Lift, drag and added mass of a hemispherical bubble sliding and growing on a wall in a viscous linear shear flow

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The three-dimensional flow around a hemispherical bubble sliding and growing on a wall in a viscous linear shear flow is studied numerically by solving the full Navier–Stokes equations in a boundary-fitted domain. The main goal of the present study is to provide a complete description of the forces experienced by the bubble (drag, lift and added mass) over a wide range of sliding and shear Reynolds numbers (0.01 \( \leq \text{Re}_b \), \( 0 \leq \text{Sr} \leq 5 \)) and shear rate (0 \( \leq \text{Sr} \leq 5 \)). The drag and lift forces are computed successively for the following situations: an immobile bubble in a linear shear flow; a bubble sliding on the wall in a fluid at rest; and a bubble sliding in a linear shear flow. The added-mass force is studied by considering an unsteady motion relative to the wall or a time-dependent radius.

Keywords: lift; drag; added mass; hemispherical bubble; shear flow; sliding motion

1. Introduction

In many practical situations sub-millimetric bubbles can be generated on a wall. Bubble formation can result from phase change in a boiling system or by desorption of gas under saturation conditions. In these situations, bubbles can interact with a liquid whose motion is sheared at the wall. In such flows, bubbles experience hydrodynamic forces due to the surrounding fluid and its sliding velocity: a drag force parallel to the wall and a lift force normal to the wall. The added-mass force has components in both directions under unsteady conditions, which can be induced by unsteady translation parallel to the wall or by a time-dependent radius in growing or collapsing situations.

The dynamics of bubbles have been extensively studied (Clift et al. 1978) and significant progress in the understanding of bubble dynamics has been observed during the last two decades thanks to the direct numerical simulation in a boundary-fitted domain and to the experimental control of the presence of contaminant in the liquid phase (Magnaudet & Eames 2000). The recent studies concerning bubble dynamics have focused on the dynamic of single spherical bubbles in an unbounded linear shear flow (Legendre & Magnaudet 1998), in a rotating flow (Candelier et al. 2004; Van Nierop et al. 2006), in a weakly turbulent flow (Merle et al. 2005), moving in the vicinity of a wall (De Vries et al. 2002; Van Nierop et al. 2006),...
Takemura et al. 2002) or interacting with a neighbouring bubble (Legendre et al. 2003). Despite these studies, the understanding of bubble dynamics in contact with a wall is still an open problem. The aim of this work is to consider the hydrodynamic interaction between a linear shear flow and a sliding hemispherical bubble.

2. Statement of the problem

Let us consider a hemispherical bubble of radius $R$ located at the origin of a Cartesian frame of reference $(e_x, e_y, e_z)$. We assume the surface tension and/or the bubble size to be large (respectively small) enough to maintain the spherical shape of the bubble. The bubble is sliding with the velocity $-U_b e_x$ on a planar wall ($y=0$) and is embedded in a steady linear shear flow (figure 1) corresponding to the undisturbed velocity field $U_x = \alpha y e_x$. In the following sections of the paper $\alpha$ will be assumed to be positive ($\alpha > 0$).

The outer fluid is Newtonian and its local velocity and pressure are denoted by $V$ and $P$, respectively. The incompressible flow around the bubble is governed by the full Navier–Stokes equations

$$\nabla \cdot V = 0, \quad \rho \left( \frac{\partial V}{\partial t} + \nabla \cdot V \right) = \nabla \cdot (-P I + \tau), \quad (2.1)$$

where $\tau$ is the viscous part of the stress tensor $\Sigma = -P I + \tau$; $\rho$ and $\mu$ denoting the fluid density and viscosity, respectively. On the wall ($y=0$) the fluid obeys a non-slip condition $V=0$, while on the bubble surface the normal velocity must vanish and the fluid experiences no tangential stress

$$V \cdot n = 0, \quad n \times (\tau \cdot n) = 0, \quad (2.2)$$

where $n$ is the outward unit normal to the bubble surface. In this study, we are interested in measuring, using direct numerical simulations, the total force $F$ experienced by the bubble

$$F = \int_S \Sigma \cdot n dS, \quad (2.3)$$

where $S=2\pi R^2$ denotes the surface of the bubble. Owing to symmetry conditions the component along the $z$-axis is zero as long as no destabilization of the wake occurs, which has been observed whatever the considered Reynolds numbers. For steady situations, the lift and drag coefficients $C_L$ and $C_D$ are usually defined through the expressions $F_D = C_D \rho U_b^2 |V(x_b) - U_b| |V(x_b) - U_b|$ and $F_L = C_L \rho U_b^2$.
(\(V(x_b) - U_b\) \(\times \omega(x_b)\) (Legendre & Magnaudet 1998), where \(\vartheta_b\) is the bubble volume; \(\Gamma\) is the projected frontal area; \(V(x_b)\) and \(\omega(x_b) = \nabla \times V\) are, respectively, the velocity and the vorticity of the unperturbed flow evaluated at the bubble location \(x_b\). For a hemisphere \(\vartheta_b = 2\pi R^3/3\) and \(\Gamma = \pi R^2/2\). In the present problem, several characteristic velocities can be chosen to define the non-dimensional coefficients. It can be the bubble sliding velocity \(U_b\), the liquid velocity in the shear flow \(\alpha y_w\) at a given distance \(y_w\) from the wall or a combination of these two velocities \(U_{\text{ref}} = U_b + \alpha y_w\). \(U_{\text{ref}}\) expresses the slip velocity between the bubble and the surrounding fluid. This point will be discussed in the paper. Under steady conditions, the solution of the problem depends only on two characteristic parameters: the sliding and the shear Reynolds numbers based on \(U_b\) and \(\alpha R\) are

\[
Re_b = \frac{2\rho U_b}{\mu} \quad \text{and} \quad Re_\alpha = \frac{2R^2\rho \alpha}{\mu}. \tag{2.4}
\]

The combination of these two Reynolds numbers is used to introduce the dimensionless shear rate

\[
Sr = \frac{Re_\alpha}{Re_b} = \frac{\alpha R}{U_b}. \tag{2.5}
\]

In the present work, \(Re_b\) and \(Re_\alpha\) varies from 0.01 to 2000 and \(Sr\) takes the values 0, 0.01, 0.1, 1, 5 and \(\infty\). The analysis of the bubble dynamics is conducted by first considering the two simplified situations: a stationary hemispherical bubble in a linear shear flow (case 1) and a hemispherical bubble translating (or sliding) on the wall in a quiescent fluid (case 2). In the following, forces and non-dimensional coefficients are denoted by using the subscripts 1 and 2, respectively.

The computations reported below have been carried out with the JADIM code presented in detail in previous studies (Magnaudet et al. 1995; Legendre & Magnaudet 1998; Legendre et al. 2003). Its ability to treat bubble hydrodynamics in shear flow has been presented in detail in Legendre & Magnaudet (1998), to which the reader is referred for a presentation of the curvilinear grid, the treatment of the curvilinear source terms involved in the momentum equation and the numerical tests concerning grid dependency and domain confinement.

3. Bubble fixed in a linear shear flow (\(U_b = 0\))

We first consider the case of a bubble fixed in a linear shear flow. The drag and lift forces are studied over a wide range of the shear Reynolds number \(Re_\alpha = 2R^2\rho \alpha/\mu\) ranging between 0.01 and 2000. As pointed out in the introduction, no solution giving the lift and drag coefficients on a half-spherical bubble on a planar wall in a simple shear flow have been derived so far. Nevertheless, the Stokes solution of the present problem can be simply derived. The velocity and pressure fields are \(V = \alpha y(e_x - R^3x/r^4)\) and \(P = -2\alpha \mu xy/r^5\), respectively. The corresponding force acting on the half bubble is

\[
F_1 = 2\pi \mu R^2 \alpha e_x,
\]

which gives the drag and lift coefficients \(C_{D1} = 4F_1 \cdot e_x/\pi \rho R^4 \alpha^2 = 16/Re_\alpha\) and \(C_{L1} = 4F_1 \cdot e_y/\pi \rho R^4 \alpha^2 = 0\), respectively. As already pointed out for a spherical bubble moving steadily in a linear viscous flow (Legendre & Magnaudet 1997, Phil. Trans. R. Soc. A (2008)
(1998), the Stokes solution induces no lift force. The lift force originates in nonlinear effects (see also Saffman (1965, 1968) and Krishnan & Leighton (1995) for the case of solid spheres).

(a) Drag force

The numerical values of the drag coefficient $C_{D1} = 4F_1 \cdot e_x/\pi \rho R^4 \alpha^2$ are plotted in figure 2. The drag coefficient is found to follow the Stokes solution $C_{D1} = 16/R_{Re}$ up to $R_{Re} = 10$. Thus, this solution obtained for creeping flow condition $R_{Re} = 10$ is valid in a larger range of the Reynolds numbers. For $R_{Re} > 10$, the behaviour of the drag coefficient progressively changes to tend asymptotically towards a constant value $C_{D1}^\infty \approx 0.137$ corresponding to a drag force controlled by the fluid inertia $F_{D1} \propto \rho \alpha^2 R^4$. By simply adding these two asymptotic behaviours, we propose a simple empirical correlation to describe the evolution of the drag force

$$C_{D1} = \frac{16}{R_{Re}} + C_{D1}^\infty,$$

i.e.

$$F_{D1} = \left[2\pi \mu R^2 + C_{D1}^\infty \frac{\pi}{4} \rho R^4 \alpha\right] \alpha e_x,$$

(3.1)

with $C_{D1}^\infty \approx 0.137$. The curve corresponding to relation (3.1) is reported in figure 2 and is found to fit the numerical results with an accuracy better than 1.5%.

(b) The lift force

Figure 3 displays the numerical values obtained for the lift coefficient $C_{L1} = 4F_1 \cdot e_y/\pi \rho R^4 \alpha^2$ in the range $0.01 \leq R_{Re} \leq 1000$. The lift coefficient is found to be always positive and the corresponding lift force acts to detach the bubble from the wall. A detailed analysis of the value of the pressure and viscous contributions to the lift force indicates that the pressure contribution is always positive while the sign of the viscous contribution changes with the Reynolds number value, typically approximately $R_{Re}=100$. For $R_{Re}>100$, viscous effects
tend to reduce the net lift force. The same trend has already been observed for a spherical bubble embedded in an unbounded linear shear flow (Legendre & Magnaudet 1998). When \( \text{Re}_a \to 0 \), figure 3 shows that the lift coefficient \( C_{L1} \) reaches a constant asymptotic value \( C_{L1}^0 \approx 1.15 \). The lift force is then directly proportional to small but non-zero inertia effects \( F_{L1} \propto \rho a^2 R^4 \) in agreement with the above discussion (\( F_{L1} = 0 \) for \( \text{Re}_a = 0 \)). The same evolution \( \propto \rho a^2 R^4 \) has been analytically demonstrated by Krishnan & Leighton (1995), who considered a rigid sphere in contact with a wall in a linear shear flow. Note that for a translating rigid sphere or fluid particle (droplet or bubble) with a velocity \( V_s \) in an unbounded linear shear flow (Saffman 1965, 1968; Legendre & Magnaudet 1997), the lift force has a different evolution \( F_{L1} \propto \mu V_s R \text{Re}_a^{1/2} \) resulting from the matching between the inner and outer expansions at the ‘Saffman’ distance \( O(R \text{Re}_a^{-1/2}) \).

When the Reynolds number increases, the lift coefficient is found to decrease. It reaches, for \( \text{Re}_a > 100 \), a constant asymptotic value \( C_{L1}^\infty \approx 0.573 \) showing a lift force controlled by inertia effects. The drag and lift forces are thus found to have the same order of magnitude at large Reynolds numbers and are linked by the relation \( F_{D1}/F_{L1} \approx 1/4 \). Finally, using the numerical values, it is possible to propose a simple empirical correlation that respects the asymptotic behaviours at low and large Reynolds numbers

\[
C_{L1} = 0.573 \frac{1 + 8 \text{Re}_a^{-1}}{1 + 4 \text{Re}_a^{-1}}.
\]

The curve corresponding to relation (3.2) is shown in figure 3 and fits the numerical results with an accuracy better than 2% except for \( 200 < \text{Re}_a < 500 \) where the discrepancy is approximately 4% because the proposed fitting does not reproduce the slight inflection of the lift coefficient evolution for these Reynolds numbers.
4. Bubble sliding in a fluid at rest ($\alpha=0$)

We now consider the situation of a bubble sliding at a constant velocity $-U_b e_x$ on a wall in a liquid at rest. The relevant parameter to describe this problem is the sliding Reynolds number $Re_b = 2\rho U_b / \mu$. The numerical simulations are performed in the reference frame moving with the bubble where the velocity of the wall and of the surrounding fluid is $U_b e_x$.

(a) The drag force

The evolution of the drag coefficient $C_{D2} = 4 F_2 \cdot e_x / \pi \rho R^2 U_b^2$ is reported in figure 4 for sliding Reynolds numbers $Re_b$ ranging from 0.1 to 1000. We observe for this flow configuration an evolution in $1/Re_b$ at low Reynolds number characteristic of a force controlled by viscous effect $F_{D2} \propto \mu RU_b$. A detailed analysis of the results shows that the relation slightly differs from the evolution of a bubble moving in an unbounded liquid. The drag coefficient is found here to be larger due to the extra dissipation induced by the rigid wall instead of a symmetry plane in an unbounded liquid. The numerical values reveal that, for $Re_b < 10$,

$$C_{D2} = \frac{28}{Re_b}, \quad \text{i.e.} \quad F_{D2} = -\frac{7}{2} \pi \mu R U_b.$$  \hspace{1cm} (4.1)

The main difference from the previous case results in the behaviour at large Reynolds numbers. In this situation, the drag is controlled by viscous effect ($C_{D2} \propto Re_b^{-1}$, i.e. $F_{D2} \propto \mu RU_b$) while it is controlled by inertia for a bubble set fixed in a linear shear flow. Indeed, we observe that the evolution for $Re_b > 50$ is given by

Figure 4. Evolution of the drag coefficient $C_{D2}$ for a hemispherical bubble sliding on a wall in a liquid at rest. Circles, numerical simulations; short-dashed line, $C_{D2} = 28/Re_b$ (relation (4.1)); long-dashed line, $C_{D2} = 48/Re_b$ (relation (4.2)); solid line, relation (4.3).
This relation is nothing more than the well-known drag evolution of a spherical bubble rising in a quiescent liquid in the limit \( \text{Re}_b \to \infty \) (Levich 1962). Indeed, the vorticity being produced at the bubble interface, the size of the layer of vorticity, diffused in the vicinity of the bubble surface and transported and diffused in its wake, tends to zero when the Reynolds number reaches important values. In consequence, the drag can be obtained by the dissipation of the viscous potential flow, the effect of the wall and that of a symmetry plane having the same role in the limit \( \text{Re}_b \to \infty \). Inspection of the numerical results shows that the evolution of the prefactor from 28 for \( \text{Re}_b \to 0 \) to 48 for \( \text{Re}_b \to \infty \) can be reproduced by the empirical evolution \( 28 + 20 \tanh(\text{Re}_b/70) \) so that the drag coefficient evolution can be simply described by the relation

\[
C_{D2} = \frac{48}{\text{Re}_b}, \quad \text{i.e. } \quad F_{D2} = -6\pi \mu R U_b. \tag{4.2}
\]

This relation, reported in figure 4, is found to describe in a very good way the evolution of the computational values of the drag coefficient.

\[
C_{D2} = \frac{28 + 20 \tanh(\text{Re}_b/70)}{\text{Re}_b}, \quad \text{i.e. } \quad F_{D2} = -\left[\frac{7}{2} + \frac{5}{2}\tanh\left(\frac{\text{Re}_b}{70}\right)\right]\pi \mu R U_b. \tag{4.3}
\]

This relation, reported in figure 4, is found to describe in a very good way the evolution of the computational values of the drag coefficient.

\( (b) \) The lift force

The simulations reveal that the bubble experiences a lift force in this flow configuration despite an unperturbed flow free of vorticity. The lift originates in the dissymmetry of the problem and cannot be expressed using a normalization based on the vorticity of the unperturbed flow. The evolution of the lift coefficient \( C_{L2} = 4 \frac{F_2 \cdot e_y}{\pi \rho R^3 U_b^2} \) is reported in figure 5 for sliding Reynolds numbers \( \text{Re}_b \).
ranging from 0.1 to 1000. Figure 5 shows that the lift coefficient increases with the Reynolds number and it evolves between two asymptotic values of order unity so that  

\[ F_{l,2}^0 \approx \frac{\rho R^2 U_b^2}{4} \]  

for the Reynolds numbers covered by our simulations. In the limit \( Re_b \to 0 \), the lift coefficient reaches the constant value  

\[ C_{l,2}^0 \approx 0.965, \quad \text{i.e.} \quad F_{l,2}^0 \approx 0.965 \frac{\pi}{4} \rho R^2 U_b^2. \]  

The lift coefficient is nearly equal to this value for \( Re_b < 10 \). A detailed inspection of the results indicates that \( C_{l,2} \) reaches this asymptotic value following the relation  

\[ C_{l,2} \approx 0.965 - 0.015 Re_b \quad \text{for} \quad Re_b < 2. \]  

The asymptotic value at large Reynolds numbers can be deduced from the potential flow solution since the vorticity is confined to a small region as revealed by the behaviour of the drag force. Using the velocity potential of the flow around a hemispherical bubble sliding on a plane \( (\Phi = -U_b \cos \theta R^2/2r^2) \), the pressure distribution at the bubble surface can be deduced as \( P = P_\infty - 9 \rho U_b^2 \sin \theta \) and integrated on the surface of the hemisphere. The corresponding force is  

\[ F_2 = 11 \pi \rho R^2 U_b^2/32 \]  

showing a contribution normal to the wall. The corresponding lift coefficient is  

\[ C_{l,2}^\infty = 11/8, \]  

which is consistent with the numerical values when \( Re_b \to \infty \). A detailed analysis of the asymptotic behaviour reveals that the lift coefficient reaches the value  

\[ C_{l,2}^\infty = 11/8 \]  

for \( Re_b \approx 1/2 \) and follows the relation  

\[ C_{l,2} \approx \frac{11}{8} \frac{2.5}{Re_b^{1/2}}. \]  

This relation, reported in figure 5, describes in a good way the numerical values for \( Re_b > 50 \). We have no argument to explain the power law \( Re_b^{1/2} \) since the viscous correction at the bubble surface is supposed to give pressure and viscous stress contributions proportional to the fluid viscosity corresponding to an evolution in \( 1/Re_b \) (Kang & Leal 1988). This particular behaviour is certainly the consequence of the shear generated on the wall by the flow going around the bubble.  

Finally, we propose the following relation to describe the evolution of the lift coefficient for the range of Reynolds numbers covered by our simulations  

\[ C_{l,2} = C_{l,2}^0 \left( 1 - 0.033 Re_b^{1/2} + 0.020 \frac{C_{l,2}^\infty}{C_{l,2}^0} Re_b \right). \]  

This simple correlation respects the two asymptotic values \( (C_{l,2}^0 \) and \( C_{l,2}^\infty \)) and reproduces our numerical simulations with a precision better than 1%.  

### 5. Bubble sliding in a linear shear flow  

In this section, we consider the general situation of a hemispherical bubble sliding at velocity \(-U_b e_x\) in a linear shear flow. As presented in the introduction, we discuss the way to decompose the drag and lift forces and more precisely to define the relevant slip velocity.  

(a) The drag force  

The numerical values of the drag coefficient \( C_D = 4 \mathbf{F} \cdot e_x / \pi \rho R^2 U_b^2 \) are reported in figure 6 versus the sliding Reynolds number \( Re_b = 2 \rho U_b / \mu \) for \( Sr = Ra / U_b = 0, 0.01, 0.1, 1 \) and 5. The evolution of the drag coefficient shows that, at low
Reynolds number, typically $Re_a \ll 1$, an evolution in $Re_b$ is observed for all the shear rates considered, the prefactor being dependent on the shear rate. For $Re_b \ll 10$, the behaviour is dependent on the value of the shear rate varying from the linear evolution in $48/Re_b$ observed in the absence of shear flow to an asymptotic constant value depending on the shear rate. Note that the normalization used for the drag force and the definition of the Reynolds number do not make it possible to report in figure 6 the results obtained for a bubble fixed in a linear shear flow. In order to find a general normalization for the evolution of the drag force, we first consider the behaviour at low Reynolds number. Taking advantage of the linearity of the creeping flow equations, the drag of a bubble sliding on a wall in a linear shear flow in the limit $Re_b \ll 1$ and $Re_a \ll 1$ may be deduced from the drag of a bubble fixed in a linear shear flow and that sliding in a quiescent fluid

$$F_D = \frac{7}{2} \pi \mu R \left[ U_b + \frac{4}{7} \alpha R \right] e_x. \tag{5.1}$$

The comparison (not reported in figure 6 for clarity) between the relation (5.1) and the numerical values shows that this relation is in very good agreement with the numerical simulation up to $Re_b = 10$. The relation (5.1) also reveals that the drag vanishes when $U_b = -4\alpha R/7$. In order to verify this result, some additional simulations were performed for $U_b$ and $\alpha$ having opposite signs. The normalized force $2F_D \cdot e_x/7\pi \mu RU_b$ is reported in figure 7 versus the velocity $U_b + 4\alpha R/7$ normalized by $U_b$ for $Re_b = 0.1$, 1 and 10. This plot confirms the linear relation between the drag and the velocity $U_b + 4\alpha R/7$, especially that the drag vanishes with $U_b + 4\alpha R/7$. In consequence, this result and the relation (5.1) show that, for $Re_b < 1$, the relevant velocity to describe the bubble dynamics is
referring to the relevant velocity of the flow has to be taken at the normal position $y_w = 4R/7 \approx 0.57R$ from the wall. Note that this position is different from that of the centre of mass of a hemisphere ($y_w = 3R/8 \approx 0.375R$).

At present we want to generalize the superposition observed at low Reynolds numbers up to larger Reynolds numbers. For this purpose, we propose to decompose the drag force as follows:

$$F \cdot e_x = \frac{\pi R^2}{4} \left[ C_{D1} \alpha R^2 + C_{D2} U_b^2 + C_{D3} \alpha R U_b \right], \quad (5.2)$$

where $C_{D1}$ and $C_{D2}$ are the drag coefficients obtained previously for a stationary bubble in a linear shear flow and for a bubble sliding in a quiescent fluid, respectively. From our numerical simulations, it is possible to calculate the contribution of the term $\rho \alpha R^3 U_b$ to the total drag. We observed that the contribution of this term is of second order for all the simulations performed so that we can reasonably consider that

$$C_{D3} \approx 0. \quad (5.3)$$

The maximum effect of this term is approximately 5% and observed at $Sr = 1$ and for $Re_b = O(10)$, since the contribution proportional to $\rho \alpha R^3 U_b$ is supposed to have its stronger influence when $\alpha R \approx U_b$, i.e. for $Sr \approx 1$. The relation (5.2) including the assumption $C_{D3} \approx 0$ is reported in figure 6 for $Sr = 0.1$, 1 and 5. We can observe that the linear superposition can be extended whatever the value of the shear rate and the Reynolds number. Consequently, the drag force of a hemispherical bubble sliding in a linear shear is

$$F_D \approx F_{D1} + F_{D2}, \quad (5.4)$$

where $F_{D1}$ and $F_{D2}$ are the drag forces given by (3.1) and (4.3), respectively.

Figure 7. Evolution of the normalized force $2F_D \cdot e_x/\pi \mu RU_b$ versus the velocity $U_b + 4\alpha R/7$ normalized by $U_b$ for circles, $Re_b = 0.1$; diamonds, $Re_b = 1$; and triangles, $Re_b = 10$; solid line, relation (5.1).
The lift force

There are at least three possible normalizations to define a lift coefficient: (i) the normalization used for a linear shear flow \( \rho \mu R^4 \alpha^2 / 4 \), (ii) the normalization used for a bubble sliding in a fluid at rest \( \rho \mu R^2 U^2_b / 4 \), and (iii) the normalization \( \rho \mu R^2 U^2_{ref} / 4 \) proposed above and based on the velocity \( U_{ref} = U_b + 4 \alpha R / 7 \). The first two normalizations do not permit representation of all the cases studied and generate a significant dispersion versus the shear rate \( Sr \). The last normalization permits the description of the drag evolution for \( Re_b!10 \) and allows us to plot in the same representation the two limit situations corresponding to \( U_b = 0 \) and \( \alpha = 0 \). The evolution of the lift coefficient \( C'_L = 4F \cdot e_y / \rho R^2 U^2_{ref} \) is reported versus the Reynolds number \( Re^* = 2R \rho |U_{ref}| / \mu \) for a hemispherical bubble sliding on a wall in a linear shear flow. Diamonds, \( Sr = 0 \); open circles, \( Sr = 0.01 \); triangles, \( Sr = 0.1 \); squares, \( Sr = 1 \); asterisks, \( Sr = 5 \); filled circles, \( U_b = 0 \); solid line, relation (4.6); short-dashed line, relation (5.5) for \( Sr = 1 \); long-dashed line, relation (3.2).

(b) The lift force

There are at least three possible normalizations to define a lift coefficient: (i) the normalization used for a linear shear flow \( \rho \mu R^4 \alpha^2 / 4 \), (ii) the normalization used for a bubble sliding in a fluid at rest \( \rho \mu R^2 U^2_b / 4 \), and (iii) the normalization \( \rho \mu R^2 U^2_{ref} / 4 \) proposed above and based on the velocity \( U_{ref} = U_b + 4 \alpha R / 7 \). The first two normalizations do not permit representation of all the cases studied and generate a significant dispersion versus the shear rate \( Sr \). The last normalization permits the description of the drag evolution for \( Re_b<10 \) and allows us to plot in the same representation the two limit situations corresponding to \( U_b = 0 \) and \( \alpha = 0 \). The evolution of the lift coefficient \( C'_L = 4F \cdot e_y / \rho R^2 U^2_{ref} \) is reported versus the Reynolds number \( Re^* = 2R \rho |U_{ref}| / \mu \) in figure 8. The lift coefficient \( C'_L \) is found to be of order \( O(1) \) for the range of Reynolds numbers \( Re^* \) covered and the shear rates reported. \( C'_L \) varies between the two limit curves corresponding to the relation (3.2) (stationary bubble in a linear shear flow, i.e. \( U_b = 0 \)) and to the relation (4.6) (bubble sliding in a fluid at rest \( \alpha = 0 \)). The non-superposition of the different evolutions suggests that the introduction of the velocity \( U_{ref} = U_b + 4 \alpha R / 7 \) is not appropriate to describe the lift mechanisms. Following the decomposition done for the drag we try to express the lift force as

\[
F_L = \rho \frac{\pi R^2}{4} \left[ C_{L,1} \alpha^2 R^2 + C_{L,2} U^2_b + C_{L,3} \alpha RU_b \right] e_y, \tag{5.5}
\]

where \( C_{L,1} \) and \( C_{L,2} \) are the lift coefficients obtained previously for a stationary bubble in a linear shear flow and for a bubble sliding in a quiescent fluid, respectively. From our numerical simulations it is possible to measure the lift contribution \( C_{L,3} \) induced by the term \( \rho \alpha R^3 U_b \). We report in figure 9 the evolution
of $C_{L3}$ versus $Re_\alpha$. $C_{L3}$ is found to have an identical evolution whatever the shear rate $Sr$ and the Reynolds number $Re_\alpha$. It evolves between two asymptotic values $C_{L3}^0 \approx 2.85$ for $Re_\alpha \to 0$ and $C_{L3}^\infty \approx 1.85$ for $Re_\alpha \to \infty$. A simple relation is proposed to describe the evolution of $C_{L3}$ between these two asymptotic limits

$$C_{L3} = C_{L3}^0 \frac{1 - 0.17Re_\alpha^{1/2} + 0.50 \frac{C_{L3}^\infty}{C_{L3}^0} Re_\alpha}{1 + 0.50 Re_\alpha}.$$  \hfill (5.6)

Finally, the lift force for a sliding hemispherical bubble in a linear shear flow can be described by the relation (5.5), the lift coefficients $C_{L1}$, $C_{L2}$ and $C_{L3}$ being given by relations (3.2), (4.6) and (5.6), respectively. The relation (5.5) is reported in figure 9 for $Sr=1$.

(c) **Note on the lift force experienced by a spherical bubble translating in a linear shear flow in contact with a wall**

As a final comment in this section, we compare our results with the lift experienced by a non-rotating rigid sphere translating in a linear shear flow in contact with a wall in the limit $Re_0$, $Re_\alpha \to 0$. Our numerical simulations have demonstrated that this limit is in fact valid for much larger Reynolds numbers, typically up to $Re = O(10)$ so that we can reasonably extrapolate this conclusion to the following discussion. The inertial lift experienced by a solid sphere moving in contact with a wall in a linear shear flow has been determined by Krishnan & Leighton (1995). Using the divergence theorem that allows the lift to be expressed as an integral of elementary creeping flow solutions, the lift can be written under the form (5.5) for a non-rotating sphere. In order to make possible the comparison between the forces, the decomposition (5.5) is maintained for the
force acting on a spherical particle. The corresponding coefficients \( C_{L1}, C_{L2} \) and \( C_{L3} \) obtained by Krishnan & Leighton (1995) for a rigid sphere are reported in Table 1. The coefficients are found to be much larger for a rigid sphere than for a hemispherical bubble.

Using an analytical argument first introduced by Legendre & Magnaudet (1997), it is also possible to deduce from the results obtained for a rigid sphere the corresponding results for a spherical bubble or droplet. Indeed, the reciprocal theorem used by Krishnan & Leighton (1995) makes it possible to express the lift force as an integral involving two terms directly proportional to the forcing exerted by the body on the flow (see also Stone et al. 2007). Since the ratio of the forcing induced by a spherical bubble to that induced by a solid sphere is \( 2/3 \), the value of the lift experienced by a bubble that \( (2/3)^2 \) times that of a rigid sphere. The corresponding values of \( C_{L1}^0, C_{L2}^0 \) and \( C_{L3}^0 \) are reported in Table 1 for a spherical bubble. These values are larger than those of a hemispherical bubble except for \( C_{L2}^0 \), the translation–translation contribution, the coefficients being of same order. For example, the lift force experienced by a stationary bubble in a linear shear flow is five times that experienced by a hemispherical bubble.

### 6. Added-mass effects

We consider in this section the added-mass force acting on a hemispherical bubble in two unsteady situations: (i) the sliding motion is unsteady and (ii) the bubble radius is time dependent. To evaluate the inertial force numerically, we employed the procedure proposed by Rivero et al. (1991) and used by Legendre et al. (1998, 2003). Briefly, this procedure consists of determining the inertial force at a given time \( t_1 \) through the difference between the total force obtained in two different situations. In the first of these, the unsteady flow is computed up to time \( t_1 + \Delta t \) and the hydrodynamic force \( F(t_1 + \Delta t) \) is evaluated. In the second computation, the unsteady forcing of the flow is set to zero at \( t=t_1 \) and the corresponding total force at time \( t_1 + \Delta t \), say \( F'(t_1 + \Delta t) \), is evaluated. It can then be proved (see Rivero et al. 1991) that the difference \( F - F' \) gives the inertia contribution at time \( t_1 \) in the limit \( \Delta t \to 0 \). We applied this method to determine the inertia force for the two unsteady situations considered below.

(a) Unsteady sliding in a shear flow

We first consider an unsteady translating motion in a steady linear shear flow. The velocity of the bubble varies linearly in time, i.e. \( U_b(t) = U_0 + At \), where \( A \) is...
a constant acceleration. The simulations are performed in a reference frame moving with the bubble so that the bubble is embedded in the flow \( \mathbf{U}_\infty(t) = (\alpha y + U_0 + At) \mathbf{e}_x \). If the flow were inviscid, the total force acting on the bubble would then reduce to the inertia force

\[
\mathbf{F}_I = -\rho \partial_y b(1 + C_M) \frac{d\mathbf{U}_\infty(t)}{dt}.
\]  

(6.1)

\( C_M \) being the so-called added-mass coefficient (Batchelor 1967, p. 409). The first contribution in the r.h.s. of (6.1) represents the force exerted by the outer flow on the volume of fluid occupied by the bubble. The second part is the added-mass force that arises because the presence of the bubble induces an acceleration of the surrounding fluid. Theoretical and numerical investigations demonstrated that the inertial force experienced by a spherical particle of arbitrary nature (i.e. a rigid particle, a drop or a bubble) moving in an unbounded flow is left unaltered by finite-Reynolds number effects. The value of \( C_M \) is equal to 1/2 whatever the relative strength of viscous effects and temporal acceleration is when compared with those of advection (see Magnaudet & Eames (2000) for a review). In other words, the contribution to the total force \( F(t) \) acting on the particle that depends directly on the instantaneous value of the acceleration \( d\mathbf{U}_\infty(t)/dt \) is identical to that predicted by irrotational theory. This result was obtained in an unbounded flow and more recently for two bubbles in interaction (Legendre et al. 2003). We want to confirm that this result is more general and also applies for a hemispherical bubble sliding on a wall in a linear viscous shear flow. Hence in the present situation, we expect to find an inertial force of the form (6.1) with \( C_M \) as predicted by irrotational theory. Considering the volume of a hemisphere \( V_b = \frac{2}{3} \pi R^3 \), the potential flow solution gives the added-mass coefficient of a hemispherical spherical body translating on a wall

\[
C_M = \frac{1}{2}. \tag{6.2}
\]

We applied the procedure of decomposition presented before for unsteady translating Reynolds number \( Re_b(t) = 2\pi \rho U_b(t)/\mu \) varying in the range 0.1–500, for the shear rate \( S_r = 0, 0.1, 1 \) and 5, and for acceleration parameter \( A_c(t) = 2RA/U_b^2(t) = 2, 10 \) and 20. The numerical values show that \( 1 + C_M \) is close to 3/2 with a variation less than 1%, generalizing the inviscid result given by (6.2).

(b) Bubble with a time-dependent radius

We now consider the situation of a bubble with a time-dependent radius \( R(t) \) in a fluid at rest. In this situation, the potential flow solution gives the inertia force

\[
\mathbf{F}_I = -\frac{3}{2} \rho \partial_t \left( C_1 \frac{\dot{R}}{R} + C_2 \dot{R} \right) \mathbf{e}_y, \tag{6.3}
\]

where \( C_1 = 3/2 \) and \( C_2 = 1 \) correspond to the respective contributions of \( \dot{R} \) and \( \ddot{R} \) to the total acceleration transmitted to the fluid for a hemisphere (Klausner et al. 1993). We note that the added mass has a contribution normal to the wall. In order to separate these two contributions, two different radial evolutions were imposed on the bubble. The first one is a linear evolution given by \( R(t) = R_0 + At \) where \( R_0 \) is the initial value of the bubble radius and \( A \) is a constant radial velocity. Under this
condition, only the acceleration transmitted to the fluid via $\dot{R} = A$ contributes to the added mass ($\ddot{R} = 0$). We have performed simulations for different radial Reynolds numbers ranging from 0.1 to 1000. The values obtained for the coefficient $C_1$ are in agreement with the value $3/2$ with a discrepancy lower than 1%. We have completed this analysis by considering a bubble growth given by $R(t) = R_0 + At^{1/2}$, which is characteristic of an evolution observed for a boiling bubble controlled by heat diffusion. For this radial growth, $\dot{R} = At^{-1/2}/2$ and $\ddot{R} = -At^{-3/2}/4$. Fixing the value $C_1 = 3/2$, we deduced from the simulations that the coefficient $C_2$ is in agreement with the value 1 with a discrepancy lower than 1%.

(c) General expression of the added-mass force

Our simulations show that the inertia force acting on a hemispherical bubble growing and sliding on a wall is given by the potential flow theory whatever the Reynolds number and the acceleration transmitted to the flow. This result extends those obtained in unbounded flows. Finally, considering the general situation of a bubble with a time-dependent radius sliding on a wall the added-mass force experienced by the bubble can be written as

$$F_I = -\rho \frac{1}{2} \frac{d\vartheta_b}{dt} U_b(t) - \frac{3}{2} \rho \vartheta_b(t) \left( \frac{3}{2} \frac{\dot{R}^2}{R} + \ddot{R} \right) n,$$

where $n$ is unit vector normal to the wall and $U_b(t)$ is the sliding velocity along the wall.

7. Conclusion

This paper has been devoted to the dynamic of a hemispherical bubble sliding on a wall in a viscous linear shear flow. For this purpose, direct numerical simulations have been performed in a boundary-fitted domain. A wide range of Reynolds numbers has been explored in order to describe situations dominated by viscous effects as well as nearly inviscid flows. In both asymptotic regimes, the evolutions of the drag and lift forces have been clearly identified as well as for intermediate Reynolds numbers. For this purpose, the lift and drag forces have been decomposed into shear–shear, translation–translation and shear–translation contributions. Correlations based on the numerical values have been proposed to describe the evolution of the corresponding coefficients. Some simulations have been performed under unsteady conditions in order to give a description of the inertia force. We have confirmed that the added-mass effect is given by the potential theory in bounded viscous situations.

References


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