An introduction to quantum turbulence

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This paper provides a brief introduction to quantum turbulence in simple superfluids, in which the required rotational motion in the superfluid component is due entirely to the topological defects that are identified as quantized vortices. Particular emphasis is placed on the basic dynamical behaviour of the quantized vortices and on turbulent decay mechanisms at a very low temperature. There are possible analogies with the behaviour of cosmic strings.

Keywords: superfluids; quantized vortex filaments; turbulence

1. Introduction

During this discussion meeting, we heard about various topological defects in cosmology and condensed matter physics. One is the cosmic string in cosmology and its analogue in condensed matter, the quantized vortex filament. We heard about the formation of this type of defect during quenching through a second-order phase transition. In condensed matter physics, recent interest in these objects has centred on their role in quantum turbulence, which is the name given to turbulence in a superfluid, for which the fluid dynamics cannot be understood in terms of classical physics.

Turbulence in classical fluids has been studied for many centuries, and it is of enormous practical importance. Yet our imperfect understanding of it presents physicists, mathematicians and engineers with a continuing challenge. The flow of a superfluid is strongly influenced by quantum effects, so that the study of quantum turbulence combines the challenges faced in classical turbulence with those associated with quantum many-body effects.

The aim of this introductory presentation is to provide a feel for the way in which quantum effects influence turbulence, to summarize how far our understanding of quantum turbulence has advanced and to generate interest in cosmological analogues. Quantum turbulence is interesting in itself, but it serves also to illustrate the basic principles governing the dynamical behaviour of quantized vortices. Corresponding principles must govern the time evolution of arrays of cosmic strings, and a comparison of the two cases would be of great interest.

Introductory reviews of quantum turbulence have been written by Vinen & Niemela (2002) and Vinen (2006). We refer the reader to these reviews for references to all but the most recent published work.

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2. Superfluidity and the quantized vortex filament

We confine our attention to the two simplest (uncharged) superfluids: the superfluid phase of liquid $^4$He and the low-temperature superfluid phase of liquid $^3$He, $^3$He-B. Both display two-fluid behaviour: a normal fluid composed of thermal excitations in the background of a superfluid component. The proportion of normal fluid falls from unity to zero as the temperature falls from the transition temperature to absolute zero. The two fluids can have separate velocity fields, $v_n(r)$ and $v_s(r)$. Superfluidity arises from Bose or BCS\(^1\) condensation: a symmetry-breaking transition involving the formation of a coherent particle field, often called the condensate wave function, $\Psi(r)$. Superflow is linked to the phase $S$ of $\Psi$,

$$v_s = (h/m)\nabla S,$$

where $m$ is the mass of a single $^4$He atom or a pair of $^3$He atoms. It follows that the flow of the superfluid component is irrotational. However, a form of rotational motion is possible in the form of a vortex filament: a line along which the amplitude of $\Psi$ vanishes and around which there is a curl-free circulation. In order that $\Psi$ be single-valued, the circulation must be quantized,

$$\Gamma = \oint v_s \cdot \, dr = n\kappa, \quad \kappa = 2\pi h/m.$$

Free vortex filaments have a single quantum, $n=1$. In $^4$He, the vortex core (the region over which $|\Psi|$ is depressed) has a radius of only approximately $5 \times 10^{-2}$ nm and is without significant structure; in $^3$He-B, the vortex core has a radius of approximately 80 nm, and it contains trapped thermal excitations to which we shall refer later.

3. A superfluid in uniform rotation

A uniform array of quantized vortex lines, as shown in figure 1, allows a superfluid to rotate in equilibrium with a containing vessel. The required uniform line spacing is $\ell=(\kappa/2\Omega)^{1/2}$, where $\Omega$ is the angular velocity of the rotating vessel. On length scales large compared with $\ell$, the flow mimics solid-body rotation, although the flow is very different on smaller scales. This illustrates two important general principles: vortex lines can be used to produce superfluid flows that mimic classical flows on large length scales; and quantum effects associated with the discrete nature of the vortex lines are important on length scales of the order of or less than the vortex spacing.

4. Quantum turbulence; large-scale properties at very low temperatures

Turbulent flow is necessarily rotational. It follows that turbulent flow of the superfluid component is possible only through the presence of vortex lines. At first sight, this suggests that quantum turbulence must be very different from its classical counterpart. However, as we have suggested, on a scale large compared with the vortex line spacing, $\ell$, the flow can be indistinguishable from a classical

\(^1\)Bardeen–Cooper–Schrieffer: the condensation involves pairs of particles, as in a superconductor.
flow. Therefore, perhaps on scales larger than $\ell$, quantum turbulence in the superfluid component might be indistinguishable from classical turbulence \textit{(quasi-classical)}. There is good experimental evidence relating to homogeneous turbulence that this is indeed the case at very low temperatures, where there is no normal fluid \textit{(Bradley et al. 2006; Walmsley et al. 2007)}: the evidence takes the form of an indirect verification that the energy spectrum for $k\ell \ll 1$ has the classical Kolmogorov form ($k$ is the wavenumber in a Fourier analysis of the turbulent velocity field)

$$E(k) = C\epsilon^{2/3}k^{-5/3}. \quad (4.1)$$

This form applies approximately to classical steady-state homogeneous turbulence in an inertial range of wavenumbers (i.e. a range in which there is no viscous dissipation), if there is dissipation $\epsilon$ per unit mass at high wavenumbers. However, we must remember that turbulence is highly nonlinear, resulting, in principle, in strong coupling between motion on different length scales, so that classical motion on a scale large compared with $\ell$ might be strongly influenced by non-classical motion on a scale less than $\ell$. In practice, such coupling appears to be largely local in $k$-space, so that quasi-classical behaviour can survive for $k\ell \ll 1$.

**5. Influence of the normal fluid**

Our discussion in §4 assumed the absence of normal fluid. In order to understand how a significant fraction of normal fluid might affect quantum turbulence, we must first discuss how the normal fluid affects the motion of a single vortex line. In the absence of normal fluid, a vortex line moves with the local superfluid velocity, this motion leading to quasi-classical behaviour on large scales. The
normal fluid consists of thermal excitations, which interact with the core of a vortex line, so that the relative motion of the line and the normal fluid leads to a frictional force on a unit length of the core of the vortex given by

\[ f = \gamma (v_n - v_L), \quad (5.1) \]

where \( v_L \) is the velocity with which the line moves and \( \gamma \) is a constant. For simplicity, we have ignored a transverse component of the force \( f \). The effect of the force is to cause the vortex to move relative to the local superfluid with a velocity given by the Magnus effect,

\[ f = \rho_s \hat{k} \times (v_s - v_L), \quad (5.2) \]

where \( \rho_s \) is the density of the superfluid component and \( \hat{k} \) is a unit vector parallel to the vortex.

It is instructive to consider the effect of this force on the motion of a vortex ring in the superfluid component. In the absence of the force, the ring propagates unchanged (ring velocity must be less than the Landau critical velocity). A force \( f \) in the direction opposite to this motion leads to a shrinkage of the ring, until the ring eventually disappears. A force in the same direction as the motion leads to the expansion of the ring, energy associated with the relative motion of the two fluids being in this case fed into an increasing length of the vortex line.

We can now understand how the force \( f \) can influence turbulence, produced, for example, by the flow through a grid. Here, we must distinguish between \(^4\text{He}\) and \(^3\text{He-B}\). In the case of \(^4\text{He}\), the normal fluid has a very small viscosity. The flow of both the fluids through a grid at a reasonable velocity will therefore generate turbulence in both the fluids. On scales large compared with \( \ell \), the superfluid turbulence will be quasi-classical, so that its statistical properties are identical with those of the normal fluid turbulence. The force \( f \) then acts simply as a force of mutual friction between the two fluids; it serves to lock together the two turbulent velocity fields. The helium behaves therefore as a single fluid with large Reynolds number, on scales greater than \( \ell \). In the case of \(^3\text{He-B}\), the normal fluid is very viscous (owing to large excitation mean free paths), and it cannot become turbulent on a laboratory scale. Mutual friction then serves simply to damp the turbulence in the superfluid component. It turns out that if \( \gamma \gtrsim \rho_s \kappa \), the quantum turbulence is killed completely. A situation in which this is observed has been described by Matti Krusius.

6. The generation of turbulence by mutual friction

The possibility that we noted of mutual friction leading to the expansion of a vortex ring suggests that the flow of the normal fluid at a sufficient velocity relative to the superfluid component might lead to the generation of turbulence in a way that has no classical analogue. In figure 2b, we show schematically a number of vortex rings. The motion of the normal fluid relative to these rings could lead to their expansion. But this expansion, by itself, will lead to their ultimate annihilation at the walls of the containing channel, so turbulence that is self-sustaining seems impossible. However, if the rings interact in such a way that reconnections (figure 2a) occur, then, as we see from figure 2b, self-sustaining turbulence does become possible. Such turbulence is in fact observed when
superfluid $^4$He carries a heat current, the heat being carried as a result of a counterflow of the two fluids (for a full discussion, see Vinen & Niemela (2002) and references therein). Reconnections often play an important role in quantum turbulence. We lack a detailed microscopic theory of reconnections in either $^4$He or $^3$He-B, although they have been shown to be theoretically possible in a weakly interacting Bose-condensed gas.

### 7. Dissipation in quantum turbulence

The physics underlying the classical Kolmogorov spectrum can be summarized as follows. A given turbulent flow at high Reynolds number generally involves motion on a wide range of length scales. The nonlinear coupling arising from the inertial term in the Navier–Stokes equation leads to a coupling between the different scales. As long as the Reynolds numbers for the different scales are large compared with unity, this coupling conserves energy. However, at a sufficiently small length scale, this condition on the Reynolds number must fail, and the motion becomes damped by viscosity. Allowing for the local nature of turbulent interactions in $k$-space, we see that the result is a flow of energy in a cascade (the *Richardson cascade*) from large scales to slightly smaller scales, to smaller scales again, until the scale is small enough for viscous dissipation. If energy is continually injected at some large scale, we can have a steady state, and it is to the inertial (energy-conserving) part of this steady state that the Kolmogorov spectrum applies. The parameter $\epsilon$ is the rate of flow of energy per unit mass down the cascade, this energy being dissipated near the length scale at which viscosity becomes important.

We see that the observed existence of a Kolmogorov spectrum in quantum turbulence requires a source of dissipation on small length scales. In $^4$He, at high temperatures, this is not a problem: the two turbulent velocity fields must decouple at scales of order $\ell$, because the superfluid flow is then strongly influenced by quantum effects, and dissipation in the superfluid component at this scale is then caused by mutual friction. But at very low temperatures, where the normal fluid has disappeared, this source of dissipation has also disappeared, and we must inquire what takes its place.

In approaching this question, we can note that, strictly speaking, viscosity is not the only source of dissipation in classical turbulence. Turbulent flows contain oscillating pressure fields, and these fields can lead to the radiation of sound. We are led therefore to inquire whether the radiation of phonons can be the source of dissipation in quantum turbulence at very low temperatures. We can get a feel for this question by calculating the radiation of sound from a quantized rectilinear vortex that rotates in an orbit of radius $r$ about an axis parallel to...
itself at angular velocity $\omega$, and then putting $r$ and $\omega$ equal to the length $\ell$ (typically 10 $\mu$m) and frequency $\kappa/\ell^2$ (typically 100 s$^{-1}$) characteristic of a vortex tangle. We find that the radiation is orders of magnitude less than is required to produce a dissipation comparable with typical values of $\epsilon$. Radiation at a rate comparable with $\epsilon$ can be achieved only at frequencies that are very much larger than $\kappa/\ell^2$, and probably of order $10^9$ s$^{-1}$. Such high frequencies can arise in vortex tangles only at very short length scales. There is, of course, a tendency for classical turbulent energy to flow towards shorter length scales, and we must therefore inquire how such a flow can occur in quantum turbulence to scales much less than the average vortex spacing $\ell$.

The answer can be seen very vividly in computer simulations of the evolution of an initially smooth tangle (figure 3). For $^4$He at temperatures above 1 K, we find that the tangle decays without a significant change in its form, the decay being due to mutual friction (the normal fluid is at rest). At zero temperature, however, where there is no normal fluid, there is hardly any obvious decay, but the form of the turbulence changes drastically; large numbers of sharp kinks appear on the lines. These sharp kinks arise from the fact that the natural evolution of the tangle leads to the occasional close approach of two lines, with a probable reconnection. Reconnections occur also at high temperatures, but the kinks are then smoothed out very rapidly by mutual friction.

These kinks are regions of very high curvature, involving already small length scales. But they are not enough. At this point, we remember that a vortex line can support a helical wave motion, called a Kelvin wave, with the dispersion relation

$$\omega = (\kappa k^2/4\pi) \ln (1/k\xi_0),$$

where $\xi_0$ is the vortex core size. The reconnections act like the plucking of a string, and lead therefore to the continual excitation of these Kelvin waves. The Kelvin waves increase in amplitude, and nonlinear interactions lead then to the transfer of energy to a wider and wider range of wavenumbers. It has been shown that the interactions are local in $k$-space, and their effect is therefore to generate a cascade that will transfer energy in steps to higher and higher wavenumbers. This process

Figure 3. Evolution of a vortex tangle at zero temperature. (a) An initially smooth tangle and (b) the same tangle after evolution for several seconds at zero temperature. (Reproduced with permission from Tsubota et al. (2000) and M. Tsubota (2000, personal communication).)
is analogous to the Richardson cascade, and it is an example of what is called \textit{wave turbulence}. The Kelvin-wave cascade has a characteristic (one-dimensional) spectrum in which the wave amplitude is given by

\[
\varphi^2(\tilde{k}) = A \rho^{-0.2} \kappa^{-0.6} \epsilon^{0.2} \tilde{k}^{-3.4}, \quad A \sim 1,
\]

where \( \rho \) is the density of the helium; \( \epsilon \) is the energy flux towards high wavenumber; and a tilde denotes a quantity relating to Kelvin waves. The Kelvin-wave cascade serves therefore to transfer energy to wavenumbers that are large enough for the radiation of phonons to match the energy flux \( \epsilon \).

We are led therefore to the following picture of steady-state homogeneous quantum turbulence in \(^4\text{He}\) at a very low temperature. Energy is fed into eddies with a size much larger than the vortex spacing \( \ell \). The energy flows in a quasi-classical Richardson cascade towards smaller scales, until it reaches the \textit{quantum length scale} \( \ell \), at which turbulent motion becomes totally non-classical. With the aid of vortex reconnections, the energy flux is then transferred to a Kelvin-wave cascade, through which it is transferred to wavenumbers high enough to allow effective dissipation by phonon emission.

In the case of superfluid \(^3\text{He}-\text{B}\), the picture is probably slightly different. As we have mentioned, the vortex core in \(^3\text{He}-\text{B}\) traps thermal excitations (Caroli–Matricon excitations, first discussed in connection with flux lines in type II superconductors). The energy-level spacing for these excitations is approximately 10 kHz, and it seems likely that the Kelvin-wave cascade is cut off at this frequency through the absorption of the Kelvin waves by the bound excitations.

Our picture of a Richardson cascade feeding a Kelvin-wave cascade is not the whole story. From an experimental point of view, there are as yet no experiments that verify directly the existence of the Kelvin-wave cascade. From a theoretical point of view, as we now indicate, there is much to be understood about the details of the crossover between the two cascades.

It can be shown that the dissipation resulting from a combination of the Kelvin-wave cascade and phonon radiation (or mutual friction at a finite temperature) is equivalent to an effective kinematic viscosity, \( \nu' \), operating on the scale \( \ell \). This effective viscosity can be measured, and measurements now exist for both \(^4\text{He}\) (Walmsley \textit{et al.} 2007) and \(^3\text{He}-\text{B}\) (Bradley \textit{et al.} 2006) at very low temperatures. In the case of \(^4\text{He}\), measurements on turbulence that incorporates a Richardson cascade cover the whole range of temperatures down to 0.1 K, and they are shown in Figure 4. At higher temperatures, the dissipation is due to mutual friction. We note that in the limit of very low temperatures, \( \nu'/\kappa \) tends to approximately \( 3 \times 10^{-3} \). It is tempting to suppose that the sudden drop in the value of \( \nu'/\kappa \) at approximately 0.8 K marks the onset of the Kelvin-wave cascade. But there are indications from experiment that, for turbulence in \(^4\text{He}\) existing only on scales of order \( \ell \) or less, the value of \( \nu'/\kappa \) suffers no drop and remains at approximately 0.2 (P. M. Walmsley & A. I. Golov 2008, private communication). In the case of \(^3\text{He}-\text{B}\), there are indications that the limiting value of \( \nu'/\kappa \) is also approximately 0.2 (Bradley \textit{et al.} 2006), even for the case when there is a Richardson cascade. These results are a challenge to theory.

Turning to the theory, we remark that, if the Richardson cascade were to join continuously to the Kelvin-wave cascade, with no appreciable length of the vortex line associated with any transition region, then we would expect \( \nu'/\kappa \) to be
in the range 0.1–1. Recent theoretical work has questioned whether such joining is realistic. L’vov et al. (2007) have suggested that the transfer of energy from the Richardson cascade to the Kelvin-wave cascade is difficult, with the result that there is a bottleneck between the cascades leading to an accumulation of excess vortex line at the lower end of the Richardson cascade. This would depress the value of $\nu’/k$. Kozik & Svistunov (2008) have argued that there must be a complicated transition region extending over rather more than a decade in wavenumber between the two cascades, and they have suggested that their analysis can account for the experimental data of figure 4. But these ideas remain difficult to understand; they are certainly controversial and have not yet gained wide acceptance. The dependence of $\nu’/k$ on the exact form of the turbulence may have its origin in the fact that the existence of the large-scale motion characteristic of the Richardson cascade requires some polarization of the vortex tangle; this polarization will certainly reduce the reconnection rate, but existing computer simulations seem to suggest that the reduction is too small to have an important effect. However, the reduction may not be small if known deviations from the classical Kolmogorov spectrum arising from ‘bunching’ of vortices are important. Our conclusions are, first, that we need direct experimental evidence for the existence and importance of the Kelvin-wave cascade, and, second, that we are still some way from an adequate understanding of the transition region between the two cascades. It would be interesting to know whether these issues have cosmological analogues. Phonon radiation from a vortex is presumably analogous to the radiation of gravitational waves by a cosmic string.

8. Summary and conclusions

We have discussed the crucial role played in quantum turbulence by topological defects in the form of quantized vortex lines. We have argued that on large length scales, encompassing large numbers of vortex lines, quantum turbulence is often closely similar to its classical counterpart, but that the structure of turbulent flow must be quite different on smaller scales, where it is dominated by the discrete nature of the quantized vortices. This discrete structure and the inviscid nature of superflow are particularly important in the dissipation of
quantum turbulence; the probable importance of Kelvin-wave cascades in this dissipation at very low temperatures is widely accepted, but, in the view of many of us, the details remain ill understood. We have focused almost exclusively on the simplest form of turbulence, one that is homogeneous, although much work is now in progress on more complicated forms, such as those generated by flow past obstacles (Hänninen et al. 2007) and those considered by Matti Krusius that involve, for example, the propagation of turbulent fronts. In comparison with classical turbulence, which has been studied for many centuries, our knowledge and understanding of quantum turbulence remains primitive. Analogies with the behaviour of cosmic strings are appealing, and we look forward to the possibility that they can lead to the fruitful exchange of ideas.

References


