{(2)}(x, t) \), and the displacement and the traction on the interface, \( u^*(x, t) \) and \( p^*(x, t) \). This system can be solved numerically by standard computational methods.

3. Validation of the method

In order to validate the proposed method, the solutions of the derived system of the boundary integral equations are compared with the analytical solutions obtained for the static case by Mossakovskii & Rybka (1964).

Consider a penny-shaped crack of the radius \( a \) under the normally incident tension–compression wave of the unit intensity and the normalized wavenumber close to zero \( (ao/c_1^{(1)}) = 0.001 \). The crack is located between steel and aluminium.

Figure 1. Maximal normal traction at the section \( S \).

Figure 2. Maximal tangential traction at the section \( S \).
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half-spaces with the following mechanical properties: \( E^{(1)} = 207 \text{ GPa} \) and \( E^{(2)} = 70 \text{ GPa} \), \( \nu^{(1)} = 0.288 \) and \( \nu^{(2)} = 0.347 \), \( \rho^{(1)} = 7860 \text{ kg m}^{-3} \) and \( \rho^{(2)} = 2700 \text{ kg m}^{-3} \).

The problem was solved numerically by the method of boundary elements. The maximal in time normal and tangential traction at the section of the interface \( S = \{ x_2 = 0, x_3 = 0 \} \) are given in figures 1 and 2, where solid lines refer to the static solution by Mossakovskii & Rybka (1964), and circles refer to the results of the current analysis. The obtained numerical results are in very good agreement with the analytical solution.

4. Conclusions

The paper concerns the application of boundary integral equations to the problem of an interface crack between two dissimilar elastic half-spaces under dynamic loading. The derived system of equations allows the evaluation of the displacements at the crack faces, and the traction and the displacements at the interface.

In reality the opposite crack faces interact with each other under dynamic loading. Under deformation of the material, the initial contact area will change in time. The shape of the contact area is unknown beforehand and must be determined as a part of the solution. The complexity of the problem is further compounded by the fact that the contact behaviour is very sensitive to the material properties of two contacting surfaces and the type of external loading (Menshykov & Guz 2006, 2007). Therefore considering this interaction would be the next natural stage of this research.

References

Ugodchikov, A. G. & Khutoryansky, N. M. 1986 *Boundary elements method in mechanics of deformable solid*. Kazan, Russia: Izd. Kazan University. [In Russian.]