Nonlinear instability of developing streamwise vortices with applications to boundary layer heat transfer intensification through an extended Reynolds analogy

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The intent of the present contribution is to explain theoretically the experimentally measured surface heat transfer rates on a slightly concave surface with a thin boundary layer in an otherwise laminar flow. As the flow develops downstream, the measured heat transfer rate deviates from the local laminar value and eventually exceeds the local turbulent value in a non-trivial manner even in the absence of turbulence. While the theory for steady strong nonlinear development of streamwise vortices can bridge the heat transfer from laminar to the local turbulent value, further intensification is attributable to the transport effects of instability of the basic steady streamwise vortex system. The problem of heat transport by steady and fluctuating nonlinear secondary instability is formulated. An extended Reynolds analogy for Prandtl number unity, $Pr=1$, is developed, showing the similarity between streamwise velocity and the temperature. The role played by the fluctuation-induced heat flux is similar to momentum flux by the Reynolds shear stress. Inferences from the momentum problem indicate that the intensified heat flux developing well beyond the local turbulent value is attributable to the transport effects of the nonlinear secondary instability, which leads to the formation of ‘coherent structures’ of the flow. The basic underlying pinions of the non-linear hydrodynamic stability problem are the analyses of J. T. Stuart, which uncovered physical mechanisms of nonlinearities that are crucial to the present developing boundary layers supporting streamwise vortices and their efficient scalar transporting mechanisms.

**Keywords:** nonlinear hydrodynamic instability; secondary instability; streamwise vortices; Reynolds analogy; heat transfer intensification in boundary layers

1. Introduction

The practical interest in intensifying surface heat transfer rates with the least penalty follows the need to reduce energy consumption via more efficient systems for energy transfer as well as the reduction in size of energy transfer equipment, such

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as a possible new generation of more efficient compact, but still macroscopic, heat exchangers. Vortices and heat transfer have been under study, with leading examples discussed by Fiebig (1996). The vortex generation process almost always involves insertion of generators of winglet type, which causes additional drag or pressure drop penalty to the system; a doubling of relative heat transfer enhancement is accompanied by an almost quadrupling of relative drag penalty (Fiebig 1996). In hydrodynamic instability generated streamwise vortices, the relative heat transfer enhanced to relative drag rise has a ratio of one to one. In this case, within certain streamwise Reynolds number ranges there exists an opportunity to enhance heat transfer for eventual practical exploitation. In this paper, I am dealing only with the fundamental fluid dynamical and heat transport problems.

This work addresses the effect of streamwise vortices on surface heat transfer rates in boundary-layer flow over slightly curved surfaces. These issues evolve out of an assemblage of past and current experimental measurements. Although no flow structures were measured, Thomann (1968) showed that surface heat transfer rates in a supersonic turbulent boundary layer over a concave wall are increased by approximately 20% over that of a flat wall. His experiments compensated for the pressure gradient so that the increase is due to surface curvature alone. He attributes this to the possible presence of counter rotating streamwise vortices of the Görtler (1940) type, which is an open flow extension of Taylor vortices. It was known at that time that Görtler vortices could also be present in turbulent boundary layers over concave walls (Tani 1962; So & Mellor 1975) and that there is a case for the existence of such vortices in supersonic turbulent boundary layers (Girgis & Liu 2000) comes as no surprise.

McCormack et al. (1970) measured significant increases in surface heat transfer rates over that of the flat plate under Görtler vortices in the downstream region. Again, simultaneous flow field measurements or observations are absent. Kottke (1986) measured heat as well as mass transfer rates on a concave wall. He inserted a variety of upstream grids and raised several interesting issues concerning the dependence on spanwise wavelength and initial disturbance levels and these are in accordance with theoretical estimates (Liu & Sabry 1991). Measurements described by Crane (1991) also show that the local surface heat transfer rates can exceed well above the flat plate values under actively developing streamwise vortices in their nonlinear region.

Overall heat transfer rates are spanwise averaged and thus a linear theory (McCormack et al. 1970), which is spanwise sinusoidal, would produce no net enhancement. In this case, the enhanced heat transfer measured must be under strongly nonlinear Görtler vortex development. It is now understood, if only from the simplified steady flow nonlinear consideration, that the enhancement comes from a predominant spanwise region about the downwash for which the temperature gradient is large and a lesser lower temperature gradient region about the upwash (Liu & Sabry 1991; Liu & Lee 1995). The steady heat transfer problem is linear under the incompressible flow assumption, but heat is transported by the velocities of the uncoupled but nonlinear momentum problem (Lee & Liu 1992). This transport, in turn, produces the spanwise unbalance of heat flux towards the wall.

More recent measurements (Momayez et al. 2004; Toé et al. 2002) indicate that the local surface heat transfer rate develops from the local laminar value to one that is significantly above the local turbulent value. This is analogous to Taylor vortex
flow, where the torque needed to drive the inner rotating cylinder increases beyond the laminar value as nonlinearity becomes important (Stuart 1963). Recently, Girgis & Liu (2006) discussed in detail the mechanics of nonlinear instability of developing streamwise vortices in the boundary layer. They showed that the skin friction attributable to the steady mean vortex flow can develop from the laminar value to that of the turbulent value on the basis of a single, most amplified spanwise mode corresponding to conditions of Swearingen & Blackwelder’s (1987) experiment. The contribution to skin friction rises beyond the local turbulent value, in the absence of turbulence, is attributed to the nonlinearities of the secondary instability (they only considered the more vigorous sinuous mode indicated by experiments). The physical mechanism is the transport of mean momentum towards the wall by the Reynolds stress of the nonlinear fluctuations. Again, the focus here is placed on the dominance of the fundamental mode (Meksyn & Stuart 1951; Stuart 1958) and the Reynolds stress mechanisms in nonlinear hydrodynamic stability (Stuart 1956a, b) that are found to be valid in numerous nonlinear instability problems in parallel and developing flows.

Owing to the intimate connection between momentum and heat transfer, what is known from momentum transfer is of great interest especially in view of the consequences of an extended Reynolds analogy that will unfold in the following. The heat transfer problem, which is linear for an incompressible flow since the nonlinear momentum and heat transfer problems are entirely uncoupled, is formulated side by side with the nonlinear momentum problem in §2, where a discussion of Reynolds average procedures appropriate to the problem precedes the statement of basic equations for the mean flow and for the nonlinear fluctuations. Physical interpretation of the nonlinear secondary instability development is given in §3. An outline of computing the simultaneous development of the mean flow and secondary instability is discussed in §4. The similarity between fluctuation streamwise velocity and temperature for Prandtl number unity is shown to be valid approximately from a detailed consideration of fluctuation quantities before averaging is discussed in §5, thus leading to the similarity between mean quantities such as the Reynolds stresses and heat fluxes and the Reynolds analogy between skin friction and heat transfer. The mechanism for heat transfer intensification is discussed in §6; this allows the use of previously computed momentum problem to be inferred for interpretation of experimental measurements carried out in air for which Prandtl number unity is a good approximation. Concluding remarks and relation to other areas of research are given in §7 as well as to possible extensions of the heat transfer problem discussed.

2. Heat transport via advection by nonlinear instability of streamwise vortices

The basic steady streamwise vortex flow was recognized by Floryan & Saric (1982) and Hall (1983, 1988) to be an upstream initial-value problem, which is parabolic in nature. This is in contrast to the linear eigenvalue problem posed originally by Görtler (1940). Day et al. (1990) showed that the local eigenvalue results for growth rates could mimic that from the linearized marching solution for steady Görtler vortices with the conclusion that the differences are modest (in the linear
region). The discussions of a neutral curve in this class of problems may be academic; the real basic issue is the receptivity problem on how disturbances would develop from upstream disturbances. With the inclusion of time-dependent secondary instability, some aspects of retaining parabolicity for the time-dependent fluctuations, which are coupled to the mean three-dimensional streamwise vortex flow through the Reynolds stresses, are discussed by Girgis & Liu (2002, 2006). I refer to Saric (1994) for a review of some basic issues.

Whether in measurements or in numerical simulations, total flow quantities are the results of such endeavours. However, averaging procedures, depending on the nature of oscillations, are essential in order to reduce aspects of the flow for purposes of physical understanding. Essential comments about averaging procedures are made first, followed by stating the simplified equations for the mean flow and fluctuations. The basis for simplification is discussed thoroughly in §3.

(a) Spatial averaging and time averaging

In the early work in dealing only with the steady (nonlinear) spatial development of streamwise vortices, the spanwise average is the appropriate ‘Reynolds average’ (Liu & Sabry 1991; Sabry & Liu 1991; Lee & Liu 1992; Liu & Lee 1995). In this case, the steady but three-dimensional boundary layer has steep normal-to-wall gradients of the dominating streamwise velocity in the downwash region between the two adjacent counter rotating streamwise vortices, and inflectional velocity profiles and high shear layers in the upwash region. The three dimensionality also brought into the picture important spanwise gradients of the mean streamwise velocity in terms of mean normal vorticity, thereby providing additional sources of wavy instability.

In the presence of time-dependent fluctuations, such as the wavy secondary instability (or turbulence), the time average, over the largest wave period, is the appropriate Reynolds average. In this case, the mean flow is time independent but three dimensional. The mean flow conservation equations appear in a similar form as those for turbulent shear flow, except that the fluctuations, prior to averaging, are calculated from nonlinear instability considerations jointly with the mean flow that they modify.

In numerical simulation as well as measurements, total signals are computed and measured, respectively. Thus, averaging appropriately brings out features and physical understanding, such as interactions and energy transfer between fluctuations and the mean flow.

In the steady flow problem (e.g. Lee & Liu 1992), the spanwise average consequences for heat transfer are described by Liu & Lee (1995), which aided the understanding of the interactions between the basic two-dimensional flow and its modification by the stresses and heat fluxes of the three-dimensional component of streamwise vortex flow.

In earlier studies of stability of parallel flows, the unstable fluctuations are periodic in the streamwise direction and the growth is in time. In this case, the Reynolds average is the spatial average over the spatial periodicity (Meksyn & Stuart 1951; Stuart 1962a, b).
Formulation of the heat transport problem

The incompressible flow assumption is made, and for which thermal convection is not important compared with inertia effects. The incompressible form of the fundamental equations can best be understood if one begins with the full compressible form subjected to a double expansion of two small parameters, one is $M^2 / T$ and the other is the relative temperature loading $D = T / T_0$, where $M$ is the free stream Mach number and $T$ is the static temperature. I refer to Panton (2005) for formal general discussions. The uncoupled momentum problem is nonlinear and is solved first; the heat transfer problem is thus linear in the incompressible approximation.

In the present class of problems, the temperature is indeed a passive scalar, advected by the active streamwise vortices and their nonlinear instability. Thus, it is essential that the basic equations of momentum and heat transport problem be considered together, even for the incompressible flow assumed.

To study the development of heat transport under time-dependent nonlinear instabilities, I begin with the Reynolds splitting procedure by representing the total flow quantity in terms of the sum of a steady mean component $q$, obtained by time averaging, and the fluctuation component $q'$ which has zero mean,

$$q + q',$$

much in the same way as in studies of turbulent shear flows. The developing mean boundary-layer equations in dimensionless form are

$$\text{continuity : } \nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

$$x\text{-momentum : } \mathbf{u} \cdot \nabla u = (Re)^{-1}\nabla^2_{yz} u - \nabla \cdot \mathbf{u}' \mathbf{w}', \quad (2.2)$$

$$y\text{-momentum : } \mathbf{u} \cdot \nabla v + u^2 / r_c = -p_y + (Re)^{-1}\nabla^2_{yz} v - \nabla \cdot \mathbf{u}' \mathbf{v}', \quad (2.3)$$

$$z\text{-momentum : } \mathbf{u} \cdot \nabla w = -p_z + (Re)^{-1}\nabla^2_{yz} w - \nabla \cdot \mathbf{u}' \mathbf{w'}, \quad (2.4)$$

$$\text{energy : } \mathbf{u} \cdot \nabla \theta = (RePr)^{-1}\nabla^2_{yz} \theta - \nabla \cdot \mathbf{u}' \theta'. \quad (2.5)$$

Length scales are normalized by $\delta_0 = \sqrt{vX_0 / U_0}$, where $v$ is the kinematic viscosity; $U_0$ is the free stream velocity; and $X_0$ is the location where the initial steady, weak Görtler vortex is inserted into the flow. The dimensionless coordinates are $x$, $y$ and $z$ in the streamwise, normal-to-wall and spanwise directions, respectively; the corresponding mean velocity components are $\mathbf{u}(u, v, w)$ and are normalized by $U_0$; the fluctuating velocity components are $\mathbf{u}'(u', v', w')$; the pressure $p$ is normalized by the free stream dynamical pressure; the Reynolds number is $Re = U_0 \delta_0 / v$; the Prandtl number is $Pr = v / \alpha$; and $\alpha$ is the thermal diffusivity. The concave plate radius is $R$, its dimensionless form is $r_c = R / \delta_0$. The dimensionless mean temperature is defined as $\theta = (T - T_W)/(T_0 - T_W)$, where $T_0$ is the free stream and $T_W$ is the wall temperature; the dimensionless fluctuation temperature is $\theta' = T'/(T_0 - T_W)$. The over bar denotes the time average over the largest period.

1 Lee & Liu (1992) showed that the downstream flow development is insensitive to $X_0$ provided that the initiating disturbance is consistent with the local mean flow conditions.

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in the flow. The Reynolds number range is assumed to be sufficiently large so as a Blasius laminar boundary layer exists upstream, and the boundary-layer thickness is small compared with the concave wall radius. The mean counter rotating vortex flow that subsequently develops is long and slender in the streamwise extent with strong streamwise velocity and relatively weak cross-sectional velocities, thus leading to the parabolized Laplacian in the cross-sectional $yz$-plane, $\nabla^2_{yz}$. The curvature effects appear only in the $y$-momentum equation (Floryan & Saric 1982; Hall 1983) as envisioned in the parallel flow formulation of Göltler (1940). By choice, the mean flow is here renormalized by the same scale as for the fluctuation equations to follow for convenience in the joint calculation of flow development downstream.

The boundary and upstream initial conditions for the mean flow are

$$\begin{align*}
y \to \infty : & \quad u = 1, \quad v_y = 0, \quad w = 0, \quad \theta = 1 \\
y = 0 : & \quad u = v = w = 0, \quad \theta = 0 \\
\mathbf{u}(x, y, 0) = & \quad \mathbf{u}(x, y, \lambda_z), \quad \theta(x, y, 0) = \theta(x, y, \lambda_z) \\
\mathbf{u}(x_0, y, z) = & \quad \mathbf{u}_G, \quad \theta(x_0, y, z) = \theta_G
\end{align*}$$

(2.6)

where $\lambda_z$ is the spanwise wavelength of the steady streamwise vortex flow; $x_0$ is the location where the weak Göltler vortex $\mathbf{u}_G$ is introduced; and $\theta_G$ is the Göltler vortex-advected mean temperature. The steady mean flow equations in the absence of secondary instability are essentially those used to compute spatial development of mushroom-like structures in terms of the total steady streamwise velocity (Lee & Liu 1992). In the presence of nonlinear instability, they are now augmented by the Reynolds stresses and heat flux, $-\mathbf{u}'\mathbf{u}'$ and $-\mathbf{u}'\theta'$, respectively, as would be expected from studies of nonlinear hydrodynamic stability of parallel shear flows (Stuart 1956a,b).

The developing fluctuation equations are

$$\begin{align*}
\text{continuity} : & \quad \nabla \cdot \mathbf{u}' = 0, \\
x\text{-momentum} : & \quad u'_t + uu'_x + v' u_y + w' u_z = -p'_x + (Re)^{-1} \nabla^2 u' + \nabla \cdot (\mathbf{u}' \mathbf{u}' - \mathbf{u}' u'), \\
y\text{-momentum} : & \quad v'_t + vv'_x = -p'_y + (Re)^{-1} \nabla^2 v' + \nabla \cdot (\mathbf{u}' \mathbf{u}' - \mathbf{u}' v'), \\
z\text{-momentum} : & \quad w'_t + ww'_x = -p'_z + (Re)^{-1} \nabla^2 w' + \nabla \cdot (\mathbf{u}' \mathbf{u}' - \mathbf{u}' w'), \\
\text{energy} : & \quad \theta'_t + uu'_x + v' \theta_y + w' \theta_z = (RePr)^{-1} \nabla^2 \theta' + \nabla \cdot (\mathbf{u}' \theta' - \mathbf{u}' \theta'),
\end{align*}$$

(2.7)–(2.11)

where subscripts indicate partial differentiation. The centrifugal force exerts its influence on the large-scale mean flow, but does not affect the short-scale secondary instability to leading order (Hall & Horseman 1991; Yu & Liu 1991). Other simplifications leading to (2.7)–(2.11) are more appropriately discussed under physical mechanisms in §3.

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The boundary and initial conditions for the fluctuations are
\[
\begin{aligned}
y \to \infty : & \quad q' \to 0 \\
y = 0 : & \quad q' = 0 \\
q'(x, y, 0) = q'(x, y, \lambda z) \\
q'(x_0, y, z) = q'_\text{linear}
\end{aligned}
\]
(2.12)
where \( q' = q'(u', v', w', \theta') \) and \( q'_\text{linear} \) is a local linear solution.

3. Discussion of physical mechanisms of developing nonlinear secondary instability

Owing to the near isotropy of the wavy secondary instability and the boundary-layer-like behaviour of the mean flow, advection is dominated by the streamwise velocity alone in the advection operator
\[
\frac{\partial}{\partial t} + u(x, y, z) \frac{\partial}{\partial x},
\]
where \( u(x, y, z) \) is in turn modified by the Reynolds stresses of the nonlinear fluctuations.

The dominant hydrodynamic instability sources are
\[
v' \frac{\partial u}{\partial y} + w' \frac{\partial u}{\partial z}.
\]
The first is the familiar normal-to-wall advection of the spanwise vorticity in parallel or nearly parallel shear flows (Stuart 1956a, b, 1958), which tends to reinforce travelling waves of an opportune phase relation as explained by Lin (1955); the second is the spanwise advection of normal-to-wall vorticity similar to the instability of a three-dimensional steady vortex flow (Sabry et al. 1990; Yu & Liu 1991, 1994). The three-dimensional nature of the streamwise velocity can be visualized locally as a two-dimensional one in the cross-sectional \( yz \)-plane bounded by the mushroom-like structure (e.g. Lee & Liu 1992). The high inflectional shear layer is provided by \( \partial u/\partial y \) in the upwash (\( \nu > 0 \)) region in the centre of the mushroom and, as we now know, predominantly supports the varicose mode of secondary instability (Sabry et al. 1990; Yu & Liu 1991; Souza et al. 2004); this is discussed and shown earlier by Stuart (1965) as a region supporting intense oscillations. The normal-to-wall mean vorticity, \( \partial u/\partial z \), provides strong inflectional shear layers when viewed in the normal-to-wall direction in the likes of a jet or a wake profile and this partially supports the sinuous mode (Sabry et al. 1990; Yu & Liu 1991). In terms of energy conversion from the mean flow to the fluctuations, there are now the dual mechanisms
\[
-\overline{u'v'} \frac{\partial u}{\partial y} \quad \text{and} \quad -\overline{u'w'} \frac{\partial u}{\partial z},
\]
brought about by the Reynolds stresses \( -\overline{u'v'} \) and \( -\overline{u'w'} \), which do work against the two rates of strain, \( \partial u/\partial y \) and \( \partial u/\partial z \), respectively.
The fluctuation temperature is simply a passive scalar, with scales imposed by the advection velocities. In terms of fluctuation temperature square (Tennekes & Lumley 1972), $\overline{\theta'^2}/2$, the corresponding conversion mechanisms from (or to) $\theta'^2/2$ are

$$-u'\theta' \frac{\partial \theta}{\partial y} - w'\theta' \frac{\partial \theta}{\partial z}.$$ 

The viscosity and heat conductivity effects are reflected by the three-dimensional Laplacian $\nabla^2$; its parabolization is discussed in §4 in connection with the discussion of the wavy characteristics of the fluctuations.

The excess Reynolds stresses in the momentum equations (2.8)–(2.10), for instance, $u'w' - u'u'$, appear on the r.h.s. The mean effects are subtracted from $u'u'$, thus now the excess stresses oscillate at the same frequency as $u'$. These are the nonlinear effect that generates harmonics, including forced subharmonics (Stuart 1960; Kelly 1967; Liu 1988). If one treats the fluctuations as representable by a single fundamental component, as anticipated in Meksyn & Stuart (1951) and subsequent followings, then these excess stresses disappear. The nonlinear effects then reduce to the advection of fluctuations by the mean flow streamwise velocity $u(x, y, z)$, which in turn is modified by Reynolds stresses of the fluctuations; the excess heat fluxes in the heat transfer problem also disappear in this approximation.

Attributed to Stuart (Ffowcs Williams et al. 1969) is a discussion of the work of Reynolds & Potter (1967). Their numerical application of the Stuart (1960) theory suggests that generation of harmonics does not play an important role compared with the more important effects of mean flow modification by the Reynolds stress of the nonlinear disturbance and the consequential modification, in turn, of the fundamental disturbance, confirming therefore, the earlier assumption of Meksyn & Stuart (1951).

4. Outline of the calculation scheme for the nonlinear momentum problem

The system of nonlinear partial differential equations for the mean field (2.1)–(2.6) is jointly solved with the nonlinear system for the fluctuations (2.7)–(2.12). The latter describes the developing secondary instability, which is coupled to the developing and continuously modified mean flow. To proceed, the fluctuation flow quantities are represented in normal mode form as strongly suggested by observations (Swearingen & Blackwelder 1987),

$$q' = \hat{q}(x, y, z)\exp i(\alpha x - \omega t) + \text{c.c.},$$

where c.c. denotes the complex conjugate; $\hat{q}(x, y, z)$ represents the complex amplitude or wave envelop $\hat{u}, \hat{v}, \hat{w}, \hat{\theta}$ for the fluctuations; the streamwise secondary instability wavenumber is $\alpha$; and the frequency is $\omega$. The wavy characteristics are assumed to be fixed by the imposed upstream initial value according to Yu & Liu (1991) and according to experiments. The mean steady flow has spanwise wavelength of $\lambda_z$. 

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This is more direct than the formal procedure suggested by Herbert (1997), where iterations are performed on the local streamwise wavenumber. Numerical application by Li & Malik (1995) showed a dominant streamwise wavelength revealed by the streakline plots (as anticipated by the observations of Swearingen & Blackwelder 1987).

Parabolization of the fluctuation system is made through the observation that the wave envelope \( q(x, y, z) \) is a slowly varying function of \( x \) compared with the rapid oscillations of the propagating waviness under the envelope. In this case, the second \( x \) derivatives are approximated by the local value in terms of \(-\alpha^2\).

The computation begins by assigning initial amplitude for the secondary instability with shape functions from Yu & Liu (1991) and with \( \alpha, \omega \) and \( \lambda_z \) fixed throughout the flow as already discussed. The Reynolds stresses are then calculated and used to modify the mean field to advance one \( \Delta x \) step. The modified mean flow subsequently supports a new fluctuating instability, and so on. The calculation of mean flow and fluctuation development follows a Meksyn & Stuart (1951) scheme for each advancing step. The physical essence of finite amplitude effect is that the basic mean flow has felt the depletion of kinetic energy owing to its production of the fluctuations, unlike linear infinitesimal disturbances that behave as if the mean flow were a reservoir of infinite energy. The finite disturbances tend to choke off their own energy supply and thus mathematically, no longer amplify exponentially as infinitesimal disturbances.

The actual application to the nonlinear momentum problem and parameter ranges is discussed in detail in Girgis & Liu (2006). Their results are used for interpretation of the heat transport problem after the Reynolds analogy is discussed in the following. For Prandtl numbers far from unity, computations of the heat transport problem along similar lines would be necessary.

### 5. An extended Reynolds analogy

Following standard definitions, the Stanton number \( St \) and the skin friction coefficient \( c_f \) are, respectively

\[
St = -k \frac{\partial T}{\partial y}|_W / \rho c_p U_0 (T_0 - T_W) \quad \text{and} \quad c_f/2 = \mu \frac{\partial u}{\partial y}|_W / \rho U_0^2,
\]

where \( \rho \) is the fluid density; \( k \) is the thermal conductivity; \( c_p \) is the heat capacity at constant pressure; and the subscript \( W \) denotes wall values. In the present context, \( St(x), c_f(x) \) are spanwise averaged but are local functions of \( x \). The viscous shear stress \( \mu \frac{\partial u}{\partial y}|_W \) reflecting streamwise resistance depends only on the mean streamwise velocity.

In the absence of secondary instability, Liu & Sabry (1991) noted the amazing similarity between (2.2) and (2.5). For \( Pr=1 \), then \( u=\theta \) provided that their respective initial and boundary conditions are also similar. Therefore, the nonlinear solution of the complete momentum problem yielding \( u \) would enable one to infer \( \theta \), and thus resulting in the Reynolds analogy

\[
St = c_f/2 \ (\text{steady flow})
\]

In the presence of nonlinear secondary instability, the problem is more intricately involved. In order to have \( u=\theta \), the Reynolds stresses and heat fluxes on the r.h.s. of (2.2) and (2.5) must also be similar, i.e. \(-\overline{u'\theta'}\).
This reduces to arguing for similarity between (2.8) and (2.11), leading to \( u' = \theta' \). The author (Liu 2005, 2007) has shown that the pressure gradient term in (2.8), which is the impediment to similarity, is of the order of the magnitude of \( u'/1 \) relative to the inertia terms. In this case, (2.8) is similar to (2.11), and, for \( Pr=1 \) and similar initial and boundary conditions, the Reynolds analogy follows:

\[
St = \frac{c_f}{2} \quad \text{(with nonlinear secondary instability)}
\]

This is referred to as an extended Reynolds analogy in that one has to show similarity between fluctuation quantities before averaging rather than considering mean quantities alone.

### 6. Heat transfer intensification

An example of measurement of surface heat transfer rate in terms of the Stanton number is shown in figure 1 taken from Momayez et al. (2004). The flow visualization indicates mushroom-like structures with a spanwise wavelength of \( \lambda_z = 1.74 \) cm, corresponding to \( A_z = 570 \). The Stanton number \( St(x) \) develops from the laminar value towards the turbulent one but rises well beyond the turbulent value apparently in the absence of developed turbulence in the boundary layer. I refer to the original papers for further details of experimental conditions. The leading edge, preceding a constant radius concave wall, is blunt corresponding to that of a NACA 0025 aerofoil. The Stanton number is obtained via a constant rate of heating applied to the wall while the response wall temperature is measured. It is more convenient to specify the wall temperature in computations (Liu & Lee 1995; Liu 2005); both procedures lead to the dimensionless Stanton number.

There are sufficient differences between conditions of the heat transfer measurements and the conditions of Swearingen & Blackwelder (1987) used here that it is more meaningful to have a side-by-side comparison for physical explanation rather than a less meaningful direct comparison. The Stanton
number inferred from the nonlinear momentum problem (Liu 2005; Girgis & Liu 2006) under the Reynolds analogy for $Pr=1$. The flat plate correlations are for laminar (lower) and turbulent (upper) boundary-layer flows. The steady flow alone, without secondary instabilities, undergoes a ‘transition’ from laminar to turbulent Stanton number (Liu & Sabry 1991; Liu & Lee 1995). The intensified heat transfer comes from the effect of ‘eddy heat flux’ of secondary instability of the sinuous mode, $A_z=460$, inferred from skin friction (Girgis & Liu 2006).

In order to explain the overshoot beyond the local turbulent value in the absence of turbulence, it is necessary to appeal to the effect of the ‘eddy heat flux’ owing to the transport effects of the secondary instability. The normal-to-wall heat flux, inferred from Reynolds shear stress $\overline{vv}'=\overline{v\theta}'$ from Yu & Liu (1994) over the $yz$ cross section, is shown in figure 3. This typifies the structural behaviour of the heat fluxes. In general, regions of $\overline{v\theta}'>0$ in the cross-sectional plane dominate, especially after spanwise averaging. The contribution from the varicose mode is shown in figure 3a; the sinuous mode heat flux is shown in figure 3b. In an actual experiment, it is most likely that both modes but of unknown strength are present. If the contours of figure 3a,b were superimposed, the resulting contour would strongly resemble in the upper boundary-layer region of $(\partial\theta/\partial y)$, consistent with typical mushroom-like structure $\theta=u$. The downwards transport of heat towards the wall by the eddy heat flux appears on the average over the spanwise direction, as if to obey the gradient transport law

$$\overline{v\theta}' \approx a_{\text{eddy}} \partial \theta / \partial y > 0,$$

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where \( \alpha_{\text{eddy}} > 0 \) is an eddy diffusivity coefficient. In this conjecture, the temperature gradient near the wall is steepened by the heat transport effect of the secondary instability as if in a turbulent flow (Tennekes & Lumley 1972). It is interesting to note that the transport effect of the heat flux is active in the upper regions of the boundary layer (figure 3) attributable to the ‘inviscid’ part of the shear flow.

The downstream development of the mushroom structure \( \theta \), inferred from \( u \) of Girgis & Liu (2006), with and without secondary instability involvement is shown in figure 4. Figure 4a shows that the entire mushroom structure is depressed towards the wall as the secondary instability develops in the streamwise direction, in contrast to figure 4b where the secondary instability is absent.

The fluctuation temperature development is also inferable from that of the streamwise fluctuation of figs. 2 and 3 of Liu (2005) and Girgis & Liu (2006), respectively. They are not shown here owing to the fact that the feature, except for stretching with boundary-layer growth, does not change significantly. Of more interest is the comparison with other computational and experimental results shown in figure 5. Although conditions are somewhat diverse, a basis for comparison is the similar magnitude of the local Görtler number, \( G_{\Theta} = R e_{\Theta} (\Theta / R)^{1/2} \), where \( \Theta \) is the local momentum thickness, which is indicative of the local mean flow. Figure 5a is from the temporal numerical simulation of Liu & Domaradzki (1993), \( G_{\Theta} = 7.23 \); the contours indicate that the sinuous mode is dominant in comparison with the varicose mode result of Girgis & Liu (2006) of figure 5b for the same \( G_{\Theta} \). Figure 5c,d is the experimental measurements of Mitsudharmadi et al. (2004) at \( G_{\Theta} = 6.12 \) and Swearingen & Blackwelder (1987) at \( G_{\Theta} = 7.23 \), respectively; the sinuous mode appears to dominate in the experiments as well. This is explained as follows: the shape of \( u'_{\text{rms}} \), interpreted as \( \theta'_{\text{rms}} \), is symmetrical about the centre of the mean mushroom shape for the sinuous mode and somewhat resembles \( -\overline{\nu' \theta'} \), inferred from \( -\overline{\nu' u'} \), of figure 3b. The varicose mode would produce a dome shape in \( \theta'_{\text{rms}} \) at the mean mushroom head (figure 3a), which is not apparent in the measurements shown in figure 5c,d as well as in the ‘natural’ numerical simulation (figure 5a). It can be concluded that the sinuous mode is the dominant effective mode, which produces the enhancement in heat transfer.
7. Concluding remarks

In studying heat transport in active hydrodynamical systems it is of utmost importance to understand the carrier of the passive scalar, especially in view of the presence of secondary instability. The mushroom structure is steepened in (a) owing to the effect of eddy heat flux. The mean temperature structure is inferred from $u=\theta$ of Girgis & Liu (2006) for $Pr=1$. (a,b)(i) $X=85$ cm, (ii) $X=90$ cm, (iii) $X=95$ cm, (iv) $X=100$ cm, (v) $X=105$ cm, (vi) $X=110$ cm.

Figure 4. A comparison of $\theta$ development (a) with the presence of secondary instability and (b) without. The mushroom structure is steepened in (a) owing to the effect of eddy heat flux. The mean temperature structure is inferred from $u=\theta$ of Girgis & Liu (2006) for $Pr=1$. (a,b)(i) $X=85$ cm, (ii) $X=90$ cm, (iii) $X=95$ cm, (iv) $X=100$ cm, (v) $X=105$ cm, (vi) $X=110$ cm.
of the possible analogy between some parts of the flow problem and that of
the passive scalar problem, such as $u_0 Z q_0$; $u Z q$; $K v_0 q_0 Z K v_0 u_0$ and so on in the
present instance. I am thus prompted to tracing the origins of the nonlinear
hydrodynamic stability ideas that are crucial to the present development.

Stuart (1960) obtained the nonlinear amplitude equation from the Navier–
Stokes equations in the weakly nonlinear instability analysis of parallel flows. He
elucidated the physical effects that contribute to the coefficient of the non-
linear term. These include (i) mean flow modification by the Reynolds stresses,
(ii) modification of the fundamental fluctuation component, and (iii) harmonic
generation. Stuart (1960) provided the means to compute these constants from
eigenfunctions of the linear theory. What is more significant is that these physical
mechanisms reflect situations well beyond the confines of the original assumptions.

In the present example of developing, strong nonlinear instability of steady
streamwise vortex flow, the mean flow modification via the Reynolds stresses is
identified here with the nonlinear mean momentum problems (2.1)–(2.4), which,
in turn, modifies the fluctuation momentum reflected in the coupled equations
(2.7)–(2.10). Harmonic generation is imbedded in the nonlinear excess Reynolds
stresses on the r.h.s. of (2.8)–(2.10). However, in the actual application, only the

Figure 5. Contours of $u'_{rms}$, interpreted as $\theta'_{rms}$ for $Pr=1$, in the cross-sectional $yz$-plane.
(a) Temporal simulation of Liu & Domaradzki (1993) at $G_\theta=7.23$; (b) simplified nonlinear spatial
calculation of Girgis & Liu (2006), sinuous mode at $G_\theta=7.23$; (c) measurements of Mitsudharmadi
et al. (2004) at $G_\theta=6.12$; and (d) measurements of Swearingen & Blackwelder (1987) at $G_\theta=7.23$.
Here, $y, z$ are dimensional, measured in cm, except (c) which is in mm.
The fundamental component is accounted for so that the excess Reynolds stresses are absent, following an earlier judicious choice in the representation of nonlinear fluctuations by the fundamental component (Meksyn & Stuart 1951; Stuart 1958).

On the other hand, the advective rate of adjustment of the nonlinear fluctuation, (2.7)–(2.10), is more complicated in spatially developing than in temporally developing flows, as on the l.h.s. of (2.8)–(2.10). The non-equilibrium effects on the r.h.s., and in general the nonlinear interaction system, are in full accordance with the physical mechanisms exposed by Stuart (1960) and remain universally robust, especially for the multidimensional developing flows of the boundary-layer type as discussed here. (I am indebted to Lord Hunt of Chesterton for sharing his thoughts on this subject (J. C. R. Hunt 1998, personal communication).)

The present class of secondary instability problems (Hall & Horseman 1991; Yu & Liu 1991, 1994; Li & Malik 1995) is identical to those of streak instability (of much smaller scale relative to Görtler vortices) in the transitional boundary layers, where streak generation is attributed to free stream turbulence. The similarity lies in the study of instability of a basic mean three-dimensional steady flow with streamwise velocity of mushroom shape (Brandt 2007). Taking into account compressibility effects, Ricco & Wu (2007) found that free stream turbulence induces very strong thermal streaks near the wall, much like the streamwise velocity streaks in an incompressible flow (Leib et al. 1999). Similar features of streaks and their secondary instability are also found in the wall region of turbulent boundary layers (Schoppa & Hussain 2002).

In the twenty-first century, one can unpretentiously say that fertile fluid mechanics research will continuously be driven by technological needs. Concerning energy efficiency of heat transfer equipment, the ‘classical’ assumption of incompressibility will have to be removed in the new generation of studies. Already, in the experiments on streamwise vorticity effects on heat transfer (Momayez et al. 2004), the relative temperature difference due to wall heating is in the vicinity of 50%. In this case, the inertia effects of density in a gaseous medium as well as variation of molecular transport properties become important. Fortunately for such applications, the Mach numbers are exceedingly low so that viscous dissipation and the work done by the pressure gradients can still be neglected in the thermodynamic energy equation. On the other extreme, in application to flight vehicles at hypersonic speeds, where the centre of pressure is most sensitive to the location of the transition region, the three dimensionality of the vehicle surface would stimulate boundary-layer instability not found on flat surfaces (M. Kloker 2008, personal communication).

Through the effects of eddy heat and mass flux from secondary instability, significant enhancement of momentum and scalar mixing has been found possible in free mixing layers (Girgis & Liu 2002; Liu 2007). It is anticipated that much more complex fluid properties and geometry would have to be taken into consideration of intensifying mixing in practical combustors and in jet exhausts in a well-controlled manner for minimizing pollutants of combustion and of aerodynamic sound.

The problem discussed here, that of heat transfer under the influence of streamwise vortices and their secondary instability, is an example that has firm connections with a leading development of fluid mechanics in the second half of the twentieth century, that of nonlinear hydrodynamic stability of shear flows.
pioneered and set firmly on its way by J. T. Stuart. He applied mathematical analyses to derive clear physical understanding of complex nonlinear processes in hydrodynamic stability. In the present context, it is clear that his mathematically derived physical insight also applies to situations well beyond the conditions from which the analyses sprung forth. In more general terms, the spirit of twentieth century fluid mechanics that emphasize rationally derived physical processes and understanding is seen here as the essential link to the furtherance of effective development of fluid mechanics in the twenty-first century. Throughout the paper, I cannot help but note how J. T. Stuart’s original ideas have come across and shaped the transition of nonlinear hydrodynamic stability in fluid mechanics into the twenty-first century, where understanding of physical mechanisms is even more important especially for the more complex situations that one will face.

I express to J. T. Stuart gratitude of an almost lifetime time scale, dating from my sabbatical leaves at the Department of Mathematics, Imperial College during 1972–1973, 1979–1980, 1987–1988 to the present, and an appreciation of his work, which continues its immense stimulation and impact. I thank the Laboratoire de Thermocinétique, École Polytechnique de l’Université de Nantes for their warm hospitality during my 2007–2008 sabbatical leave when the manuscript was prepared, in particular, I thank H. Peerhossaini and ‘l’equipe thermodiffusie et ecoulement complexes et énergie’ for stimulating my interest in intensified heat transfer under streamwise vortices. L. Momayez helped with figures 1 and 2 for which I am grateful. Editor Xuesong Wu has rendered immense help towards the presentation of the final manuscript, for which I am most grateful. Some aspects of this work were presented at the 15th International Couette–Taylor Workshop, LMPG, Université du Havre, 9–12 July 2007. The unspoken efforts of Publishing Editor S. Abbott and Production Editor M. Williams are most appreciated.

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Instability of streamwise vortices


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