Wave breaking and shock waves for a periodic shallow water equation

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This paper is devoted to the study of a recently derived periodic shallow water equation. We discuss in detail the blow-up scenario of strong solutions and present several conditions on the initial profile, which ensure the occurrence of wave breaking. We also present a family of global weak solutions, which may be viewed as global periodic shock waves to the equation under discussion.

Keywords: wave breaking; shock waves; peakons; shallow water equation

1. Introduction

The nonlinear partial differential equation
\[ u_t - u_{txx} + 4u u_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R} \tag{1.1} \]
was recently derived by Degasperis & Procesi (1999) as one of the three equations within the family
\[ m_t + um_x + bu_x m + \gamma u_{xxx} = c_0 u_x, \tag{1.2} \]
satisfying asymptotic integrability, a necessary condition for complete integrability. In equation (1.2), the quantity \( m = u - \alpha^2 u_{xx} \) stands for a momentum variable with a positive parameter \( \alpha \). Also \( c_0 \) and \( \gamma \) are positive constants and \( b \) is an arbitrary real parameter. Setting \( b=3 \) in equation (1.2), rescaling the independent variables, shifting the dependent variable and applying a Galilean transformation of the form \( x \mapsto x + ct \), we get, in fact, equation (1.1).

The other two cases are the Camassa–Holm (CH) equation
\[ u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.3} \]
obtained similarly for \( b=2 \), and the Korteweg–de Vries (KdV) equation
\[ u_t - 6uu_x + u_{xxx} = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{1.4} \]
which follows formally from equation (1.2) by letting \( \alpha \to 0 \). Each of the above equations (1.1), (1.3) and (1.4) models the unidirectional irrotational free surface flow of a shallow layer of an inviscid fluid moving under the influence of gravity over a flat bed. In these models, \( u(t,x) \) represents the wave’s height above the flat bottom, \( x \) is proportional to the distance in the direction of propagation and \( t \) is proportional to the elapsed time.

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It is well known that the KdV equation is a completely integrable system and that its solitary waves are solitons (McKean 1979; Drazin & Johnson 1989). The Cauchy problem of the KdV equation has been the subject of many studies, and a satisfactory theory is now available. A remarkable result of Tao (2002) says that the KdV equation is globally well posed for initial data belonging to $L^2(\mathbb{R})$ (see also Kenig et al. 1993). It is further known that the KdV equation does not accommodate the phenomenon of wave breaking (by wave breaking, we understand that the wave remains bounded on the whole interval of existence, but its slope becomes unbounded as time approaches the finite positive blow-up time). The lack of wave breaking may be viewed as a certain shortcoming of the KdV equation, as already remarked implicitly 30 years ago by Whitham: ‘Although both breaking and peaking, as well as criteria for the occurrence of each, are without doubt contained in the equations of the exact potential theory, it is intriguing to know what kind of simpler mathematical equation could include all these phenomena’ (cf. Whitham 1974). Of course, several shallow water equations comprising blow-up phenomena have been discussed so far. But, either these equations are not integrable or they do not have solitons or peakons.

Against this background, it becomes clear that the CH equation gave a considerable impetus to the theory, since this equation shows a variety of important properties. In fact, Camassa & Holm (1993) discovered that it is formally integrable (in the sense that there is an associate Lax pair), and that its solitary waves are peakons, i.e. solitons that have a peak at their crest (see also Constantin & Strauss 2000; Constantin & Molinet 2001). The CH equation was actually obtained much earlier by Fokas & Fuchssteiner (1981) as a bi-Hamiltonian system with infinitely many conservation laws. After the work of Camassa & Holm (1993), it was shown by Constantin & McKean (1999) and Constantin (2001) that for a large class of initial data, the CH equation is in fact an integrable infinite-dimensional Hamiltonian system (see also Beals et al. 2000). It is quite striking that the CH equation not only possesses global classical solutions but also models wave breaking. Indeed, while certain initial profiles develop into global waves (Constantin & Escher 1998a), others lead to wave breaking in finite time (cf. Constantin 1997; Constantin & Escher 1998b). Moreover, it is known that wave breaking is the only way in which solutions to the CH equation may develop a singularity (cf. Constantin & Escher (2000) for the periodic case and Constantin (2000) for the case of solutions decaying at infinity).

The existence of both peakons and wave breaking asks for the notion of a weak solution to the CH equation. The first approach to study weak solutions is based on the observation that equation (1.3) can be reformulated as the non-local conservation law (cf. Constantin & Escher 1998c; Constantin & Molinet 2000),

$$u_t + \left[ \frac{u^2}{2} + (1 - \partial_x^3)^{-1} \left( u^2 + \frac{u_x^2}{2} \right) \right]_x = 0. \quad (1.5)$$

It is worthwhile to mention that the weak solutions constructed in the latter papers are unique in the very same class of functions, in which existence is shown.

Thereafter, Xin & Zhang constructed weak solutions to equation (1.3) as weak limits of classical solutions of a viscous regularized version of equation (1.3) (cf. Xin & Zhang 2000, 2002). However, uniqueness of these solutions can only be guaranteed if a suitable assumption on the initial data is fulfilled, which presently rules out wave breaking of classical solutions. We also refer to the recent papers of Wahlen (2006a, b) for a discussion of aspects related to the work of Xin & Zhang (2000, 2002).
A third approach to weak solutions to the CH equations was recently proposed by Bressan & Constantin (2007). Introducing a new set of independent and dependent variables, it is possible to convert the CH equation into a semilinear system, which can be solved globally for all initial data belonging to $H^1$. Returning to the original variables, one gets a global weak solution, which is conservative in the sense that the total energy is preserved for almost every time.

Degasperis et al. (2002) proved that the Degasperis–Procesi equation is formally integrable by constructing a Lax pair for equation (1.1). It is also shown in Degasperis et al. (2002) that equation (1.1) has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and that it admits exact peakon solutions which are analogous to the Camassa–Holm peakons. The DP equation can also be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH equation (cf. Johnson 2002). Moreover, Dullin et al. (2001) showed that the DP equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation.

Despite the above-mentioned common properties of the CH and DP equations, these two equations are truly different. One of the most important features of equation (1.1) is the fact that it has not only peaked solitons (Degasperis et al. 2002), i.e. solutions of the form $u_c(t, x) = c \exp(-|x - ct|)$, with $c > 0$, but also shock waves of the form (cf. Coclite et al. submitted; Lundmark in press)

$$u_c(t, x) = -\frac{1}{t + c} \text{sgn}(x) \exp(-|x|), \quad c > 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (1.6)$$

Another significant difference becomes clear by comparing the isospectral problem in the Lax pair for both equations. Letting again $m := u - u_{xxx}$, we have the third-order equation

$$\psi_x - \psi_{xxx} - \lambda m \psi = 0 \quad (1.7)$$

as the isospectral problem for equation (1.1) (cf. Degasperis et al. 2002), while in the case of equation (1.3), the second-order equation

$$\psi_{xx} - \frac{1}{4} \psi - \lambda m \psi = 0 \quad (1.8)$$

is the relevant spectral problem in the Lax pair (cf. Camassa & Holm 1993).

In this paper, we study the periodic initial value problem for the Degasperis–Procesi equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

$$u(t, x + 1) = u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}. \quad (1.9)$$

We first investigate the blow-up scenario of classical solutions to equation (1.9). Based on a new conservation law, we show that the first singularity of classical solutions must occur as a wave breaking, and shock waves possibly appear afterwards. Then, we present several criteria which imply that a strong solution produces wave breaking in finite time. In particular, we show that any solution emerging from an odd initial data blows up in finite time. This is in clear contrast to the situation on the line, because it is shown in Escher et al. (2006) that there are global classical solutions with odd initial data.
Second, we introduce the notion of weak solutions to equation (1.9) and we show that for any \( c > 0 \), the function

\[
  u_c(t, x) = \begin{cases} 
    \frac{\sinh(x - [x] - 1/2)}{t \cosh(1/2) + c \sinh(1/2)}, & x \in \mathbb{R} \setminus \mathbb{Z}, \quad c > 0, \\
    0, & x \in \mathbb{Z}
  \end{cases}
\]

is a global weak solution such that \( u_c \in L_\infty(\mathbb{R}_+ \times \mathbb{S}) \) and

\[
  \lim_{x \to 0^+} u_c(t, x) = - \lim_{x \to 0^-} u_c(t, x) = - \frac{1}{t \coth(1/2) + c}.
\]

Thus, \( u_c(t, x) \) may be viewed as a shock wave for problem (1.9).

2. Wave breaking of strong solutions

Throughout the remainder of this note, we identify 1-periodic functions with functions over the circle \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \). Moreover, given \( s \in \mathbb{R} \), we write \( H^s(\mathbb{S}) \) for the classical Sobolev spaces over \( \mathbb{S} \).

Using Kato’s theory on quasilinear \( m \)-accretive evolution equations, it can be shown that, given \( u_0 \in H^s(\mathbb{S}) \) with \( s > 3/2 \), there is a unique maximal \( T = T(u_0) > 0 \) and a strong solution

\[
  u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}))
\]

of the initial value problem (1.9) (cf. Yin 2004).

In order to describe the blow-up scenario of strong solutions, we first derive an \( a \) \textit{priori} bound of the \( L_\infty \)-norm of strong solutions. In order to do so, let

\[
  y(t, x) := (1 - \partial_x^2) u(t, x), \quad v(t, x) := (4 - \partial_x^2)^{-1} u(t, x),
\]

for \( (t, x) \in [0, T) \times \mathbb{S} \) and define

\[
  E_2(u(t)) := \int_{\mathbb{S}} y(t, x)v(t, x)dx \quad \text{for} \quad t \in [0, T).
\]

Using equation (1.9), integration by parts shows that \( dE_2(u(t))/dt \equiv 0 \), i.e. \( E_2 \) is a conservation law for the DP equation. From this conservation law, we easily get an \( L_2 \)-bound on \( u \). In fact, Parseval’s formula yields

\[
  E_2(u(t)) = 4 \sum_{n=-\infty}^{\infty} \frac{1 + 4\pi^2 n^2}{4 + 4\pi^2 n^2} |\hat{u}_n(t)|^2, \quad t \in [0, T),
\]

where \( \hat{u}_n(t) \) stands for the \( n \)th Fourier mode of \( u(t, \cdot) \). Thus,

\[
  \|u(t)\|_{L_2}^2 = \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 \leq 4 \sum_{n=-\infty}^{\infty} \frac{1 + 4\pi^2 n^2}{4 + 4\pi^2 n^2} |\hat{u}_n|^2 \leq 4E(u(t)) = 4E(u(0)) = 4 \sum_{n=-\infty}^{\infty} \frac{1 + 4\pi^2 n^2}{4 + 4\pi^2 n^2} |(\hat{u}_0)_n|^2
\]

\[
  \leq 4 \sum_{n=-\infty}^{\infty} |(\hat{u}_0)_n|^2 = 4\|u_0\|_{L_2}^2 / 4.
\]

Similarly, one shows that \( \|u(t)\|_{L_2}^2 \geq \|u_0\|_{L_2}^2 / 4 \).

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Let \( G(x) := (\cosh(x - [x] - (1/2))/(2 \sinh(1/2)) \) denote the kernel of \((1 - \partial_x^2)^{-1}\), i.e. \((1 - \partial_x^2)^{-1} f = G * f\) for all \( f \in L^2(\mathbb{S})\). Then, a direct calculation shows that
\[
0_t + uu_x = -\partial_x G \ast \left( \frac{3}{2} u^2 \right) = -3G \ast (uu_x).
\]

On the other hand, using Lagrangian coordinates \( q(t, x) \), the material derivative \(0_t + uu_x\) of \(u\) is given by
\[
(u_t + uu_x)(t, q(t, x)) = \frac{du(t, q(t, x))}{dt},
\]
where \( q(\cdot, x) \) is the solution of the initial value problem
\[
\begin{align*}
q_t &= u(t, q), \quad t \in [0, T), \\
q(0, x) &= x, \quad x \in \mathbb{R}.
\end{align*}
\]
Using explicit estimates of the kernel \( G \), one can show that
\[
\left| \frac{du(t, q(t, x))}{dt} \right| \leq \frac{3\kappa}{4} \|u(t, x)\|_{L^2}^2,
\]
where \( \kappa = \coth(1/2) \) (cf. Escher et al. in press). Integrating the above inequality and recalling equation (2.1), we get
\[
-3\kappa \|u_0\|_{L^2}^2 t + u_0(x) \leq u(t, q(t, x)) \leq 3\kappa \|u_0\|_{L^2}^2 t + u_0(x).
\]
It is not difficult to see that, given \( t \in [0, T) \), the mapping \( q(t, \cdot) \) is an increasing diffeomorphism on \( \mathbb{R} \). Thus, we conclude that
\[
\|u(t, \cdot)\|_{L^s} = \|u(t, q(t, \cdot))\|_{L^s} \leq 3\kappa \|u_0\|_{L^2}^2 t + \|u_0\|_{L^s}, \quad t \in [0, T).
\]
This shows that any strong solution stays bounded on bounded time-intervals, so that blow-up must occur in higher derivatives. On the other hand, letting \( \delta := s - 3/2 \) and \( \gamma := 1/2 - 2\delta \), theorem 2.7.1 in Triebel (1983) implies the following embeddings for the natural phase space \( C([0, T), H^s(\mathbb{S})) \):
\[
C([0, T], C^{1+\gamma}(\mathbb{S})) \hookrightarrow C([0, T), H^s(\mathbb{S})) \hookrightarrow C([0, T), C^{1+\delta}(\mathbb{S})).
\]
This means that a \( C^{1+\gamma} \-a \text{ priori} \) bound prevents blow-up phenomena in finite time and suggests that blow-up must occur as wave breaking, i.e. in the first derivative. This is in fact true, as the following result shows.

Theorem 2.1 (Escher et al. 2006). Given \( u_0 \in H^s(\mathbb{S}) \) with \( s > 3/2 \), blow-up of the strong solution \( u = u(\cdot, u_0) \) in finite time \( T < +\infty \) occurs if and only if
\[
\liminf_{t \uparrow T} \left\{ \min_{x \in \mathbb{S}} [u_x(t, x)] \right\} = -\infty.
\]
In this case, we have
\[
\lim_{t \to T} \left( \min_{x \in \mathbb{S}} [u_x(t, x)](T - t) \right) = -1,
\]
while the solution \( u \) remains bounded.
It is worthwhile to mention that the blow-up rate of the periodic DP equation is $-1$, whereas the blow-up rate for the periodic CH equation is $-2$ (cf. Constantin & Escher 2000).

We conclude this section by presenting several sufficient conditions on the initial data which imply wave breaking of strong solutions.

**Theorem 2.2 (Escher et al. 2006).** Assume that $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$ is not constant.

(a) If $u_0$ is even or if

$$\left|\min_{x \in \mathbb{S}} u_0'(x)\right| \geq \left|\max_{x \in \mathbb{S}} u_0'(x)\right|,$$

then for sufficiently large $n$, the solution to problem (1.9) with initial data $u_0(nx)$ blows up in finite time.

(b) If $u_0$ is odd or if $\int_{\mathbb{S}} u_0(x)dx = 0$ or $\int_{\mathbb{S}} u_0^3(x)dx = 0$, then the corresponding solution to problem (1.9) blows up in finite time.

Part (a) of the above theorem is the consequence of a more general result saying that if the slope of the initial data is sufficiently negative, then wave breaking occurs. It can also be shown that the steeper the slope of the initial data is, the quicker the solution blows up.

The second part of theorem 2.2 is based on the fact that if a solution has a zero at any instant, then wave breaking must occur. Since each of the conditions in (b) is preserved in time, the result follows straightforwardly.

3. Shock waves

In this section we will present an example of a periodic shock wave to problem (1.9). One of the important features of the Degasperis–Procesi equation on the line is that it has not only peakons (Degasperis et al. 2002), i.e. solutions of the form

$$u_c(t, x) = ce^{-|x-ct|}, ~ c > 0,$$

but also shock waves (Lundmark in press), which are given by

$$u_c(t, x) = -\frac{1}{t+c} \text{sgn}(x)e^{-|x|}, ~ c > 0.$$

It is known (cf. Yin 2004) that the Degasperis–Procesi equation on the circle admits periodic peakons of the form

$$u_c(t, x) = c \frac{\cosh(x-ct-[x-ct]-\frac{1}{2})}{\sinh(\frac{1}{2})}, ~ x \in \mathbb{R}, ~ t \geq 0.$$

In fact, the above peakons can be obtained as the most natural extension of $u_c(t, x) = ce^{-|x-ct|}, c > 0$, to a periodic function on $\mathbb{R}$ (Constantin & Escher 1998d),

$$\sum_{n=-\infty}^{\infty} ce^{-|x-ct+n|} = c \frac{\cosh(x-ct-[x-ct]-\frac{1}{2})}{\sinh(\frac{1}{2})}, ~ x \in \mathbb{R}, ~ t \geq 0, ~ c > 0.$$
If we mimic this procedure for shock waves, we get

\[
\sum_{n=-\infty}^{\infty} - \frac{1}{t + c} \text{sgn}(x + n)e^{-|x+n|} \\
= - \frac{1}{t + c} \left( \sum_{n=\infty}^{-|x|} \text{sgn}(x + n)e^{-|x+n|} + \sum_{n=-|x|}^{\infty} \text{sgn}(x + n)e^{-|x+n|} \right) \\
= - \frac{1}{t + c} \left( - \sum_{n=-\infty}^{-|x|} e^{x+n} + \sum_{n=-|x|}^{\infty} e^{-x-n} \right) \\
= - \frac{1}{t + c} \left( \frac{e^{x-[x]-1}}{1-e^{-1}} + \frac{e^{-x+[x]}}{1-e^{-1}} \right) = - \frac{1}{t + c} \frac{\sinh(x-[x]-\frac{1}{2})}{\sinh(\frac{1}{2})}, \quad x \in \mathbb{R}\setminus\mathbb{Z}.
\]

Surprisingly, none of the functions of the family

\[
\hat{u}_c(t, x) = \begin{cases} 
\frac{\sinh(x-[x]-\frac{1}{2})}{(t+c)^{-1} \sinh(\frac{1}{2})}, & x \in \mathbb{R}\setminus\mathbb{Z}, \quad t \geq 0, \\
0, & x \in \mathbb{Z}, \quad t \geq 0,
\end{cases}
\]

with \(c>0\), is a weak solution to equation (1.9) in the following sense.

**Definition 3.1.** A function \(u_c \in L_\infty(\mathbb{R}_+ \times \mathbb{S})\) satisfying the identity

\[
\int_{\mathbb{R}_+} \int_{\mathbb{S}} \left(u\psi_t + \left[u^2/2 + G * (3u^2/2)\right]\psi_x\right) dx dt + \int_{\mathbb{S}} u_0(x)\psi(0, x) dx = 0
\]

for all \(\psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{S})\) is called a global weak solution to equation (1.9).

However, the following result is true.

**Theorem 3.1.** Given \(c>0\), let

\[
u_c(t, x) = \begin{cases} 
\frac{\sinh(x-[x]-1/2)}{t \cosh(1/2) + c \sinh(1/2)}, & x \in \mathbb{R}\setminus\mathbb{Z}, \quad t \geq 0, \\
0, & x \in \mathbb{Z}, \quad t \geq 0.
\end{cases}
\]

Then \(u_c\) is a global weak periodic solution of equation (1.9).

For the proof of theorem 3.1, we refer to Escher et al. (2006).
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References


