Application of interval iterations to the entrainment problem in respiratory physiology

BY JACQUES DEMONGEOT* AND JULES WAKU

TIMC-IMAG Laboratory, University J. Fourier of Grenoble, 38706 La Tronche, France

We present here some theoretical and numerical results about interval iterations. We consider first an application of the interval iterations theory to the problem of entrainment in respiratory physiology for which the classical point iterations theory fails. Then, after a brief review of some of the main aspects of point iterations, we explain what is meant by the term ‘interval iterations’. It consists essentially in replacing in the point iterations the function to iterate by a set-valued map. We present both theoretical and numerical aspects of this new type of iterations and we observe the dynamical behaviours encountered, such as fixed intervals and interval limit cycles. The comparison between point and interval iterations is carried out with respect to a parameter $\varepsilon$, which determines the thickness of a neighbourhood around the function to iterate. We will finally focus our attention on the Verhulst and Ricker functions largely used in population dynamics, which exhibit various asymptotic behaviours.

Keywords: interval iterations; respiratory entrainment; set-valued map; invariant domain; intervals limit cycles; population dynamics

1. Introduction

It is well known (May 1976; May & Oster 1976; Demongeot et al. 1997; Murray 2002) that a first-order difference equation (e.g. the logistic one-dimensional equation for a single species) allows the description of complex dynamical behaviours in population growth modelling in many contexts and several disciplines, such as in biology, economy and social sciences (Schaffer et al. 1986; Demongeot & Leitner 1996; Demongeot et al. 1997; Stenseth et al. 1997; Demongeot & Waku 2005). The case of $n$-dimensional flows (system of difference equations for $n$ species) will not be treated in the following, but could be considered as a natural generalization of the techniques here proposed. Let the basic equation be

$$x_{t+1} = f(x_t). \quad (1.1)$$

The variable $x_t$ can be referred to as the ‘population’ at time $t$; $f$ is usually a nonlinear function, containing one or more adjustable parameters, which tune the nonlinear behaviour of the considered system. Population here means people,
animals, insects, bacteria or any self-reproducing organisms that grow over time and fluctuate erratically. So equation (1.1) can model, for example, the growth of a population from one generation to another. \( x_t \) can be considered as the population size at time \( t \) (demography), the fraction of population infected at time \( t \) (epidemiology), the number of bits of information that can be remembered after a lapse of time \( t \) (telecommunications), the number of people to have heard a piece of information/rumour after time \( t \) (sociology), the phase of a periodic signal in physiology, etc. Depending on the domains of application, parameters have different significance. For example, when \( x_t \) is a population size, one of the parameters can be the growth rate and another the maximum number of individuals in the area of study.

The function \( f \) can also represent the relationship between \( x_{t+1} \) and \( x_t \), a discrete variable linked to a dynamical system trajectory or to a time series of empirical observations (e.g. a Poincaré map or an empirical phase response curve), and equation (1.1) can exhibit complex dynamical behaviours including fixed points, cycles of points of arbitrary \( 2^m \) order and chaos when \( f \) is as a function of the first argument a non-invertible, uni-modal map. These complex dynamical behaviours appear, in general, when \( f \) is continuous and uni-modal, from a stable point, through a sequence of bifurcations, into stable cycles of period 2, \( 2^2, 2^3, 2^4, \ldots, 2^n, \ldots \) and finally into a regime of chaos. This regime is characterized by the rapid loss of predictive power owing to the property that trajectories arising from nearby initial conditions diverge exponentially fast at the average. We can note that the sequence of bifurcations follows Sarkovskii’s \( \omega \)-order (Holmgren 1996) (respectively, Cosnard’s (1983) \( c \)-order) for uni-modal (respectively, multi-modal) maps. When the empirical data used for identifying the function \( f \) come from a noisy experimental process, we propose in the following to replace \( f \) by a set-valued map whose graph (Aubin & Frankowska 1990; Chen et al. 2005) is bounded by the graphs of the functions \((1 - \varepsilon)f\) and \((1 + \varepsilon)f\). The point iterations dynamics of \( f \) is then replaced by a new process called interval iterations, which just corresponds to the iterations of this set-valued map. The present paper is devoted to new results concerning the bifurcations observed for the interval iterations process of some set-valued maps defined from \( C^1 \) uni-modal maps \( f \) on \([0,1]\), applied in respiratory physiology and population dynamics.

2. Physiological motivation

Respiratory rhythm is controlled by bulbar centres that are characterized by an autonomous periodic free run and a particular ability to be forced by peripheral stimulations such as stretch receptor excitations at the end of each lung inflation (Pham Dinh et al. 1983). Let us denote the central respiratory variable by \( R \), which is equal to 1 during central inspiration (corresponding to the phrenic nerve activity), and 0 during central expiration (corresponding to the phrenic silence). Let \( P \) denote the peripheral respiratory variable, which is equal to 1 when the lungs are inflated (by an external pump with period \( T \)) and 0 when they are deflated. Respiratory regulatory properties imply that the delay \( x_{i+1} \) (modulo \( T \)) between the end of the \((i+1)\)th central inspiration and that of the \((i+1)\)th peripheral inflation depends only on the previous delay \( x_i \). Figure 1 shows the placement of the \( R \) signal (black bars) on the \( P \) signal. In figure 2, the
Figure 1. Relationships between respiratory variable $R$ indicating the phases of inspiration with bold black bars (respectively expiration without bars) when $R$ equals 1 (respectively, 0) and pump variable $P$ indicating the phases of lung inflation (respectively, deflation) by an external pump when $P$ equals 1 (respectively, 0). $t_i^T$ denotes the delay between the $i$th inspiration and the $i$th inflation.

Figure 2. (a) Parametric space $(I, T)$, with indications of the experiments reported in figure 4 (black points 1–5) and of the observed $p/q$ entrainment; (c) experimental recording of the $R$ activity keeping only the bold black bars of figure 1 (displayed modulo $T$) showing an empirical $4/3$ entrainment; theoretical landscape showing (b) Arnold’s tongues in qualitative agreement with (a) experimental tongues, but quantitative discrepancies (e.g. a width of the harmonic 1/1 region larger in the theoretical than in the empirical case).

curve giving $x_{i+1}$ as function $f(x_i)$ of $x_i$ is called the respiratory phase response curve. Forcing the respiratory centres to have $p$ inspirations during $q$ pump cycles is called the $p/q$ entrainment that corresponds to the existence of a cycle of order $p$ for iterations of $f$ and to the occurrence of $(p–q)$ changes between

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the upper and the lower parts of the graph of $f$ during this cycle. The interval iterations of the set-valued map centred on the central cubic function (the width of this map surrounding $f$ corresponding to the contour lines of the empirical distribution containing 95% of the experimental points) gives the following results: a regularization occurs from $\varepsilon = 0.056$ to 0.2, with a $4/3$ (followed by a $6/5$, for $\varepsilon > 0.2$) entrainment (figure 5). The experimental phase response curve in figure 2 has been built from the data recorded in rabbits (Demongeot et al. 1987). We see in the table in figure 5 a dramatic change concerning the order of the observed entrainments: if we start from the initial point 0.25 or a short range interval $[0.5 - \varepsilon, 0.5 - \varepsilon]$, then we get a complex dynamics similar to that observed in point iterations; but when the length of the initial interval increases, then we observe a simplification of the cycles which tends to be a fixed interval or a cycle of two intervals, which is compatible with experimental observations where cycle order is systematically smaller than that predicted by point iterations. In practice, when the a priori knowledge on the noise is slight, we start with a large initial interval and hence we get a reduced number of bifurcations.

In Demongeot et al. (1997), we called ‘chaos’ (Hoppensteadt 1993) what we observed to be a unique fixed interval corresponding to the closure of the set of the chaotic point iterations values ($\varepsilon = 0$) (figure 3). The threshold $\varepsilon_c$ from which bifurcations to high-order cycles disappear is $\varepsilon_c = 0.025$ in figure 5. The theoretical bifurcation Arnold’s tongues landscape (figure 2b) fits qualitatively experimental data (Demongeot et al. 1987) and interval iterations regularize point iterations, replacing high-order cycles by their nearest sufficiently large bifurcation tongue (figure 4). The main interest of the interval iterations is to give, starting from different initial intervals, the same result on the entrainment fraction $p/q$, with the same uncertainty $\varepsilon = 0.1$ as on the experimental curve. This qualitative fit allows us to avoid at this step the model refutation, which could be the case, by examining only point iterations corresponding to a behaviour at the edge of chaos (figure 5).

Figure 6 shows the estimates of the empirical distribution of the data deviations with respect to the central interpolated curve (a cubic spline) given in figure 2: this distribution is not normal and has a skewness (Fisher $\gamma_1$)
Figure 4. Experimental records (retaining only the bold black bars of inspirations) corresponding to the empirical Arnold’s tongue landscape (figure 2b), with the indication of the parametric circumstances 2, 4 and 5 (top right).

Figure 5. Set-valued map extracted from experimental data in the case of the empirical entrainment $4/3$ (figure 2) for $\varepsilon = 0.1$, confining 95% of the (a) observed phase shift responses, where the solid (respectively dashed and dotted) line represents the cubic spline approximation (respectively the lower and upper curves of the 95% confidence cylinder) of the observed points, and (b) same map modulo $T = 1$, from $[0, 1]$ to $[0, 1]$; order of the interval limit cycles simulated with $\varepsilon$ increasing from 0 to 0.15, in the case of the initial interval $I(0) = [1/2 - \varepsilon, 1/2 + \varepsilon]$ or $I(0) = \{0.25\}$ (middle) and $I(0) = [0.25, 0.80]$ (bottom).

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parameter of skewness equals 0.31) and a kurtosis (Fisher $\gamma_2$ parameter of kurtosis equals 0.6) incompatible with the Gaussian hypothesis ($p = 0.4$ for both the skewness and the kurtosis test of normality). This strong evidence against normality prevents the use of the additive noised point iterations approaches (Homburg & Young 2006). We can make explicit other arguments against this approach.

**Proposition 2.1.** Let us consider the iterations on $I = [0, 1]$ by $f$ of a random variable $X_0$ valued in $I$, with a noisy innovation at each time step: $X_t = (1 + Y_{t-1})f(X_{t-1})$, where $f$ is the logistic function $f(x) = rx(1-x)$, $Y_0$ is a random variable independent of $X_0$ with mean 0 and variance $\epsilon^2$ (e.g. a variable uniform on $[-\sqrt{3}\epsilon, \sqrt{3}\epsilon]$), and $Y_{t-1}$ has the same density function as $Y_0$ and is independent of the $X_s$’s for times $s < t$ and of the $Y_s$’s for $s < t - 1$. When converging to an equilibrium measure, the probability distribution of $X_t$ has its variance $V(X_t)$ of order $O(r^2 t)$.

**Proof.** It is possible to prove the concentration towards an equilibrium measure for the associated dynamics in the space of probability distributions (Tong 1983; Wu & Shao 2004). $Y_{t-1}f(x_t - 1)$ can be considered as an additive random noise. If $X$ and $Y$ are independent, $Z = (1 + Y)f(X)$ has a variance $V(Z)$ verifying

$$
\epsilon^2 E(f^2(X)) \leq (1 + \epsilon^2) E(f^2(X)) - E^2(f(X)) = V(Z) = (1 + \epsilon^2) V(f(X)) + \epsilon^2 E^2(f(X)) \leq (1 + \epsilon^2) E(f^2(X)) \leq (1 + \epsilon^2) m^2,
$$

where $E$ denotes the expectation and $m = \sup_{x \in I} f(x)$. If $f$ is the logistic function $f(x) = rx(1-x)$, we have more precisely

$$
V(f(X)) = E(f^2(X)) - E^2(f(X)) = r^2[E(X^2(1-X)^2) - E^2(X(1-X))]
= r^2[E(X^2) - 2E(X^3) + E(X^4) - E^2(X) + 2E(X)E(X^2) - E^2(X^2)],
$$

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and, from Jensen’s inequality $-E(X^3) \leq -E^3(X)$, we get
\[ V(f(X)) \leq r^2[V(X) + 2E(X)V(X) + V(X^2)] \]
\[ \leq 2r^2 V(X)(1 + E(X)) \]  
(2.1)

and
\[ r^2\varepsilon^2 E(X^2(1 - X)^2) \leq V(Z) = E((1 + Y)^2f(X)^2) - E^2((1 + Y)f(X)) \]
\[ = \varepsilon^2 E(f(X)^2) - E^2(f(X)) \]
\[ = \varepsilon^2 V(f(X)) + (\varepsilon^2 - 1)E^2(f(X)) \leq 2\varepsilon^2 r^2 V(X)(1 + E(X)) \]
\[ \leq 4\varepsilon^2 r^2. \]  
(2.3)

Because the upper and the lower bounds of $V(x_t)$ are of the order $O(r^{2t})$, the variance of $x_t$ is not controlled in the additive noise scheme, if $r > 1$. The ignorance about the exact distribution of deviations to $f$ pushes us to choose a uniform random variable for $X_0$ and $Y_0$ and for the $Y_t$’s on the iterated intervals $I(t)$’s, especially in the vicinity of the point iteration case ($\varepsilon$ small). We will in the following denote by $F_\varepsilon(I)$ the image of the interval $I$ by a set-valued map whose graph is a neighbourhood $V_\varepsilon(f)$ of the graph of $f$ (Aubin & Frankowska 1990)

\[ F_\varepsilon(I) = [\inf\{y; x \in I, (x, y) \in V_\varepsilon(f)\}, \sup\{y; x \in I, (x, y) \in V_\varepsilon(f)\}]. \]

By choosing interval iterations, we do not manage the uncertainty as in Markov iterations of an initial probability distribution on $I$ by a deterministic transfer function $f$ leading to a stationary distribution (Dobbs 2007), but nevertheless the interval iteration we propose deals with an absence of knowledge on $f$ coming from an empirical intrinsic—but unknown and non-classical—noise. Comparable studies have been done in set-valued iterations theory (Nadler 1969; Smajdor 1985; Przytycki 1987) giving general conditions of convergence to attractors (De Farias & van Roy 2000; Graczyk et al. 2004; Kleptsyn & Nalskii 2004; Cheban & Mammana 2005; Desheng & Kloeden 2005; Kloeden & Valero 2005; Kamran 2007), but never solving the precise problem of their bifurcations in comparison with point iterations as reference when the uncertainty on $f$ vanishes.

Another type of iterations consists of adding a noise to the second member of equation (1.1): $X_{t+1} = f(X_t) + W(t)$, where $\{W(t)\}_{t \geq 0}$ is a uniform or Gaussian random process (Demongeot et al. 1987; Chan & Tong 1994). The succession of the ‘images’ of an interval in such a dynamics has analogies with interval iteration, but the initial uniform probability distribution is replaced by a non-uniform one since the first iteration, except if $X_0$ is uniform and if $W(t)$ depends on $X_t$ and is uniform on the interval $[(1 - \varepsilon)f(X_t), (1 + \varepsilon)f(X_t)]$. For example, near a stable fixed point $\beta/(1 - \alpha)$ of $f$, we can linearize $f$ as $f(x) \approx \alpha x + \beta$ and the local random iterations are

\[ X_t = (1 + Y_{t-1})f(X_{t-1}) \approx (1 + Y_{t-1})(\alpha X_{t-1} + \beta), \]

with $Y_{t-1}$ uniform on $[-\sqrt{3}\varepsilon, \sqrt{3}\varepsilon]$ and independent of the $X_t$’s for $s < t$ and the $Y_s$’s for $s < t - 1$. In this case, $X_t$ has the same asymptotic distribution as the random variable $\beta/(1 - \alpha) + Z$, where $Z = \beta \sum_{i \in N} \alpha^i Y_i$, i.e. a circular distribution on $[-\sqrt{3}\varepsilon\beta/(1 - \alpha), \sqrt{3}\varepsilon\beta/(1 - \alpha)]$ (Lévy 1939).

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3. Point iterations: basic definitions

To study the dynamics of a discrete system or to solve equations numerically, we consider an initial point and successively see what it becomes after a time interval. The sequence of the so-obtained points gives the dynamical behaviour of the initial point during iterations. When we consider a continuous univariate function $f$ defined on a bounded interval $I = [a, b]$ into itself, we denote the iterates of $f$ by the notation $f^0(x) = x; f^1(x) = f(x), \ldots, f^n(x) = f(f^{(n-1)}(x))$, for $n = 2, 3, \ldots$. Then we say that $x^*$ is a fixed point of period $m$ or order $m$ if and only if

$$(i)f^{om}(x^*) = x^* \quad \text{and} \quad (ii)f^{oj}(x^*) \neq x^* \quad \text{for} \quad j \leq m.$$

The orbit of a point $y$ iterated by the map $f$ is the set of points $W(y) = \{f^{oj}(y), \forall j = 0, 1, 2, \ldots\}$. The local behaviour of a fixed point $x^*$ of $f$ is determined by its multipliers defined by

$$\lambda(x^*) = \frac{df(x)}{dx}|_{x=x^*}.$$

The study of the behaviour of a discrete dynamical system consists of characterizing the multipliers of its fixed points and of its limit cycles. Then we have the classification

- $|\lambda(x^*)| < 1 \Rightarrow x^*$ is an attracting equilibrium or stable fixed point;
- $|\lambda(x^*)| = 0 \Leftrightarrow x^*$ is a super-stable fixed point;
- $|\lambda(x^*)| = 1 \Leftrightarrow x^*$ is a neutral fixed point; and
- $|\lambda(x^*)| > 1 \Rightarrow x^*$ is a repulsive equilibrium or unstable fixed point.

To be more precise, if $x^*$ is stable, any small perturbation from its equilibrium decays to zero, monotonically if $0 < \lambda(x^*) < 1$, or with decreasing oscillations if $-1 < \lambda(x^*) < 0$. On the other hand, if $x^*$ is unstable, any perturbation is monotonically amplified if $\lambda(x^*) > 1$, or by growing oscillations if $\lambda(x^*) < -1$. Bifurcations occur when $|\lambda(x^*)|$ passes through 1. So with $\lambda(x^*) = -1$, we have a pitchfork bifurcation, while with $\lambda(x^*) = 1$, we have a fold bifurcation.

Let $x_0, x_1, x_2, \ldots, x_{m-1}$ be the elements of an $m$-cycle $C$ of $f$, i.e. $C = (x_0, x_1, x_2, \ldots, x_{m-1})$ verifies

$$\forall i = 1, \ldots, m - 1, \quad f(x_{i-1}) = x_i \quad \text{and} \quad f(x_{m-1}) = x_0,$$

then all these points are fixed points of period $m$ of $f$. We can easily check, thanks to a chain of differentiations of $f$, that they have the same cycle multiplier that is defined by

$$\lambda^{(m)}(C) = f'(x_0)f'(x_1)\ldots f'(x_{m-1}) = \prod_{i=1, \ldots, m-1} f'(x_i) \quad (3.1)$$

with $f'(x_i) = df(x)/dx$ taken at $x = x_i$. For such a periodic orbit, we have the same classification as for fixed points replacing the point multiplier by $\lambda^{(m)}(C)$: $C$ is stable if $|\lambda^{(m)}(C)| < 1$; if $|\lambda^{(m)}(C)| = +1$, the cycle $C$ is called neutral; if $|\lambda^{(m)}(C)| > 1$, $C$ becomes unstable and is in general replaced by a new stable cycle of period $2^k+1$, with $m = 2^k$; the maximum of stability, called super-stability, is observed for the cycle $C$ such as $\lambda^{(m)}(C) = 0$.

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4. Interval iterations

Interval iterations can be considered as the natural generalization of point iterations. In this new iterations type, the entity to iterate is a compact interval with non-empty interior instead of a point. If we denote the upper (respectively, lower) boundary of the function \( f \) to iterate by \( f^+ = (1 + \varepsilon) f \) (respectively, \( f^- = (1 - \varepsilon) f \)), then after the \( (i+1) \)th iteration, we obtain from the interval \( I(i) \) another interval \( I(i + 1) \) defined by \( I(i + 1) = \inf_{x \in I(i)} f(x) \), which means that \( I(i + 1) = \inf_{x \in I(i)} (f(x) + \varepsilon) \), \( \sup_{x \in I(i)} f(x) \) from \( I(0) \) as \( [x_0, x_1] \subseteq I = [a, b] \), then \( I(1) = \inf_{x \in I(0)} f(x) \), \( \sup_{x \in I(0)} f(x) \). We can also define the interval iterations process by considering the set-valued application \( S_\varepsilon \) from \( I \) to the set of subsets of \( I \), denoted \( P(I) \), defined by

\[
\forall x \in I, \quad S_\varepsilon(x) = [f_\varepsilon(x), f^\varepsilon(x)] \quad \text{and} \quad I(i + 1) = \cup_{x \in I(i)} S_\varepsilon(x).
\]

By allowing a sufficient regularity (usually \( C^1 \)) to the map \( f \) on the interval \( I \), we can consider two possible situations,

(i) if \( x_0 = x_1 \), then \( I(0) = \{ x_0 \} \), \( I(1) = [f_\varepsilon(x_0), f^\varepsilon(x_0)] = [(1 - \varepsilon) f(x_0), (1 + \varepsilon) f(x_0)] \), and

(ii) if \( x_0 \neq x_1 \), the existence of \( I(1) \) follows from the fact that \( I(0) = [x_0, x_1] \) is a compact interval and that \( f_\varepsilon \) (respectively, \( f^\varepsilon \)) are continuous functions.

It has been already shown (Demongeot & Waku 2005) that the notions of fixed points and cycles of points can be generalized into those of fixed intervals and cycles of intervals. We will study hereafter the stability of this new type of iterations. For point iterations simulations, we begin the process with an initial point \( x_0 \) and successively compare the point \( x_i \) with \( x_{i+1} \) as \( i \) increases until two consecutive points coincide under a fixed stop criterion, when the iteration converges. The new approach adopted here involves, instead iterating points, iterating intervals. If the map under consideration is \( f \), then a neighbourhood centred on this function can be defined by the neighbourhood \( V_\varepsilon(f) \) defined using five different methods,

(i) \( V_\varepsilon(f) = \{(x, y) \in I^2 / y \in S_\varepsilon(x) \} \) (\( \varepsilon \)-vertical neighbourhood),

(ii) \( V_\varepsilon(f) = \{(x, y) \in I^2 / \sup_{z \in f^{-1}(y)} d(x, z) \leq \varepsilon \} \) (\( \varepsilon \)-vertical neighbourhood),

(iii) \( V_\varepsilon(f) = \cup B((x, f(x)), \varepsilon(x)) \cap I^2 \), with \( B((x, f(x)), \varepsilon(x)) \) the open ball centred in \( (x, f(x)) \) with radius \( \varepsilon(x) \) (orthogonal tubular neighbourhood),

(iv) \( V_\varepsilon(f) = \{(x, y) \in I^2 / \exists \rho \in [r - \varepsilon, r + \varepsilon] \text{ and } y = f_\rho(x) \} \), where \( \rho \) is a parameter (\( 2\varepsilon \)-diagonal parametric neighbourhood), and

(v) \( V_\varepsilon(f) = \) the subset of \( I^2 \) delimited by the contour line of a probability distribution \( g \) on \( I^2 \) containing \( (1 - \varepsilon) \) per cent of the mass of \( g \), whose projection of its set of maxima (crest line) is \( f \) (supposed to be an application). Given a set of data, \( g \) could be replaced by a smooth estimation of the presence probability distribution of the experimental points. If the marginal repartition function \( G_x \) of \( y \) (considered as a random variable) is continuous for any \( x \in I \), then we can take \( V_\varepsilon(f) = \{(x, y) \in I^2 / y \in [h_x(\varepsilon/2), H_x(1 - \varepsilon/2)] \} \), where we denote, if \( G_x \) is not invertible: \( h_x(\xi) = \inf_{z; G_x(z) = \xi} z \) and \( H_x(\xi) = \sup_{z; G_x(z) = \xi} z \). All the definitions above account for the existence of a certain uncertainty.
about the knowledge of \( f \), whose empirical observations are supposed to lie in \( V_\varepsilon (f) \). The first (respectively, second) definition of \( V_\varepsilon (f) \) expresses this uncertainty as a statistical regression with \( x \) (respectively, \( y \)) as a control variable. The third definition is used if uncertainty is \( x \)-dependent, and the fourth when uncertainty lies on the parameter of the function, if the variation coefficient of the data (the ratio between standard deviation and mean) remains constant. The fifth definition corresponds to a neighbourhood in the observation space of the experimental data. We will in the following use definition (1.1).

5. Dynamical properties

**Lemma 5.1.** (After Demongeot et al. (1997).) Let \( f \) be a \( C^1 \) uni-modal map of the interval \([0, 1]\) and \( I \) an invariant interval for the interval iterations process in which \( f \) has its unique maximum \( f(x_c) \) at \( x_c \), then \( I = F_\varepsilon (I) \supset [f_\varepsilon (f^\varepsilon (x_c)), f^\varepsilon (x_c)] \).

**Proof.** (i) By definition, \( \sup (f^\varepsilon (I)) = f^\varepsilon (x_c) \), hence (figure 1): \( \sup I = \sup (f^\varepsilon (I)) = f^\varepsilon (x_c) \). (ii) We also have: \( \inf (I) \leq f_\varepsilon (f^\varepsilon (x_c)) \), because \( f^\varepsilon (x_c) \in f^\varepsilon (I) \subset I \) and \( \forall x \in I \), \( \inf (I) \leq f_\varepsilon (x) \).

**Theorem 5.2.** Let \( f \) be a \( C^1 \) uni-modal map of the interval \([0, 1]\) (i.e. \( f(x) \) increases from 0 to \( f(x_c) \), where \( x_c \) is the location of the maximum of \( f \), then decreases to 0). If \( I \) denotes the stable fixed interval, supposed to be unique, of the interval iterations process of the set-valued map defined by \( \forall x \in [0, 1], S_\varepsilon (x) = [f_\varepsilon (x), f^\varepsilon (x)] \), then \( I \) contains the stable fixed points \( x^*_\varepsilon \) and \( x^{*\varepsilon} \) of, respectively, \( f_\varepsilon \) and \( f^\varepsilon \) supposed to exist and to be unique. We have

(i) if \( x^*_\varepsilon < x^{*\varepsilon} \), then \( I = [x^*_\varepsilon, x^{*\varepsilon}] \);
(ii) if \( x^*_\varepsilon < x_c < x^{*\varepsilon} \), then \( I = [\inf(x^*_\varepsilon, f_\varepsilon (f^\varepsilon (x_c))), f^\varepsilon (x_c)] \); and
(iii) if \( x_c \leq x^*_\varepsilon < x^{*\varepsilon} \), then there exists,

— either an interval \([x_1, x_2]\), for which \( x_1 > x_c \), \( f_\varepsilon (x_2) = x_1 \), \( f^\varepsilon (x_1) = x_2 \) and \( x_1 \) (respectively, \( x_2 \)) is a stable fixed point of \( f_\varepsilon \) (respectively, \( f^\varepsilon \)), with \( f_\varepsilon (f^\varepsilon (x_c)) > x_c \), then \( I = [x_1, x_2] \),

— or an interval \([x_1, x_2]\) with the same characteristics as above, but with \( f_\varepsilon (f^\varepsilon (x_c)) \leq x_c \), then we have a cycle of intervals,

\[
I_1 = [f_\varepsilon (f^\varepsilon (x_c)), x_2], \quad I_2 = [x_1, f^\varepsilon (x_c)] \quad \text{if we start from } I(0) = [x_c, x_c],
\]

— or an interval \([x_1, x_2]\) with the same characteristics as above, but with \( x_1 \leq x_c \), then

\[
I_1 = [f_\varepsilon (f^\varepsilon (x_c)), f^\varepsilon (x_c)],
\]

— or, more generally, a sequence of \( 2^m \) points in \([0, 1]\), \( x_1, \ldots, x_{2^m} \) as extremities of \( 2^{m-1} \) intervals constituting a limit cycle of order \( 2^{m-1} \). The order of apparition of these intervals in interval iterations will be described in propositions 5.4 and 5.5.

**Proof.** (i) If \( x^*_\varepsilon \) and \( x^{*\varepsilon} \) are less than \( x_c \), they are stable fixed points of, respectively, \( f_\varepsilon \) and \( f^\varepsilon \) and the result comes from the fact that \( f_\varepsilon \) and \( f^\varepsilon \) are
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$f^\varepsilon = f(1 + \varepsilon)$

$f^\varepsilon = f(1 - \varepsilon)$

Figure 7. Existence of an interval $[x_1, x_2]$, for which $x_1 > x_c$, $f_\varepsilon(x_2) = x_1$, $f^\varepsilon(x_1) = x_2$ and $x_1$ (respectively, $x_2$) is a stable fixed point of $f_\varepsilon \circ f^\varepsilon$ (respectively, $f^\varepsilon \circ f_\varepsilon$).

Figure 8. The first three steps of logistic interval iterations for $k = 2$, $r = 2.7$, $\varepsilon = 0.15$ and $I_0 = \{1/2\}$.

increasing between 0 and $x_c$; if we start from $I(0) = [x_\varepsilon^*, x_\varepsilon^*]$, then the iterated intervals $I(i)$ converge to $[x_\varepsilon^*, x_\varepsilon^*]$ because

- $f^{\varepsilon, k}(x_\varepsilon^*) \leq x_\varepsilon^*$ and $f^{\varepsilon, k}(x_\varepsilon^*) \geq x_\varepsilon^*$, $\forall k \geq 2$;
- $f^{\varepsilon, k}(x_\varepsilon^*)$ and $f^{\varepsilon, k}(x_\varepsilon^*)$ converge, respectively, to $x_\varepsilon^*$ and $x_\varepsilon^*$ when $k$ tends to infinity; and
- $I \supset \bigcup_{k \in \mathbb{N}} \left([x_\varepsilon^*, f^{\varepsilon, k}(x_\varepsilon^*)] \cup [f^{\varepsilon, k}(x_\varepsilon^*), x_\varepsilon^*]\right)$.

(ii) The result comes from the fact that $f^\varepsilon(x_c)$ is the maximum value of $f^\varepsilon$ on $[0,1]$ and from lemma 5.1.

(iii) The results come from the fact that $x_1$ (respectively, $x_2$) is a stable fixed point of $f_\varepsilon \circ f^\varepsilon$ (respectively, $f^\varepsilon \circ f_\varepsilon$), and from lemma 5.1 (figures 7 and 8).

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the partial order defined on $x \subset X$. Let $x$ configuration corresponding to the limit cycle of intervals $[x, 1]$ belonging to a limit cycle of order $l$. Then, if $\epsilon$ is sufficiently small, $x$ verifies

\[ w_0(f) = \tilde{w}_0(f) = Id_{[0, 1]} < w_1(f) = f^\epsilon \circ f^\epsilon \circ f^\epsilon < \tilde{w}_1(f) = f^\epsilon \circ f^\epsilon \circ f^\epsilon < \cdots \]

\[ < \tilde{w}_n(f) = w_{n-1}(f) \circ f^\epsilon \circ w_{n-1}(f) \circ f^\epsilon \circ w_{n-1}(f) \circ f^\epsilon \circ \tilde{w}_{n-1}(f). \]

**Proposition 5.4.** Let $f$ be $C^1$ uni-modal map of $[0, 1]$ on itself and $x$ in $[x, 1]$ belonging to a limit cycle of order $2^k$ for the interval iterations of $f$. Then, if $\epsilon$ is sufficiently small, $x$ verifies

(i) if $k = 2p$, then $x = f^\epsilon \circ w_0(f)(x)$, and

(ii) if $k = 2p + 1$, then $x = f^\epsilon \circ f_0(f) \circ f_0(f) \circ \tilde{w}_p(f)(x)$.

**Proof.** Let the case $p = 0$: the only possible configuration corresponding to an invariant interval $[x_1, x_2]$ is given in figure 9a and we have: $x = f^\epsilon \circ w_0(f)(x)$. In case $p = 1$, the point $x_1$ (respectively, $x_2$) has bifurcated in two points, $x_1$ and $x_2$ (respectively, $x_3$ and $x_4$), and the only possible configuration corresponding to the limit cycle of intervals $([x_1, x_2], [x_3, x_4])$ is given in figure 9b and we have $x_1 = f^\epsilon \circ f^\epsilon \circ f_0(f)(x_4) = f^\epsilon \circ \tilde{w}_1(f)(x_3)$. The other possible configurations are $x_1 = f^\epsilon \circ f^\epsilon \circ f_0(f_0)(x_4)$, $x_1 = f^\epsilon \circ f^\epsilon \circ f_0(f_0)(x_4)$ and $x_1 = f^\epsilon \circ f^\epsilon \circ f_0(f_0)(x_4)$ (figure 9c–e), all incompatible with the dynamics. Indeed,

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$x_1$ needs to be the ancestor of $x_i$, because $f^r(x_1)$ is the rightmost among the iterated extremities of intervals, and $x_1$ (respectively, $x_2$) needs to be the ancestor of $x_2$ (respectively, $x_3$) for the same kind of argument. The case where $p$ is any integer strictly more than 1 is proved by using an induction based on the fact that the case $p = 0$ is obtained from the case $p - 1$ from a doubling period bifurcation. It is true when $\varepsilon = 0$, because it comes from the point iterations dynamics of the functions $f \ C^1$ uni-modal map of the interval [0,1] on itself, with $f(0) = f(1) = 0$ (Collet & Eckmann 1980), and it remains true for $\varepsilon$ sufficiently small by using an argument of continuity; for large $\varepsilon$, we use simulations in the case of the logistic function $f(x) = rx(1 - x)$.

**Proposition 5.5.** The functional $w$-order induces a punctual $w$-order on the indices set of the points visited on [0,1]. If $\{x_1, x_2, \ldots, x_{2k}\}$ represents, in the natural order of [0,1], the extremities of the $2^{k-1}$ intervals observed after $(k - 1)$ bifurcations, then let $i_k(m)$ be the index of the $(m - 1)$th iterate of $x_{2k}$ (for $m$ integer). Then the punctual $w$-order verifies

\[
k = 0, \quad i_0(m) = 1, \quad \forall m \in \mathbb{N}, \quad m \geq 1
\]
\[
k = 1, \quad i_1(m - 1) = 2, \quad i_1(m) = 1, \quad \text{for} \ m = 2p, \quad \forall p \in \mathbb{N}, \quad p \geq 1
\]
\[
\text{for } k \geq 2, \text{ we have a recurrence between } i_k', s \text{ and } i_k'' s,
\]
\[
i_k(1) = 2^k - i_{k-2}(2) + 1,
\]
\[
i_k(2) = i_{k-2}(2) = 1,
\]
\[
i_k(3) = 2^{k-1} + i_{k-2}(2),
\]
\[
i_k(4) = 2^{k-1} - i_{k-2}(2) + 1,
\]
\[
j = 1
\]
\[
i_k(5) = 2^k - i_{k-2}(1) + 1,
\]
\[
i_k(6) = i_{k-2}(1),
\]
\[
i_k(7) = 2^{k-1} + i_{k-2}(1),
\]
\[
i_k(8) = 2^{k-1} - i_{k-2}(1) + 1,
\]
\[
\vdots
\]
\[
i_k(8j - 7) = 2^k - i_{k-2}(2j) + 1,
\]
\[
i_k(8j - 6) = i_{k-2}(2j),
\]
\[
i_k(8j - 5) = 2^{k-1} + i_{k-2}(2j),
\]
\[
i_k(8j - 4) = 2^{k-1} - i_{k-2}(2j) + 1,
\]
\[
j = 2^{k-3}
\]
\[
i_k(8.2^{k-3} - 7) = 2^k - i_{k-2}(2^{k-2}) + 1,
\]
\[
i_k(8.2^{k-3} - 6) = i_{k-2}(2^{k-2}),
\]
\[
i_k(8.2^{k-3} - 5) = 2^{k-1} + i_{k-2}(2^{k-2}),
\]
\[
i_k(8.2^{k-3} - 4) = 2^{k-1} - i_{k-2}(2^{k-2}) + 1,
\]
\[
\vdots
\]
\[
i_k(8.2^{k-3} - 3) = 2^k - i_{k-2}(2^{k-2} - 1) + 1,
\]
\[
i_k(8.2^{k-3} - 2) = i_{k-2}(2^{k-2} - 1),
\]
\[
i_k(8.2^{k-3} - 1) = 2^{k-1} + i_{k-2}(2^{k-2} - 1),
\]
\[
i_k(8.2^{k-3}) = 2^{k-1} - i_{k-2}(2^{k-2} - 1) + 1.
\]

**Proof.** The above formulae can be easily proved by recurrence. Notice that we have $\forall k \geq 1, 1 \leq j \leq 2^{k-3}, i_k(8j - 7) + i_k(8j - 6) = i_k(8j - 5) + i_k(8j - 4) = i_k(8j - 3) + i_k(8j - 2) = i_k(8j - 1) + i_k(8j) = 2^k + 1$.

The following examples are given for the logistic function.

(i) For $k = 2$, the punctual $w$-order gives (table 1)
\[
i_2(1) = 4, \quad i_2(2) = 1, \quad i_2(3) = 3, \quad i_2(4) = 2.
\]
for any integer \( n \geq 0 \), defined by the following recurrence relations:
\[
A_{131271; http://research.att.com/sequences (registered in the Encyclopedia of Integer Sequences under reference by considering the integer sequences we are calling median spiral order

\[\text{Phil. Trans. R. Soc. A} 4730\]  
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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10}
\caption{The succession of even integers met in the sequence \( a(6, \cdot) \).}
\end{figure}

Table 1. Interval limit cycle of order 2, with \( I(0) = [0.500, 0.500] \), \( \epsilon = 0.0003 \) and \( r = 3.600 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( f_2(x_1) )</th>
<th>( f_3(x_2) )</th>
<th>( f_3(x_3) )</th>
<th>( I_1 )</th>
<th>( I_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3238</td>
<td>0.6012</td>
<td>0.7881</td>
<td>0.9001</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(ii) For \( k = 3 \), the punctual \( w \)-order gives (table 2)
\[
i_3(1) = 8, \quad i_3(2) = 1, \quad i_3(3) = 5, \quad i_3(4) = 4, \quad i_3(5) = 7, \quad i_3(6) = 2, \quad i_3(7) = 6, \quad i_3(8) = 3.
\]

(iii) For \( k = 4 \), the punctual \( w \)-order gives (table 3)
\[
i_4(1) = 16, \quad i_4(2) = 1, \quad i_4(3) = 9, \quad i_4(4) = 8, \quad i_4(5) = 13, \quad i_4(6) = 4, \quad i_4(7) = 12, \quad i_4(8) = 5, \quad i_4(9) = 15, \quad i_4(10) = 2, \quad i_4(11) = 10, \quad i_4(12) = 7, \quad i_4(13) = 14, \quad i_4(14) = 3, \quad i_4(15) = 11, \quad i_4(16) = 6.
\]

More precisely, we can show that the recurrence can be established by considering the integer sequences we are calling median spiral order sequences (registered in the Encyclopedia of Integer Sequences under reference A131271; http://research.att.com/~njas/sequences/?q=demongeot&sort=1&fmt =0&language=english&go=Search), defined by the following recurrence relations: for any integer \( k \), let \( n = 2^{k-2} \) and consider the sequence
\[
\forall m = 1, \ldots, n/2, \quad a(k, 2m-1) = a(k-1, m) \quad \text{and} \quad a(k, 2m) = n + 1 - a(k, 2m-1).
\]

For \( 2 \leq k \leq 7 \), we have, for example
\[
a(2, \cdot) = \{1\}, \quad a(3, \cdot) = \{1, 2\}, \quad a(4, \cdot) = \{1, 4, 2, 3\}, \quad a(5, \cdot) = \{1, 8, 4, 5, 2, 7, 3, 6\},
\]
\[
a(6, \cdot) = \{1, 16, 8, 9, 4, 13, 5, 12, 2, 15, 7, 10, 3, 14, 6, 11\},
\]
\[
a(7, \cdot) = \{1, 32, 16, 17, 8, 25, 9, 24, 4, 29, 13, 20, 5, 28, 12, 21, 2, 31, 15, 18, 7, 26, 10, 23, 3, 30, 14, 19, 6, 27, 11, 22\}.
\]

The succession of the even integers respects the following scheme (similar to those exhibited in Cosnard (1983); figure 10).
Table 2. Interval limit cycle of order 4, with $I(0) = [0.500, 0.500]$, $\varepsilon = 0.0003$ and $r = 3.520$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3726</td>
<td>0.3736</td>
<td>0.5108</td>
<td>0.5133</td>
<td>0.8228</td>
<td>[0.3726, 0.3736]</td>
</tr>
<tr>
<td>$x_5 = f^k(x_1)$</td>
<td>$x_4 = f^k(x_2)$</td>
<td>$x_7 = f^k(x_3)$</td>
<td>$x_2 = f^k(x_4)$</td>
<td>$x_6 = f^k(x_5)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Interval limit cycle of order 8, with $I(0) = [0.480, 0.4800]$, $\varepsilon = 0.0003$ and $r = 3.560$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3483</td>
<td>0.3496</td>
<td>0.3716</td>
<td>0.3755</td>
<td>0.4907</td>
<td>0.4990</td>
<td>0.5488</td>
<td>0.5523</td>
<td>[0.4907, 0.4990]</td>
</tr>
<tr>
<td>$x_9 = f^k(x_1)$</td>
<td>$x_8 = f^k(x_2)$</td>
<td>$x_{13} = f^k(x_3)$</td>
<td>$x_{12} = f^k(x_4)$</td>
<td>$x_5 = f^k(x_5)$</td>
<td>$x_7 = f^k(x_7)$</td>
<td>$x_2 = f^k(x_8)$</td>
<td>$x_1 = f^k(x_9)$</td>
<td></td>
</tr>
</tbody>
</table>

Then we have

$$\forall j = 1, \ldots, k - 1, \quad i_k(2^{j+1} - 3) = 2^k + 1 - a(k, j), \quad i_k(2^{j+1} - 2) = a(k, j),$$

$$i_k(2^{j+1} - 1) = 2^{k-1} + a(k, j), \quad i_k(2^{j+1}) = 2^{k-1} + 1 - a(k, j).$$

For $k = 5$, we have, for example (the sequence $a(5, \cdot)$ is indicated in bold),

$$\{i_k(j)\}_{j=1}^n = \{32, 1, 17, 16, 25, 8, 24, 9, 29, 4, 20, 13, 28, 5, 21, 12, 31, 2, 18, 15, 26, 7, 23, 10, 30, 3, 19, 14, 27, 6, 22, 11\}.$$

These results indicate that the $i_k(j)$’s are iterated by exchanging intervals from four sets of close intervals, whose extremities belong, respectively, to the four sets: $\{i_k(2j - 2)\}_{j=2}^{2^k - 1}, \{i_k(2j)\}_{j=2}^{2^k - 1}, \{i_k(2j - 1)\}_{j=2}^{2^k - 1}$ and $\{i_k(2j - 3)\}_{j=2}^{2^k - 1}$. 

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We leave the reader to check the consistency of the recurrent formulae of proposition 5.5 with the examples given in tables 1–3. We can make more precise the relationship between \( x_1 \) and \( x_2 \), respectively fixed points of \( f_\varepsilon \) of \( \varepsilon \) and \( f_\varepsilon^{of} \), verifying

\[
x_2 > x_1 > x_c, \quad f_\varepsilon(x_2) = x_1 \quad \text{and} \quad f_\varepsilon^{of}(x_1) = x_2,
\]

as in case iii(a) of theorem 5.2, because if \( \exists \tau > 0/\forall y \in [x_1 - \tau, x_2 + \tau], -1 < f'(y) < 0 \), then \( f \) is locally invertible, and there exists \( \eta \) given by Rolle’s theorem, such as \( 0 < \eta < x_2 - x_1 \) and \( x_1 = x_2 - e/f'(x_2 - \eta) \), where \( e = f(x_2) - f(x_1) = x_1 - x_2 + \varepsilon(f(x_1) + f(x_2)) \). The interval \([x_1, x_2] \) is then a stable fixed interval for the interval iterations dynamics. More generally, for proving the existence of fixed intervals for \( f^{om} \) (\( m=1 \) in the previous example), we can use numerous fixed set theorems proven for the set-valued mappings (figure 11) (Nadler 1969; Smajdor 1985; Przytycki 1987; De Farias & van Roy 2000; Reich & Zaslavski 2002; Graczyk et al. 2004; Kleptsyn & Nalskii 2004; Cheban & Mammana 2005; Desheng & Kloeden 2005; Kloeden & Valero 2005; Kamran 2007; Wlodarczyk et al. 2007). These theorems involve in general a local Lipschitzian property of the set-valued mapping, for an adapted metric.

**Proposition 5.6.** Let us consider the case of the set-valued logistic mapping \( f(x) = rx(1 - x) \) and use as set distance the Hausdorff metric \( H \). Then, if we start from \( I(0) = [x, y] \), with \( 1/2 < x < x^* < y < 1 \), where \( x^* = f(x^*) \), there exists a stable fixed interval \( I^* \) to which converge the iterates \( I(t) = f(I(t - 1)) \) of \( I(0) \).

**Proof.** We have \( I(1) = [(1 - \varepsilon)ry(1 - y),(1 + \varepsilon)rx(1 - x)] \), \( I(2) = [(1 - \varepsilon^2)r^2x(1 - x)(1 - (1 + \varepsilon)rx(1 - x)),(1 - \varepsilon^2)r^2y(1 - y)(1 - (1 - \varepsilon)ry(1 - y))] \), and by

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Application of interval iterations

\[ I_1 = [0.4907, 0.4990] \]
\[ I_2 = [0.8896, 0.8901] \]
\[ I_3 = [0.3483, 0.3496] \]
\[ I_4 = [0.8080, 0.8096] \]
\[ I_5 = [0.3716, 0.3755] \]
\[ I_6 = [0.8802, 0.8816] \]
\[ I_7 = [0.8312, 0.8349] \]
\[ I_8 = [0.8080, 0.8096] \]

Figure 12. Graphical example of the limit cycle of order 8 corresponding to table 3.

Table 4. Limit cycle orders of interval iterations starting with the interval \( I(0) = [0.25, 0.25] \).

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>0</th>
<th>0.001</th>
<th>0.005</th>
<th>0.010</th>
<th>0.025</th>
<th>0.050</th>
<th>0.075</th>
<th>0.100</th>
<th>0.1250</th>
<th>0.150</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.50 cycle order</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.10 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.20 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.30 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.40 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.50 cycle order</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.60 cycle order</td>
<td>752</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.70 cycle order</td>
<td>1458</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.80 cycle order</td>
<td>3625</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

supposing that \( y - x^* > x^* - x \) and \( \varepsilon \) is sufficiently small, we have

\[
H(I(i), I(i + 1)) - H(I(i + 1), I(i + 2)) = (f^\varepsilon(x) - y) - (f^\varepsilon(y) - f^\varepsilon(x)) = (\beta - \alpha) + (f^\varepsilon(\beta) - f^\varepsilon(\alpha)),
\]

where \( \alpha = y \) and \( \beta = f^\varepsilon(x) \); but \( f^\varepsilon \) is a local Lipschitzian contraction, if \( 2 < r < 3 \), because

\[
f^\varepsilon(x^*_\varepsilon) = (1 - \varepsilon)(2 - r).
\]

Then the set-valued application is a set-valued contraction mapping and the existence of a stable fixed interval has been proved in Nadler (1969).

We can obtain the results of tables 1–3 and figure 12 by using either interval iteration simulations or Newton iterations or continued fractions method (Lange 1986) which gives with a controlled accuracy the roots of polynomials. For example, in the logistic case, if \( \varepsilon = 0.0003 \) and \( r = 2.8 \), the fixed point of the function \( f(x) = rx(1 - x) \) is equal to 0.6428571429 and the invariant interval is \([0.6418799286, 0.6438345399]\). By looking at simulated dynamical behaviours (tables 4–6), we observed (table 4), for \( \varepsilon \) and \( I(0) \) small, the same first doubling bifurcations as for point iterations. When \( \varepsilon \) increases, \( I(0) \) remaining narrow, then we observe progressively only one doubling bifurcation (table 5) and when \( \varepsilon \) is small but \( I(0) \) wide, we observe always the fixed interval: \( I = [f^\varepsilon(f^\varepsilon(x_c)), f^\varepsilon(x_c)] \) because \( f^\varepsilon(f^\varepsilon(x_c)) < x_c \) and \( x^*_\varepsilon > x_c \) (table 6 and case iii(c) of theorem 5.2).

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Table 5. Limit cycle orders of interval iterations starting with the interval $I(0) = [0.50 - \varepsilon, 0.50 + \varepsilon]$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.0001</th>
<th>0.005</th>
<th>0.010</th>
<th>0.025</th>
<th>0.050</th>
<th>0.075</th>
<th>0.100</th>
<th>0.1250</th>
<th>0.150</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.50 cycle order</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.10 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.20 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
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<tr>
<td>3.30 cycle order</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<tr>
<td>3.50 cycle order</td>
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<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<tr>
<td>3.60 cycle order</td>
<td>752</td>
<td>2</td>
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<td>3.70 cycle order</td>
<td>1458</td>
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<tr>
<td>3.80 cycle order</td>
<td>3625</td>
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<td>1</td>
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<td>1</td>
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Table 6. Limit cycle orders of interval iterations starting with the interval $I(0) = [0.25, 0.80]$.

<table>
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<tr>
<th>$r$</th>
<th>0.0001</th>
<th>0.005</th>
<th>0.010</th>
<th>0.025</th>
<th>0.050</th>
<th>0.075</th>
<th>0.100</th>
<th>0.1250</th>
<th>0.150</th>
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<td>3.70 cycle order</td>
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</tr>
<tr>
<td>3.80 cycle order</td>
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<td>1</td>
<td>1</td>
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</table>

6. Demographic application

(a) Verhulst model

The equation defined by

$$N_{t+1} = N_t(r - dN_t)$$

is called the ‘logistic difference equation’. It is well known that, when $d = 0$, the population of size $N_t$ at time $t$ grows (respectively, decreases) purely geometrically if $r > 1$ (respectively, $r < 1$) (May 1976; Murray 2002). When $d \neq 0$, the growth curve presents a hump whose steepness is turned by $r$. The canonical form of equation (6.1) is obtained by defining the normalized size $X_t = dN_t/r$

$$X_{t+1} = rX_t(1 - X_t).$$

In equation (6.2), $X_t$ lies in the interval $[0,1]$ and the positive constant $r$ is the rate of growing/decay of the population. To improve the model, let $K$ be the maximum of the population taken into consideration, then

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1.0

\( b \)

\( a \)

0.8

0.6

0.4

0.2

0 0.2 0.4 0.6 0.8 1.0

\( x \): normalized size of population

\( f(x) \): unimodal function

Figure 13. Verhulst model: examples of curves with \( K = 1 \). (a) \( r = 1.5 \) and (b) \( r = 3.5 \).

Table 7. Stability of the fixed point \( x^*_2 \) in the Verhulst model.

| \( r = 2 \) | \( x^* \) is super-stable |
| \( r = 1 \) or \( r = 3 \) | \( x^* \) is neutral |
| \( 1 < r < 3 \) | \( x^* \) is attractant or stable |
| \( r < 1 \) or \( r > 3 \) | \( x^* \) is repulsive or unstable |

\[ X_{t+1} = rX_t \left( \frac{1 - X_t}{K} \right), \tag{6.3} \]

which has the interest of indicating explicitly that the maximum value of the population is \( K \), depending on the available resources. \( K \) is the carrying capacity of the environment. For this equation (6.3), the variable lies naturally in \([0, K]\) and equation (6.3) is sometimes referred as the Verhulst equation. When dealing with equilibrium conditions, one can note that the logistic growth has two stationary states, namely \( x^*_1 = 0 \) and \( x^*_2 = K(1 - 1/r) \); \( x^*_1 \) is always unstable and \( x^*_2 \) is stable if and only if \( 1 < r < 3 \). From this observation, one says that the carrying capacity \( K \) determines essentially the size of the steady-state population while \( |2 - r| \) is a measure of the rate at which it is reached (figure 13). If we consider the logistic function \( f(x) = rx(1 - x) \), which describes the behaviour of the logistic model, we have two fixed points, i.e. \( x^*_1 = 0 \) and \( x^*_2 = 1 - 1/r \), having, respectively, multipliers equal to \( \lambda(x^*_1) = r \) and \( \lambda(x^*_2) = 2 - r \). One can easily see that \( x^*_1 \) is attracting for \( 0 < r < 1 \), the only realistic values where the positivity of the function is respected; \( x^*_1 \) is neutral for \( r = 1 \) (we then observe a fold bifurcation owing to the non-stability of \( x^*_1 \) for \( r > 1 \)) and repulsive for \( r > 1 \). The classification of proposition 2.1 yields also for \( x^*_2 \) in regard of the parameter \( r \) as can be seen in table 7.

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We can see that \( f \) is defined onto the interval \([0,1]\). So the region where the study of this function really has an interest is the one with values of \( r \) between 0 and 4. This is simply obtained by considering that the value of the function at its maximum has to be in the interval of definition of the function. Detailed studies in the unstable fixed points region defined by \( r \) are interesting because it is where the unpredictable dynamical behaviour is encountered. The interval generally considered as the invariant region of this map is \([x_L, x_U] = \left[ \frac{r^2(4 - r)}{16}, \frac{r}{4} \right]\), because the critical point is \( x_c = \frac{1}{2} \) (see also Demongeot & Waku 2005). By considering interval variations, we observe in tables 4 and 5 that if \( I(0) \) is a singleton or a narrow interval for \( r \) larger than 3 and \( \varepsilon = 0.001 \), then the invariant interval becomes a cycle of order 2 of intervals and after \( r = 3.5 \) a cycle of order 4 of intervals. The cascade of doubling bifurcations we always observe dramatically changes for \( \varepsilon > 0.01 \) (only one bifurcation is observed) and for \( \varepsilon > 0.1 \) (no more bifurcations are observed). If \( I(0) \) becomes sufficiently wide we cannot obtain any cycle and there is always a unique stable interval equal to \( [f_\varepsilon(f_\varepsilon(x_c)), f_\varepsilon(x_c)] \).

\( b \) Ricker model

The equation defined by

\[
N_{t+1} = N_t \exp \left[ r \left( \frac{1 - N_t}{K} \right) \right]
\]

(6.4)

represents the ‘Ricker model’ for the growth of a single species in ecology literature (May & Oster 1976; Reich & Zaslavski 2002). Ricker introduced equation (6.4) to describe fish populations from the Pacific coast of Canada. This equation has also been used to describe epidemic systems. The Ricker model (figure 14) is considered to be biologically and ecologically more realistic than
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Table 8. Stability of the fixed point $x^*$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$x^*$ stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 1$</td>
<td>$x^*$ is super-stable</td>
</tr>
<tr>
<td>$0 &lt; r &lt; 2$</td>
<td>$x^*$ is attractant or stable</td>
</tr>
<tr>
<td>$r = 2$</td>
<td>$x^*$ is Lyapunov stable</td>
</tr>
<tr>
<td>$r &gt; 2$</td>
<td>$x^*$ is repulsive or instable</td>
</tr>
</tbody>
</table>

the logistic model. If $K = 1, f(x) = x \exp[r(1 - x)]$, which describes the behaviour of the Ricker model related to equation (6.4). We observe two fixed points, $x^*_1 = 0$ whose multiplier is $\lambda(x^*_1) = \exp(r)$ and $x^*_2 = 1$, with $\lambda(x^*_2) = 1 - r$. As in §7a, we can summarize the dynamical behaviour of the second fixed point in table 8. As the critical point is $x_c = 1/r$, the general invariant region of this map is $[\exp(2r - 1 - e^{r-1})/r, \exp(r - 1)/r]$. By considering interval iterations, we observe the same qualitative features as for the logistic model, i.e. the same cascade of doubling period bifurcations starting at $r = 2$, for $\varepsilon < 0.01$ as for $\varepsilon = 0$, plus a systematic fixed interval for $\varepsilon$ more than 0.1 (table 6). We can finally remark that some uni-modal growth curves in population dynamics show pitfalls in upper and lower bounds of the population size; in these cases, the precise corresponding interval iterations must be obtained (Demongeot & Waku 2005).

7. Conclusion

After a physiological example justifying the introduction of interval iterations and a brief review about point iterations, we defined interval iterations by using instead of function $f$ a couple of functions $f_\varepsilon = (1 - \varepsilon)f$ and $f^\varepsilon = (1 + \varepsilon)f$ corresponding to the lower and upper bounds of a set-valued mapping of width $2\varepsilon f$ containing $f$. The interval iterations have the advantage that they can be reduced to the classical point iterations when $\varepsilon$ tends to 0 and so can be considered as their natural extension. In this paper, we have carried out some calculations on interval iterations showing essentially that, when the parameter $\varepsilon$ is small, the bifurcations of point iterations are respected. When $\varepsilon$ takes large values, depending on the shape of $f$, a stable fixed interval is quickly reached after a few iterations. The main interest of interval iterations is to better take into account the uncertainty about the function $f$ to iterate, owing to multiple errors about the exact location of experimental points. Then the truly observed bifurcations in experiments are more comparable to those observed during simulated interval iterations of the set-valued mapping and then the qualitative fit with empirical data becomes, in general, better.

References


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