This paper presents an overview of the current state of the art in the analysis of discontinuity-induced bifurcations (DIBs) of piecewise smooth dynamical systems, a particularly relevant class of hybrid dynamical systems. Firstly, we present a classification of the most common types of DIBs involving non-trivial interactions of fixed points and equilibria of maps and flows with the manifolds in phase space where the system is non-smooth. We then analyse the case of limit cycles interacting with such manifolds, presenting grazing and sliding bifurcations. A description of possible classification strategies to predict and analyse the scenarios following such bifurcations is also discussed, with particular attention to those methodologies that can be applied to generic n-dimensional systems.

Keywords: piecewise smooth systems; bifurcations; discontinuity-induced bifurcations

1. Introduction

While the behaviour of systems described by smooth vector fields has a long and distinguished history (e.g. Guckenheimer & Holmes 1983; Kuznetsov 2004), an understanding of the dynamics of systems that are not smooth is still in its infancy. Such systems fall into broad (overlapping) classes: impacting, piecewise smooth (PWS) and hybrid (systems with a mixture of continuous and discrete time dynamics). They are often used to capture the behaviour of systems that contain some sort of discontinuous events. Examples of such systems include devices with impacts and friction in mechanics, walking and hopping machines in robotics, power electronic circuits in electrical engineering, control systems and more recently gene regulatory networks and neurons in computational neuroscience and biology (e.g. Brogliato 1999; Leine 2000; de Jong et al. 2004; Coutinho et al. 2006; di Bernardo et al. 2008a; Polynikis et al. 2009).

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Once such a model is available, it is often essential in applications to characterize its bifurcations. But most of the existing theoretical and numerical tools deal with smooth dynamical systems.

In contrast, the current theory of bifurcations in dynamical systems that are not smooth is still incomplete. Most of the results available deal with PWS dynamical systems, and we will focus on this class of systems in this review. PWS systems are described by vector fields changing configuration according to the location of their trajectory in phase space. Specifically, the phase space of a PWS system can be divided into several regions associated with different functional forms of the system vector field. A given PWS system of interest can then be classified in terms of its degree of smoothness across switching manifolds dividing one region from the other.

It is possible to have discontinuous state trajectories across such manifolds, as in the case of impacting systems, or continuous states but discontinuous vector fields, as in the case of so-called Filippov systems (Filippov 1988). A further possibility is that of piecewise smooth continuous (PWSC) vector fields that are continuous across the phase-space boundaries but possess a discontinuous Jacobian.

PWSC, Filippov and impacting systems can all exhibit a wide range of nonlinear phenomena including bifurcations and chaos (di Bernardo et al. 2008a). In addition to classical bifurcations, PWS systems can exhibit unique phenomena. These events, better known as discontinuity-induced bifurcations (DIBs), have been found to occur when an invariant set of the system, e.g. an equilibrium or limit cycle, crosses or hits tangentially one of the switching manifolds in phase space. When this occurs, the system can exhibit dramatic transitions, including sudden transitions from periodic solutions to large-scale chaotic attractors that have been observed in a large number of examples (with possibly the most famous experimental observation of such a phenomenon being the ‘jump to chaos’ studied in power electronic converters or mechanical impact oscillators; see Nordmark (1991) and Banerjee & Verghese (2001) for further details).

Over the past few years, much research effort has been spent classifying and investigating these phenomena (for a review of the available results, see di Bernardo et al. (2008b) or the books by Mosekilde & Zhusubalyev (2003), Leine & Nijmeijer (2004) and di Bernardo et al. (2008a)). Initially, the analysis focused on the classification of the possible bifurcation scenarios that occur when fixed points in maps collide with switching manifolds (so-called border-collision (BC) bifurcations, or more simply border collisions). The pioneering work by Feigin (1970, 1974, 1995) established analytical conditions for piecewise linear continuous (PWLC) maps to classify BCs involving fixed and two-periodic points. A complete classification of all possible scenarios in the one- and two-dimensional case was then presented by Nusse & Yorke (1992, 1994, 1995) and Banerjee & Grebogi (1999). A classification of the simplest scenarios in n-dimensions was reported in di Bernardo et al. (1999).

An equally intensive research effort concerned the analysis of DIBs of equilibria and limit cycles in continuous-time PWS systems. In the case of equilibria, interesting scenarios were highlighted in di Bernardo et al. (2008c), including the possibility of discontinuity-induced Hopf bifurcations. In general, such bifurcations, also termed boundary equilibrium bifurcations (BEBs), were studied...
Review. Piecewise smooth systems

in low-dimensional PWSC or Filippov systems (e.g. Küpper & Moritz 2001; Zou & Küpper 2001a; di Bernardo et al. 2002a; Leine & van Campen 2002; Leine & Nijmeijer 2004; Zou et al. 2006).

In the case of limit cycles, the main DIB phenomenon which has been shown to affect dramatically the system behaviour is the well-known grazing bifurcation (e.g. Nordmark 1992, 2001; Chin et al. 1994; di Bernardo et al. 2001b). Grazing is said to occur when a limit cycle hits tangentially one of the switching manifolds in phase space. As discussed later, this can explain the sudden transition to chaos from periodic solutions often observed in practical applications.

In all of these cases, the lack of a consistent theory for PWS systems means that the analysis is carried out by using local linearization and so-called discontinuity mappings (§7).

The aim of this paper is to present a review of the current state of the art on the analysis of bifurcations and chaos in PWS dynamical systems, highlighting the many pressing open problems that still need to be resolved in order to achieve a general framework for such phenomena. Given the large number of results available in the literature, we will focus here on those that can be applied to generic n-dimensional PWS systems.

2. Classifying piecewise smooth systems

To begin with, we should point out that even the definition of a PWS system is something that needs careful consideration. There are several different definitions proposed in the literature. From these then follow different ideas of what constitutes a solution, an orbit, topological equivalence and, hence a bifurcation. We do not have the space to discuss all the different aspects here. The reader is referred to the excellent account recently given in Colombo (2009).

To give a flavour of the issues involved, we give the following summary of the various definitions of a PWS system. Filippov (1988) has a weak definition of PWS, allows regions of phase space to be empty and boundaries between regions to have cusps. This allows for greater complication with little extra gain. Broucke et al. (2001) use a definition that equates a PWS system to a special type of hybrid automaton (e.g. Branicky et al. 1998). But neither Filippov (1988) nor Broucke et al. (2001) give definitions that deal with hysteresis or impact. di Bernardo et al. (2008a) give a definition that allows for impacts, but do not give conditions for behaviour when two boundaries intersect, allow for an infinite number of finite time impacts and boundaries to intersect with zero angle and also do not deal with hysteresis. Simic et al. (2005) are even more general, but still omit sliding and hysteresis. Finally, Colombo (2009) himself gives another definition that is closer to a hybrid automaton than that due to Broucke et al. (2001), but which unifies the three broad categories of impacting, PWS and hybrid systems.

As far as the notion of solution is concerned, most authors accept the one due to Filippov (1988). But the idea of an orbit and topological equivalence differ between authors. For example, both Filippov (1988) and Kuznetsov et al. (2003) take orbits to be forward time solutions. Their definitions of topological equivalence between two systems differ, however (Filippov (1988) requires the direction of time to be preserved). Teixeira (1990, 1993) restricts his definition...
of topological equivalence to points close to the switching manifold and Broucke et al. (2001) adopt a viewpoint that only allows one orbit to pass through each point in state space.

Given the number of different definitions, it seems sensible here to focus on one set of definitions and develop the theory accordingly. Hence, we will focus our attention on bimodal PWS systems of the form

\[
\dot{x} = \begin{cases} 
F_1(x, \mu), & \text{if } H(x, \mu) > 0, \\
F_2(x, \mu), & \text{if } H(x, \mu) < 0,
\end{cases} 
\tag{2.1}
\]

where \( x \in D \subseteq \mathbb{R}^n, \mu \in \mathbb{R} \), \( F_1, F_2 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) and \( H : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) are sufficiently smooth functions of both their arguments throughout \( D \). We will label as \( \Sigma \) the switching manifold (or discontinuity boundary) defined by \( H(x) = 0 \). (Note that, in general, it is assumed the vector fields either are defined on \( \Sigma \) or are set-valued functions.)

Locally, \( \Sigma \) divides \( D \) into the two regions \( D_1 \) and \( D_2 \) where the system is smooth and defined by the vector fields \( F_1 \) and \( F_2 \), respectively:

\[
D_1 = \{ x \in D : H(x, \mu) > 0 \}
\]

and

\[
D_2 = \{ x \in D : H(x, \mu) < 0 \}.
\]

We assume that both the vector fields \( F_1 \) and \( F_2 \) are defined over the entire local region of phase space under consideration, i.e. on both sides of \( \Sigma \). Thus, the flows \( \Phi_i, i = 1, 2 \), generated by each of the vector fields can be defined as the quantities that satisfy

\[
\frac{\partial}{\partial t} \Phi_i(x, t) = F_i(\Phi_i(x, t)), \quad \Phi(x, 0) = x. \tag{2.2}
\]

Here, we assume that such flows can be expanded as a Taylor series about the switching manifold.

Then, the degree of smoothness of the system vector field can be defined as follows.

**Definition 2.1** (di Bernardo et al. 2008a, ch. 2, p. 73). The degree of smoothness of system (2.1) at a point \( x_0 \in \Sigma \) is equal to \( r \) if the Taylor series expansions of \( \Phi_i(x_0, t) \) and \( \Phi_j(x_0, t) \) with respect to \( t \), evaluated at \( t = 0 \), agree up to terms of \( O(t^{r-1}) \). That is, the first non-zero partial derivative with respect to \( t \) of the difference \([\Phi_i(x_0, t) - \Phi_j(x_0, t)]\)\( |_{t=0} \) is of order \( r \).

We investigate the possible DIBs of equilibria for three different classes of PWS dynamical systems: (i) systems with degree of smoothness equal to 2 or PWSC, (ii) systems with degree of smoothness equal to 1 or Filippov systems, and (iii) systems with degree of smoothness equal to 0 or impacting systems.

Note that, in the case of PWSC systems, the Jacobian is discontinuous across \( \Sigma \) while both the states and the vector field are continuous (degree of smoothness \( r = 2 \)). Therefore, we assume the following continuity assumption to be satisfied

\[
F_1(x, \mu) = F(x, \mu) \quad \text{and} \quad F_2(x, \mu) = F(x, \mu) + G(x, \mu)H(x, \mu), \tag{2.3}
\]

for some smooth function \( G : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n \), so that when \( H(x, \mu) = 0 \), then \( F_1 = F_2 \).
Filippov systems are instead characterized by having $F_1(x) \neq F_2(x)$ when $x \in \Sigma$ ($r = 1$). It is well known that if $\langle \nabla H, F_1 \rangle \langle \nabla H, F_2 \rangle < 0$, then these systems can exhibit so-called sliding motion (Filippov 1988). This solution type is characterized ideally by infinitely many switchings forcing the trajectory to remain (slide) on $\Sigma$. In general, suppose that the direction of the system vector field points simultaneously towards $\Sigma$ from both of its sides. Then, when the system trajectory hits the switching surface, it will be constrained to evolve on it, until the direction of the vector field on one side or the other changes. Hence, by studying the gradient of the vector field in a neighbourhood of $\Sigma$, we can identify regions $\tilde{\Sigma} \subset \Sigma$ where sliding is possible, which we will term sliding regions. As detailed in Utkin (1992), such regions can be defined analytically. Moreover, the system dynamics within the sliding region can be studied by looking at an appropriate reduced order system. This can be obtained by applying Utkin’s equivalent control method (Utkin 1992) or Filippov’s convex method (Filippov 1988). Specifically, sliding can occur through points satisfying $H(x, \mu) = 0$, where the system follows the sliding vector field defined by

$$F_s(x, \mu) = (1 - \lambda) F_1(x, \mu) + \lambda F_2(x, \mu),$$

where $\lambda = \lambda(x, \mu)$ is chosen to keep $H = 0$. The values of $\lambda$ are restricted to $0 < \lambda < 1$.

Finally, impacting systems are systems whose state variable $x$ is itself discontinuous ($r = 0$). In general, these systems take the form

$$\dot{x} = F(x, \mu) \quad \text{if} \quad H(x, \mu) > 0 \quad (2.4)$$

with impact at the surface $\Sigma$ defined by $H(x, \mu) = 0$, where the impact law is defined by a reset map $R(x^-, \mu)$, taking the form

$$x^+ = R(x^-, \mu) = x^- - G(x^-, \mu) H_x F(x^-, \mu). \quad (2.5)$$

As is well known in the literature (e.g. Budd & Dux 1994; Osorio et al. 2005; A. B. Nordmark 2007, personal communication), these systems can exhibit chattering (also known as the Zeno phenomenon in the hybrid systems literature; Zhang et al. 2000). Chattering corresponds to the accumulation of an infinite sequence of events in finite time.

3. Bifurcations of piecewise smooth systems

PWS systems can exhibit the same classical bifurcations as smooth systems. In each of the phase-space regions, the corresponding vector field is smooth and well defined everywhere and hence, under parameter variations, saddle nodes, Hopf and all other classical bifurcations can be observed to occur. Here, we focus on those bifurcation phenomena that are unique to PWS systems or DIBs. Colombo (2009) distinguishes between weak and strong DIBs. The former occur in limit cycles where the Poincaré map is smooth near the bifurcation point. Traditional methods can then be used to study bifurcations in these systems, even though some of the resulting bifurcations are not standard (e.g. early work by Hogan (1989)). We are concerned here with strong DIBs, where the Poincaré map is not smooth near the bifurcation point.
The current working definition of a DIB as presented in di Bernardo et al. (2008a, ch. 2, p. 98) can be given by first extending the concept of topological equivalence to PWS systems.

**Definition 3.1.** Two PWS dynamical systems of the form (2.1) with, possibly, smooth reset maps $R$ and $\tilde{R}$ applying at the switching manifolds $\Sigma$ and $\tilde{\Sigma}$, respectively, are said to be *piecewise topologically equivalent* if

— they are topological equivalent, i.e. there exists a homeomorphism $h$ that maps the orbits of the first system onto orbits of the second one, preserving the direction of time;
— the homeomorphism $h$ can be chosen so as to map each of the switching manifolds of the first system onto each of the manifolds of the second one.

**Definition 3.2.** A system of the form (2.1) is said to undergo a DIB at a parameter value $\mu = \mu_0$ if there exists an arbitrarily small perturbation that leads to a loss of piecewise topological equivalence.

We can now list and classify the main type of DIBs.

(a) *Discontinuity-induced bifurcations*

DIBs in PWS maps and flows can be classified heuristically as follows in terms of the system invariant set (equilibrium, limit cycle etc.) which is involved in the phenomenon of interest.

— *BCs of fixed points*. These bifurcations occur in PWS discrete-time systems or maps and correspond to a fixed point (or periodic point) of the map being perturbed away from the map switching manifold under parameter variations.
— *Boundary equilibrium bifurcations* (BEBs). Similar to BCs but involving equilibria of PWS flows.
— *Grazing bifurcations of limit cycles*. A grazing bifurcation is said to occur whenever a limit cycle (or a section of it) intersects tangentially one of the switching manifolds in phase space. Grazing can occur in all classes of PWS flows and can be associated with dramatic bifurcation scenarios.
— *Sliding bifurcations of limit cycles*. These bifurcations can only occur in Filippov systems. In this case, a limit cycle develops an intersection (tangential or transversal) with a sliding region, i.e. a region on one of the system switching manifolds where sliding is possible.
— *DIB of tori or strange attractors*. Finally, it is possible for other invariant sets, such as tori or strange attractors, to become involved in a DIB, e.g. by developing tangencies with one of the switching manifolds in phase space. The study and understanding of these events is still an open problem in the existing literature. Some preliminary results can be found in Zhusubaliyev et al. (2002, 2008), Benadero et al. (2003), Zhusubaliyev & Mosekilde (2006), Szalai & Osinga (2009) and Simpson & Meiss (2008).

In what follows, we will look at each of the phenomena listed above, briefly describing how they can be classified in §7.
4. Border collisions and boundary equilibrium bifurcations

(a) Border collisions

We start by considering the case of piecewise linear maps. We focus our attention to a region \( D \subset \mathbb{R}^n \) of phase space which is chosen so that, by an appropriate choice of local coordinates, the map under investigation can be described as

\[
x \mapsto \begin{cases} 
G_1(x, \mu) & \text{if } H(x) > 0, \\
G_2(x, \mu) & \text{if } H(x) < 0,
\end{cases}
\]  

(4.1)

where \( H(x) = 0 \) defines a smooth boundary \( \Sigma \) which, as for PWS flows, separates \( D \) into two regions denoted by \( D_1 \) and \( D_2 \), i.e.

\[
\Sigma := \{ x \in D : H(x) = 0 \},
\]  

(4.2)

\[
D_1 := \{ x \in D : H(x) > 0 \}
\]  

(4.3)

and

\[
D_2 := \{ x \in D : H(x) < 0 \}.
\]  

(4.4)

We assume that \( G_1(x, \mu) \in C^k \) if \( x \in D_1 \), \( G_2(x, \mu) \in C^k \) if \( x \in D_2 \) and \( G_1(x, \mu) = G_2(x, \mu) \) when \( x \in \Sigma \), i.e. the map is smooth to order \( k \) in each of the subregions \( D_1 \) and \( D_2 \) while it is continuous on \( \Sigma \).

Let \( \hat{x} \) be a fixed point of such a mapping that we assume depends continuously on the parameter \( \mu \in (-\epsilon, \epsilon) \). We say that \( \hat{x} \) is a border-crossing fixed point if it crosses \( \Sigma \) transversally, as \( \mu \) is varied between \(-\epsilon \) and \( \epsilon \). Without loss of generality, we assume that (i) \( \hat{x} \in \Sigma \) for \( \mu = 0 \), (ii) \( \hat{x} \in D_1 \) for \(-\epsilon < \mu < 0 \), and (iii) \( \hat{x} \in D_2 \) for \( 0 < \mu < \epsilon \).

Definition 4.1 (BC—after Nusse & Yorke (1992)). We say that a fixed point \( \hat{x} \) undergoes a BC bifurcation for \( \mu = 0 \) if \( \hat{x} \) is a border-crossing point, and linearizing the map about \( \hat{x} \) in \( D_1 \) and \( D_2 \) yields

\[
\frac{\partial G_1}{\partial x} (\hat{x}, \mu) \big|_{\mu=0} \neq \frac{\partial G_2}{\partial x} (\hat{x}, \mu) \big|_{\mu=0}.
\]  

(4.5)

Condition (4.5) implies that the Jacobian of the map about the fixed point is discontinuous at the border-crossing point.

BC bifurcations can organize several types of bifurcation scenarios in a PWS system. The number of references in the literature is vast and a complete list cannot be reported here for the sake of brevity. Examples include Feigin (1970), Nusse & Yorke (1992, 1995), Zhusubaliyev et al. (2001), Jain & Banerjee (2003), Avrutin & Schanz (2004), Kowalczyk (2005), Sushko et al. (2005), Roy & Roy (2008), Glendinning & Wong (2009) and Pring & Budd (2010). Understanding the relationship between the properties of the PWS map and the scenario observed at a BC is the aim of classification strategies, which will be presented in §7.

(b) Boundary equilibrium bifurcations

As pointed out in Leine (2000) (see also di Bernardo et al. 2008c), equilibria of PWS flows can also interact with the switching manifolds as parameters are varied. Suppose that \( \hat{x} \) is an equilibrium point of equation (2.1), which we assume
depends continuously on the parameter $\mu \in (-\varepsilon, \varepsilon)$. We say that $\hat{x}$ is a border-crossing equilibrium if it crosses transversally $\Sigma$, as $\mu$ is varied between $-\varepsilon$ and $\varepsilon$. Without loss of generality, we assume that (i) $\hat{x} \in \Sigma$ for $\mu = 0$, (ii) $\hat{x} \in F_1$ for $-\varepsilon < \mu < 0$, and (iii) $\hat{x} \in F_2$ for $0 < \mu < \varepsilon.$

**Definition 4.2 (BEB).** We say that a stationary point $\hat{x}$ undergoes a BEB for $\mu = 0$ if $\hat{x}$ is a border-crossing equilibrium, and linearizing the system about $\hat{x}$ in $D_1$ and $D_2$ yields

$$\frac{\partial F_1}{\partial x}(\hat{x}, \mu)|_{\mu=0} \neq \frac{\partial F_2}{\partial x}(\hat{x}, \mu)|_{\mu=0}.$$ (4.6)

It is worth mentioning here that the definition$^1$ given above is not exactly equivalent to the one reported in Leine (2000). There, the DIB of an equilibrium is defined as the point at which the eigenvalues of the system are set-valued and contain a value on the imaginary axis. Thus, the definition given here is weaker than the one contained in Leine (2000).

5. Grazing bifurcations in piecewise smooth flows

Up to now, we have looked at DIBs of stationary points in maps and flows. We now move to the case of DIBs of limit cycles in flows. In this case, sudden changes in the qualitative dynamics of a PWS system are often associated with tangential intersections (grazings) of a system periodic orbit with a switching manifold (e.g. Casas et al. 1996; Piiroinen et al. 2004; Dankowicz & Zhao 2005; Thota & Dankowicz 2006).

We say that a periodic orbit $\hat{x} = \hat{x}(t)$ of equation (2.1) is a grazing orbit if, for some time $t = t^*$, $x(t)$ hits tangentially the switching manifold $\Sigma$ (defined by $H(x) = 0$) at the point $x^* = x(t^*)$, which is termed the grazing point. At the grazing point, the following conditions are satisfied for $i = 1, 2$:

$$H(t^*) = 0,$$ (5.1)

$$\nabla H(x^*) \neq 0,$$ (5.2)

$$\left\langle \nabla H(x^*), \frac{\partial \Phi_i}{\partial t}(x^*, t^*) \right\rangle = \left\langle \nabla H(x^*), F_i^* \right\rangle = 0$$ (5.3)

and

$$\left. \frac{d^2 H(\Phi(x^*, t))}{dt^2} \right|_{t = t^*} = \left\langle \nabla H^*, \frac{\partial F_i^*}{\partial x} \right\rangle + \left\langle \frac{\partial^2 H}{\partial x^2} F_i^*, F_i^* \right\rangle > 0.$$ (5.4)

The first two conditions state that $H$ correctly defines the switching manifold. The third condition states that at the grazing point, $x = x^*$, the vector field is tangent to $\Sigma$, i.e. $(\partial H/\partial t)(x^*) = 0$. The final condition then ensures that the curvature of the trajectory on both sides of the switching manifold is of the same sign with respect to $H$ (without loss of generality, we assume this sign to be positive). Heuristically, this means that the system trajectory has a local minimum at the grazing point.

$^1$Note that, in Leine (2000), these bifurcations are termed discontinuous bifurcations.
In addition, we assume that no sliding motion can take place on \( \Sigma \). A necessary condition to avoid sliding is that, under the flow of system (2.1) sufficiently close to the grazing point, the boundary \( \{ H = 0 \} \) should never be simultaneously attracting (or repelling) from both sides \( D_1 \) and \( D_2 \); that is
\[
(\nabla H, F_1)(\nabla H, F_2) > 0, \quad x \in D
\]
for some \( \epsilon > 0 \). This assumption will be removed in §6 where grazings in systems with sliding are considered.

**Definition 5.1 (grazing bifurcation).** A periodic orbit \( \hat{x}(t) \) is said to undergo a grazing bifurcation for \( m = 0 \) if (i) it is a grazing orbit for \( m = 0 \) at the grazing point \( \hat{x}^* = \hat{x}(t^*) \), (ii) for \( \epsilon > 0, \delta > 0 \) sufficiently small, \( \hat{x}(t), t \in (t^* - \delta, t^* + \delta) \) does not cross \( \Sigma \) if \( m \in [-\epsilon, 0[, \) and (iii) \( \hat{x}(t), t \in (t^* - \delta, t^* + \delta) \) crosses \( \Sigma \) transversally for \( m \in ]0, \epsilon[ \).

As we will see in §7, the properties of the system vector field across the boundary are crucial in the determination of the bifurcation scenarios following a grazing bifurcation.

Another case, reported in the literature and given the name of corner collision, is the grazing intersection of a periodic orbit with a switching manifold that is itself non-smooth (corner). This is the case, for instance, in many power electronics systems, where the switching manifold is often defined by non-smooth signals (e.g. sawtooth signal). Corner collisions have recently been detected also in impacting systems and, most notably, can be used to explain the bifurcation phenomena often detected in cam-follower mechanical systems, as detailed in Alzate et al. (2007), Osorio et al. (2005, 2008) and Budd & Piironen (2006).

### 6. Sliding and chattering bifurcations

As mentioned above, PWS systems can also exhibit a peculiar type of solution, the so-called sliding motion (Filippov 1988). Sliding either can be introduced ad hoc for control purposes by appropriately designed switching controllers (Utkin 1992) or can be an inherent feature of the system, e.g. owing to the presence of friction (Oestreich et al. 1996; Galvanetto 1997; Blazejczyk-Okolewska et al. 1999; Dankowicz 1999; di Bernardo et al. 2003).

We say that a limit cycle of system (2.1) undergoes a sliding bifurcation when grazing occurs on the boundary of the region in phase space where sliding is possible.

Up to very recently, analytical and numerical investigations had shown that there were four fundamental types of sliding bifurcations (figure 1), as reported in di Bernardo et al. (2001d, 2002b) and also independently in Feigin (1994).

But very recent work by Jeffrey & Hogan (submitted) has shown that the picture is more complicated. By using singularity theory of scalar functions, they derived a complete classification of sliding bifurcations. The key idea was to attribute sliding bifurcations to geometric properties of the switching manifold, namely to points where there are folds, cusps and two-folds. In this way, they uncovered a further four sliding bifurcations, all catastrophic in nature. One of these has been observed in a model of a superconducting resonator Jeffrey et al. (2010).
Figure 1. Four possible bifurcations involving collision of a segment of the trajectory with the boundary of the sliding region: (a) transversal intersection or sliding–crossing; (b) switching–sliding; (c) grazing–sliding; (d) adding–sliding. (Reproduced from di Bernardo et al. (2008a) with permission from Springer-Verlag.)

Sliding bifurcations have been shown to give rise to complex phenomena including deterministic chaos and can be used to explain the formation and metamorphosis of stick-slip oscillations in friction oscillators (Oestreich et al. 1996; di Bernardo et al. 2003) and relay feedback systems (di Bernardo et al. 2001d).

In impacting systems, chattering (or Zeno) bifurcations have also been proposed to occur when the accumulation point of an infinite chattering sequence (a Zeno point in the hybrid systems literature; Zhang et al. 2001) crosses one of the phase-space boundaries under parameter variations. Such a bifurcation, first noted by A. B. Nordmark (2007, personal communication), is now being studied in cam-follower mechanical devices (Osorio et al. 2005, 2008). In these devices, it has been noted that it is indeed the presence of a complete chattering sequence becoming incomplete as the Zeno accumulation point crosses itself that the switching manifold causes intricate bifurcation structures. Chattering can also organize the elaborate structure of the basins of attraction of different coexisting asymptotic solutions in impacting systems as noted in Budd & Dux (1994).

7. Classification of discontinuity-induced bifurcations

Having outlined the main features of each of the principal DIBs, we now propose a set of simple defining and non-degeneracy conditions for the various types of DIBs that allows the prediction of the simplest possible scenarios following a particular DIB. We will focus our attention on those techniques that can be applied to generic $n$-dimensional PWS maps and flows. We start with the case of BCs.
(a) Border-collision bifurcations

The first methodology to classify the possible scenarios following a BC bifurcation in PWLC maps is due to Feigin (1970, 1974, 1978). The methodology was later expounded in greater detail in di Bernardo et al. (1999). In a different context, a classification strategy for PWLC maps was also proposed by Nusse & Yorke (1992, 1994, 1995) (see also Banerjee & Grebogi 1999). Later extensions include the case of PWLC maps with a gap (Jain & Banerjee 2003; Hogan et al. 2007).

In his methodology, Feigin addresses the problem of classifying the simplest possible scenarios following a BC in a generic $n$-dimensional PWLC map. Because of its generality, the strategy can be used to distinguish between the three following scenarios at a BC.

— *Persistence*, where the border-colliding fixed point persists under parameter variations, crossing transversally the switching manifold.
— *Non-smooth fold*, where a stable and an unstable fixed point emerge from the BC point.
— *Non-smooth period doubling*, where a stable (or unstable) period-2 point emerges from the BC point.

The occurrence of each of these three scenarios can be easily detected by using the following classification theorem (for a proof, see di Bernardo et al. (2008a)).

Consider the following PWLC map:

\[
\begin{align*}
    x & \rightarrow \begin{cases}
    A_1 x + B \mu, & c^T x > 0, \\
    A_2 x + B \mu, & c^T x < 0,
    \end{cases}
\end{align*}
\]  

(7.1)

where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, $A_1$ and $A_2$ are matrices of appropriate dimensions satisfying the continuity assumption

\[
A_2 = A_1 + G c^T,
\]

(7.2)

with $G$ being an arbitrary vector in $\mathbb{R}^n$, and $B$ and $c^T$ are vectors of the appropriate dimensions.

**Theorem 7.1 (Feigin 1970, 1974).** Let $p_1(\lambda)$ be the characteristic polynomial of matrix $A_1$ and $p_2(\lambda)$ the characteristic polynomial of $A_2$ in equation (7.1). Moreover, let

\[
\begin{align*}
    \sigma_1^+ & := \text{number of real eigenvalues of } A_1(\alpha_i) \text{ greater than } 1; \\
    \sigma_2^+ & := \text{number of real eigenvalues of } A_2(\beta_i) \text{ greater than } 1; \\
    \sigma_1^- & := \text{number of real eigenvalues of } A_1 \text{ less than } -1; \\
    \sigma_2^- & := \text{number of real eigenvalues of } A_2 \text{ less than } -1.
\end{align*}
\]
Assume that the following non-degeneracy conditions are satisfied:

\[
\begin{align*}
\det(I - A_1) &\neq 0, \\
\det(I + A_1) &\neq 0, \\
c^T(I - A_2)^{-1}B &\neq 0, \\
1 - c^T(I - A_1)^{-1}G &\neq 0 \\
\text{and} \\
1 - c^T(I + A_1)^{-1}G &\neq 0.
\end{align*}
\]

Then, at a BC, we have:

**persistence** if either

\[
1 - c^T(I - A_1)^{-1}E > 0 \tag{7.3}
\]

or, equivalently,

\[
p_1(1)p_2(1) > 0 \tag{7.4}
\]

or

\[
\sigma_1^+ + \sigma_2^+ \text{ is even;} \tag{7.5}
\]

**non-smooth fold** if

\[
1 - c^T(I - A_1)^{-1}G < 0 \tag{7.6}
\]

or, equivalently,

\[
p_1(1)p_2(1) < 0 \tag{7.7}
\]

or

\[
\sigma_1^+ + \sigma_2^+ \text{ is odd;} \tag{7.8}
\]

**non-smooth period doubling** if

\[
1 + c^T(I + N_1)^{-1}E < 0 \tag{7.9}
\]

or, equivalently,

\[
p_1(-1)p_2(-1) < 0 \tag{7.10}
\]

or

\[
\sigma_1^- + \sigma_2^- \text{ is odd.} \tag{7.11}
\]

(b) Boundary equilibrium bifurcations

BEBs in flows can be studied similarly to BCs by extending to PWSC flows the classification strategy proposed by Feigin for BCs. The idea is once again to look at the characteristic polynomials of the matrices of the piecewise linear system obtained by linearizing the PWSC flow of interest about the boundary equilibrium point.

Without loss of generality, we assume that the BEB takes place at \(x = 0\) when \(\mu = 0\), and we consider a sufficiently small neighbourhood of the origin, say \(D \subset \mathbb{R}^n\), which is divided by \(\Sigma\) into the two disjoint subsets \(D_1\) and \(D_2\) where the vector field is smooth.
Linearizing the PWS flow given by equation (4.5) with respect to $x$ and $\mu$ on both sides of $\Sigma$, we then have

$$\dot{x} = \begin{cases} A_1 x + B \mu & \text{if } c^T x > 0, \\ A_2 x + B \mu & \text{if } c^T x < 0, \end{cases}$$

(7.12)

where

$$A_1 = \frac{\partial F_1}{\partial x} \bigg|_{x=0}, \quad A_2 = \frac{\partial F_2}{\partial x} \bigg|_{x=0}$$

(7.13)

and

$$B = \frac{\partial F_1}{\partial \mu} \bigg|_{\mu=0} = \frac{\partial F_2}{\partial \mu} \bigg|_{\mu=0}.$$ 

(7.14)

Note that since we are assuming the vector field to be continuous but non-smooth across $\Sigma$, i.e. when $x_1 = 0, \mu = 0$, the matrices $A_1 = [a_{ij}^{(1)}]$ and $A_2 = [a_{ij}^{(2)}]$ are such that $a_{ij}^{(1)} = a_{ij}^{(2)}$ if $j \neq n$.

We can now start a classification of the possible behaviour at a discontinuous bifurcation by analysing the piecewise linear flow (7.12). As already done for BCs, it is possible to give conditions on the eigenvalues of $A_1$ and $A_2$ that identify different dynamical scenarios. In particular, let $x^*$ and $x^{**}$ be two equilibria of the subsystems $\dot{x} = A_1 x + B \mu$ and $\dot{x} = A_2 x + B \mu$, respectively, and let $\sigma_i^+$ and $\sigma_i^-$ be the number of real eigenvalues of $A_i$ greater and less than zero, respectively, $i = 1, 2$. To describe what happens as the system parameter $\mu$ is varied, in the simplest case, the two following conditions can be derived.

— **Persistence.** An equilibrium exists on one side of the boundary (corresponding to the fixed point $x^*$), and is converted at the discontinuous bifurcation point ($\mu = 0$) into a stationary point of the same type existing on the other side of the boundary (corresponding to $x^{**}$), if

$$\sigma_1^- + \sigma_2^+ \text{ is even.}$$

(7.15)

— **Non-smooth fold.** The two equilibria, $x^*$ and $x^{**}$, both exist on one side only of the boundary, collide on the border at a discontinuous bifurcation, $\mu = 0$, and then vanish on the other side, if

$$\sigma_1^- + \sigma_2^+ \text{ is odd.}$$

(7.16)

The strategy presented here is valid for $n$-dimensional systems. An extension of this strategy to other classes of PWS systems including impacting flows has been presented in di Bernardo et al. (2008c), where discontinuity-induced Hopf bifurcations are also discussed. These phenomena are associated with the emergence of a family of limit cycles from the BEB point whose amplitude scales linearly with the parameter variation (see Zou & Küpper 2001b; di Bernardo et al. 2008c).

8. Classification of grazing and sliding bifurcations

We now look at the problem of classifying grazing and sliding bifurcations of limit cycles.
The starting point is to note that grazing of limit cycles can be associated with border-crossing fixed points of appropriately defined normal-form maps. Without loss of generality, assume that a periodic orbit \( p(t) \) undergoes a grazing bifurcation for \( m = 0 \) at the grazing point \( p^* = p(0) = 0 \). Let \( \Pi \) be some appropriately chosen Poincaré hyperplane transversal to the system flow on one side of the phase-space boundary \( \Sigma \) (figure 2).

Assume \( \Gamma \) to be the manifold on \( \Pi \) that is mapped by the flow to the grazing set \( L_g \) on \( \Sigma \), i.e. the set of point on \( \Sigma \) associated with all trajectories that graze. Grazing orbits will intersect \( \Pi \) on \( \Gamma \). Moreover, \( \Gamma \) divides the hyperplane \( \Pi \) into the two regions \( \Pi_1 \) and \( \Pi_2 \) associated with trajectories that cross and do not cross the boundary, respectively. We can now define the Poincaré map, \( \mathcal{P} : \Pi \mapsto \Pi \), generated by the system flow from \( \Pi \) back to itself.

Let \( x^* \in \Gamma \) be the fixed point on \( \Pi \) corresponding to the grazing orbit \( p(t) \) for \( \mu = 0 \). We now introduce a system of local coordinates on \( D \) with the origin located at \( x^* \), such that the sign of the \( n \)th coordinate determines whether a given point, \( x = (x_1, x_2, \ldots, x_{n-1}) \), is in the region \( \Pi_1 \) or \( \Pi_2 \), i.e. \( x_n > 0 \Leftrightarrow x \in \Pi_1 \), \( x_n < 0 \Leftrightarrow x \in \Pi_2 \) and \( x_n = 0 \Leftrightarrow x \in \Gamma \).

We assume that \( \mathcal{P} \) is continuous in a neighbourhood of \( x^* \) and such that its dependence on \( \mu \) is sufficiently smooth. Then, \( x^* \) is a border-crossing fixed point of \( \mathcal{P} \) in the sense that \( x^* \in \Pi_1 \) if \( \mu \in [-\epsilon, 0[ \), \( x^* \in \Pi_2 \) if \( \mu \in ]0, \epsilon] \) and \( x^* \in \Gamma \) if \( \mu = 0 \).

Since trajectories emerging from \( x^* \) may either cross the boundary \( \Sigma \) or not, \( \mathcal{P} \) is accordingly obtained by considering the two submappings \( \mathcal{P}_1 : \Pi_1 \mapsto \Pi \) and \( \mathcal{P}_2 : \Pi_2 \mapsto \Pi \) describing the system motion in the case that it crosses the boundary \( (\mathcal{P}_1) \) or not \( (\mathcal{P}_2) \). Namely, \( \mathcal{P} \) is the PWS ‘normal-form’ mapping given by

\[
x \mapsto \begin{cases} 
\mathcal{P}_1(x, \mu) & \text{if } x \in \Pi_1, \\
\mathcal{P}_2(x, \mu) & \text{if } x \in \Pi_2.
\end{cases} \quad (8.1)
\]

Thus, a periodic orbit of a PWS continuous-time system, such as equation (2.1), undergoing a grazing bifurcation is associated with a border-crossing fixed point of an appropriately defined local PWS map.
Note that it is often assumed in the literature on non-smooth bifurcations that the study of a grazing bifurcation in a continuous-time system of ODEs can be reduced to the study of the BC of the fixed point of its corresponding PWS normal-form map. According to the definition of a BC, this is only true if the normal-form mapping has a discontinuous Jacobian on $\Sigma$, i.e. if

$$\left. \frac{\partial P_1}{\partial x}(x^*, \mu) \right|_{\mu=0} \neq \left. \frac{\partial P_2}{\partial x}(x^*, \mu) \right|_{\mu=0}.$$

As reported in di Bernardo et al. (2001c), this is not true in general, but only in a limited number of cases such as when the switching manifold is itself non-smooth (corner collision; Leine 2000; di Bernardo et al. 2001a).

A particularly useful concept in the derivation of normal-form maps for grazing and sliding bifurcations is that of discontinuity mapping (Dankowicz & Nordmark 2000).

Specifically, take a trajectory such as the one depicted in figure 3 and suppose that it intersects some Poincaré section, $\Sigma_1$, at some time $t_s < 0$ and a second Poincaré section, $\Sigma_2$, at some time $t_f > 0$ (figure 3). In order to compute such a trajectory from $\Sigma_1$ to $\Sigma_2$, we would compute the first segment (from $\Sigma_1$ to $\tilde{x}$) using flow 1. Then, we would consider the second flow (from $\tilde{x}$ to $\hat{x}$) to take into account the fact that the system has crossed the switching manifold. Finally, we would use again flow 1 to compute the third segment of the trajectory (from $\hat{x}$ to $\Sigma_2$). Alternatively, we could use flow 1 to compute the trajectory until it reaches a given reference section (e.g. the plane $\Pi$ in figure 3), even if it crosses the switching manifold. At this point, we would apply an appropriate correction (the discontinuity mapping) to take into account the fact that the manifold has been crossed. Finally, we would apply again flow 1 to compute the final part of the trajectory from the corrected initial point on $\Pi$ to the desired Poincaré section $\Sigma_2$. In this sense, the discontinuity mapping represents the correction brought about by the presence of the switching manifold.

To carry out the normal-form map derivation for grazing bifurcations, one can then consider perturbations of a grazing or sliding trajectory close to the bifurcation point. Discontinuity maps can be used to characterize the bifurcation scenario observed. In general, grazing bifurcations can be shown to be associated with normal-form maps of the form (see Dankowicz & Nordmark 2000; di Bernardo et al. 2001b):

$$x \mapsto \begin{cases} A_1x + B\mu, & \text{if } c^T x < 0, \\ A_2x + D(c^T x)^\gamma + B\mu, & \text{if } c^T x > 0, \end{cases} \quad (8.2)$$

where $A_1, A_2, B, c^T$ and $D$ are appropriate matrices or vectors and

- $\gamma = 1/2$ if $F_1^* \neq F_2^*$ (i.e. the vector field is discontinuous across $\Sigma$);
- $\gamma = 1$ if $\Sigma$ is non-smooth (corner collision);
- $\gamma = 3/2$ if $F_1^* = F_2^*$ but $F_1^x \neq F_2^x$ (i.e. the system has discontinuous Jacobian across $\Sigma$).

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Sliding bifurcations can also be studied by deriving appropriate normal-form maps similarly to that done for grazings. Results reported in di Bernardo et al. (2002b) show that the sliding normal-form maps are still of the form (8.2) but, in this case, \( \gamma = 1, \gamma = 2 \) or \( \gamma = 3 \) according to the topology of the sliding bifurcation detected in the system (one of those depicted in figure 1). Recently, the existence of ‘catastrophic’ sliding bifurcations was also reported (see the work by Jeffrey & Hogan (submitted)). Such bifurcations involve the presence of repelling sliding motion and cannot be studied using discontinuity maps. A geometric approach is instead taken in Jeffrey & Hogan (submitted) to classify all possible types of sliding bifurcations including the ‘regular’ cases analysed in di Bernardo et al. (2002b).

To classify grazings or sliding bifurcations associated with piecewise linear maps, one can then use the classification strategy for BCs presented in §7. Results concerning the square-root case (\( \gamma = 1/2 \)) were presented in the literature on low-dimensional impact oscillators and were recently generalized to \( n \)-dimensional systems in Fredriksson & Nordmark (2000). Some results on maps characterized by \( \gamma > 1 \) can be found in Halse et al. (2003).

9. Conclusions

We have discussed the occurrence of DIBs in PWS dynamical systems. We reviewed recent results concerning the main types of bifurcations for PWS maps and flows, proposing a way of organizing them into a consistent theory of
bifurcations in PWS systems. We have seen that PWS systems can exhibit a multitude of different bifurcation phenomena that cannot be understood using the standard theory of bifurcations in smooth systems. These include interactions between fixed points of maps or equilibria and limit cycles of flows with the system discontinuity sets.

We discussed a possible classification strategy for BCs of fixed points in maps. We then showed that bifurcations of limit cycles in flows can be studied by using appropriate non-smooth normal-form maps that turn out to be piecewise linear in some cases (namely the corner collision and the grazing–sliding scenarios). When this occurs, it is possible to use Feigin’s classification strategy and its extensions for one-dimensional and two-dimensional systems in order to correctly predict the bifurcation scenario observed. This is particularly relevant in the area of non-smooth circuits and systems where bifurcations of this type are often observed and left unexplained.

The theory of bifurcations in PWS systems is far from being complete. Many challenges and open problems remain to be addressed. Firstly, on a fundamental level, we do not have yet proper rigorous extensions to PWS of concepts such as codimension, topological equivalence and invariant manifolds which are now well assessed for their smooth counterparts. Also, a classification of all possible bifurcations in these systems is still incomplete. Most notably, DIBs of equilibrium sets, Zeno or chattering points and other attractors such as tori or strange attractors are still waiting to be properly studied. Another important aspect missing from the theory is the generalization of the results presented above to other classes of systems with discontinuities; for example, the general hybrid systems formulation where both state-dependent and time-dependent switchings can occur. Also, the case of DIBs of higher codimension is still an open problem. A preview of possible cases can be found in Kowalczyk et al. (2006) and a more detailed analysis for some specific cases in Kowalczyk & di Bernardo (2005) and Colombo & Dercole (2010). From a biological viewpoint, this seems to be essential for the application of the theory of PWS to analyse, for example, gene regulatory networks that are typically characterized by a large number of parameters (de Jong et al. 2004; Polynikis et al. 2009). Thus, much work is still needed to achieve the ultimate goal: a unified and general analytical framework to classify bifurcations in PWS and, more generally, hybrid dynamical systems.

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