Review

Delay stabilization of periodic orbits in coupled oscillator systems

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We study diffusively coupled oscillators in Hopf normal form. By introducing a non-invasive delay coupling, we are able to stabilize the inherently unstable anti-phase orbits. For the super- and subcritical cases, we state a condition on the oscillator’s nonlinearity that is necessary and sufficient to find coupling parameters for successful stabilization. We prove these conditions and review previous results on the stabilization of odd-number orbits by time-delayed feedback. Finally, we illustrate the results with numerical simulations.

Keywords: delay; stabilization; coupled oscillators; Pyragas control

1. Introduction

Coupled nonlinear oscillators are models for complex systems in different fields of science ranging from engineering to neurology. Understanding and controlling the behaviour of these oscillators is therefore a central issue in nonlinear science. The control of resonance effects and synchronization is of particular importance. For instance, cognitive brain functions (binding problem) and pathological brain conditions, such as epilepsy and Parkinson’s disease (Schiff et al. 1994; Rosenblum et al. 2001; Rosenblum & Pikovsky 2004; Popovych et al. 2005), are related to the synchronization of neurons and neural populations. Time delays are always present in coupled systems owing to the finite signal-propagation time. These time lags give rise to complex dynamics and have been shown to play a key role in the synchronization behaviour of neural systems (Hauschildt et al. 2006; Gassel et al. 2007; Dahlem et al. 2009; Schöll et al. 2009). Coupled lasers exhibit similar phenomena to coupled neurons and have attracted much attention owing to their importance in telecommunication applications (Wünsche et al. 2005; Fischer et al. 2006; Klein et al. 2006; Shaw et al. 2006; Landsman & Schwartz 2007; Flunkert et al. 2009). For many technological and medical applications, non-invasive methods of control are desirable as they have

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no side effects and do not compromise the performance of the controlled system. On the other hand, non-invasive control methods also provide an effective tool for studying complex systems because they allow us to uncover unstable dynamical structures in experiments (Sieber et al. 2008). Stabilization of unstable periodic orbits has thus been identified as a task of central importance, even in the context of chaotic dynamics (Ott et al. 1990; Boccaletti et al. 2000; Gauthier 2003; Schöll & Schuster 2008). Delayed-feedback control, as proposed by Pyragas (1992), has proven to be a powerful non-invasive method for the stabilization of unstable periodic orbits (Schöll & Pyragas 1993; Baba et al. 2002; Beck et al. 2002; von Loewenich et al. 2004) and unstable steady states (Hövel & Schöll 2005; Schikora et al. 2006; Dahms et al. 2007, 2008) in dynamical systems. It has, for instance, been successfully applied in spatially extended systems (Unkelbach et al. 2003; Stegemann et al. 2006; Postlethwaite & Silber 2007; Dahlem et al. 2008, in press; Schneider et al. 2009; Kyrychko et al. in press) and even in noise-driven systems (Janson et al. 2004; Pomplun et al. 2005; Flunkert & Schöll 2007; Pototsky & Janson 2007; Prager et al. 2007; Hizanidis & Schöll 2008; Majer & Schöll 2009).

The basic idea is as simple, as it is often effective. Let \( z = z(t) = z(t + p) \), \( t \in \mathbb{R} \), \( z \in \mathbb{R}^n \), denote a periodic orbit with minimum period \( p > 0 \) of a system,

\[
\dot{z} = F(z),
\]

Then, consider a linear time-delayed feedback,

\[
\dot{z}(t) = F(z(t)) + B(z(t − τ) − z(t)),
\]

involving a time delay \( τ > 0 \). For delays \( τ = np \) that are an integer multiple \( n \) of the minimum period \( p \), any control force \( B(z(t − τ) − z(t)) \) vanishes along the given periodic orbit \( z(t) \). The instability of \( z(t) \) in equation (1.1), however, may change and may in fact be overcome for suitable choices of the control matrix \( B \) in equation (1.2) (Fiedler et al. 2007).

The above idea can be implemented and tested easily in experimental systems, even in the absence of specific models \( F(z) \). It is therefore perhaps not surprising that the literature on this Pyragas method has grown to about 1000 publications. For some recent surveys on theoretical and experimental aspects, see, for example, Schöll & Schuster (2008).

Severe restrictions for the applicability of the method were believed to exist (Just et al. 1997; Nakajima 1997; Nakajima & Ueda 1998a; Harrington & Socolar 2001; Pyragas et al. 2004; Pyragas & Pyragas 2006). It was commonly believed that unstable periodic orbits with an odd number of real Floquet multipliers larger than unity could never be stabilized by delayed-feedback control. Recently, this alleged odd-number theorem has been refuted by a counter example (Fiedler et al. 2007, 2008a,b,c; Just et al. 2007; Kehrt et al. 2009); see §4 for a summary of these results.

In this paper, we study two coupled Hopf normal-form oscillators. We generalize the delayed-feedback control of Pyragas to a delay coupling in order to stabilize in-phase and anti-phase periodic orbits in the coupled system and thus control specific prescribed synchronization patterns of the oscillators. For anti-phase solutions, we advocate a suitable half-period delayed feedback, which is analogous to a suggestion made by Nakajima & Ueda (1998b) in the context of single Duffing and Lorenz equations. To be specific, and to enable
complete analytical results, we consider planar oscillators with an additional rotational equivariance; see equation (2.2). Such planar truncated Stuart–Landau oscillators are mathematically motivated and justified by normal-form analysis. In particular, our claims remain valid under perturbations breaking the rotational normal-form symmetry and, as well, under perturbations by higher-order terms.

This paper is organized as follows. In §2, we introduce the model of two diffusively coupled Hopf normal-form oscillators and investigate the in-phase and anti-phase solutions. Section 3 describes our non-invasive delay coupling and states the stabilization theorems—the main results of this work. In §4, we review previous results on the stabilization of subcritical Hopf bifurcation, in planar normal form, by non-invasive delayed feedback as a preparation for the proofs of the theorems. Section 5 discusses the characteristic equation of our control system (3.2), linearized at $z_1 = z_2 = 0$. Section 6 proves the theorems formulated in §3. In §7, we provide some numerical illustrations. Finally, §§8–10 conclude with a discussion of our results.

2. Model of two diffusively coupled oscillators

Our model system (1.1) takes the following specific form:

\[ \dot{z}_1 = f(z_1) + a \cdot (z_2 - z_1) \]  
and \[ \dot{z}_2 = f(z_2) + a \cdot (z_1 - z_2). \]  

Here, the planar vectors $z_1, z_2 \in \mathbb{R}^2 \cong \mathbb{C}$ describe the state of the respective oscillator, and

\[ f(z) = (\lambda + i + \gamma |z|^2)z, \]  

where real parameter $\lambda$ and fixed complex $\gamma$ describe the nonlinear dynamics of each separate oscillator. Note that $f(z)$ is chosen to coincide with the normal form for Hopf bifurcation, truncated at third order. The angular frequency at the Hopf bifurcation is normalized to unity. The scalar $a > 0$ is the diffusive coupling constant. The individual oscillator $z_i$ undergoes Hopf bifurcation of a periodic orbit,

\[ z_1(t) = z_2(t) = z_+(t) = r_+ \exp\left(\frac{2\pi i t}{p_+}\right), \]

with amplitude $r_+^2 = -\lambda / \text{Re} \gamma$ for $\lambda \text{Re} \gamma < 0$, as $\lambda$ increases through the bifurcation point $\lambda = 0$. The Hopf bifurcation is subcritical for fixed $\text{Re} \gamma > 0$, and supercritical for $\text{Re} \gamma < 0$. The minimum period $p_+$ depends on the amplitude $r_+$ via

\[ \frac{2\pi}{p_+} = 1 + r_+^2 \text{Im} \gamma. \]

Owing to symmetry, the coupled oscillator dynamics (2.1) possesses dynamically invariant subspaces. Let

\[ z_\pm = \frac{1}{2} (z_1 \pm z_2) \]
denote the average and the asynchrony of the two oscillators. Then,

\[ \dot{z}_+ = \frac{1}{2} (f(z_+ + z_-) + f(z_+ - z_-)) \]  
(2.6a)

and

\[ \dot{z}_- = \frac{1}{2} (f(z_+ + z_-) - f(z_+ - z_-)) - 2az_- \]  
(2.6b)

are equivalent to equations (2.1). Note how \( \dot{z}_- = 0 \) for \( z_- = 0 \); hence, the *in-phase* subspace

\[ Z_+ := \{ (z_+, z_-) | z_- = 0 \} \]  
(2.7)

characterized by \( z_1 \equiv z_2 \), is dynamically invariant. Because \( f(-z) = -f(z) \) is an odd nonlinearity, the *anti-phase* subspace

\[ Z_- := \{ (z_+, z_-) | z_+ = 0 \} \]  
(2.8)

where \( z_1 \equiv -z_2 \), is likewise invariant.

Specifically, the in-phase dynamics on \( Z_+ \) coincides with equation (2.2) and features Hopf bifurcation of the periodic orbit \( z_+(t) \) with minimum period \( p_+ \), as in equations (2.3) and (2.4). The anti-phase dynamics on \( Z_- \), by contrast, is given by

\[ \dot{z}_- = f(z_-) - 2az_- \]  
(2.9)

Therefore, anti-phase Hopf bifurcation occurs at \( \lambda = 2a \) and generates periodic orbits

\[ z_-(t) = r_- \exp \left( \frac{2\pi it}{p_-} \right) \]  
(2.10)

with amplitude \( r_-^2 = -(\lambda - 2a)/\text{Re}\, \gamma \) and minimum period \( p_- \) according to

\[ \frac{2\pi}{p_-} = 1 + r_-^2 \text{Im}\, \gamma. \]  
(2.11)

Compared with in-phase Hopf bifurcation at \( \lambda = 0 \), the anti-phase Hopf bifurcation point \( \lambda = 2a \) has shifted to the right, by \( 2a \), but the bifurcation direction coincides with the in-phase solutions. Indeed, the dynamics in \( Z_+ \) and \( Z_- \) only differ by a shift of \( 2a \) in the bifurcation parameter \( \lambda \).

The above elementary observations have an important consequence for the stability properties of the bifurcating anti-phase periodic orbits \( z_-(t) \). As \( \lambda = 2a > 0 \), at Hopf bifurcation, the unstable dimension of \( z_-(t) \) is at least 2, as inherited from the Hopf bifurcation point itself. Specifically, the unstable dimension is 3 in the subcritical case \( \text{Re}\, \gamma > 0 \), and 2 in the supercritical case \( \text{Re}\, \gamma < 0 \). As usual, the unstable dimension denotes the number of Floquet multipliers strictly outside the complex unit circle, counting algebraic multipliers. See Diekmann *et al.* (1995), Fiedler *et al.* (2007), Just *et al.* (2007) and Schöll & Schuster (2008) for the mathematical centre manifold machinery behind this result on exchange of stability at bifurcations. See also figure 1.

### 3. Stabilization by delay—theorems

We now attempt to stabilize the unstably born anti-phase periodic solution \( z_-(t) \) by a delayed-control term that is adapted to the specific symmetry \( z_+ \equiv 0 \) of \( z_-(t) \). Recall that \( z_-(t - p_-/2) = -z_-(t) \); see equation (2.10). Therefore, \( z_+ \equiv 0 \)
implies the following spatio-temporal symmetry with respect to exchanging the two subsystems

\[ z_1(t) = z_2\left(t - \frac{p_-}{2}\right) \]  \hspace{1cm} (3.1a)

and

\[ z_2(t) = z_1\left(t - \frac{p_-}{2}\right). \]  \hspace{1cm} (3.1b)

Indeed, the oscillators switch their roles after half a period in the anti-phase case. This, in turn, motivates us to seek a stabilization of the solution \( z_- (t) \) in the form of delayed coupling

\[ \dot{z}_1 = f(z_1) + a \cdot (z_2 - z_1) + b \cdot (z_2(t - \tau) - z_1) \]  \hspace{1cm} (3.2a)

and

\[ \dot{z}_2 = f(z_2) + a \cdot (z_1 - z_2) + b \cdot (z_1(t - \tau) - z_2) \]  \hspace{1cm} (3.2b)

for complex parameter \( b \in \mathbb{C} \). Omitted arguments of \( z_1 \) and \( z_2 \) indicate evaluation at time \( t \) here. Note that the delay \( \tau \) is non-invasive for

\[ \tau = \frac{1}{2} np_- \]  \hspace{1cm} (3.3)

i.e. for integer multiples \( n \) of half the minimum period \( p_- \). Such a delayed coupling would be invasive on in-phase solutions \( z_+ \) because \( z_2(t - \tau) - z_1(t) = z_1(t - \tau) - z_2(t) \neq 0 \), for half-period delays \( \tau \), unless \( p_+ = p_-/2 \).

Next, we state stabilization theorems for the supercritical and subcritical cases. The respective proofs follow in §6.
Figure 2. Stabilization region $\Lambda_a$ (shaded) of complex control coefficients $b \in \mathbb{C} \setminus \mathbb{R}$ for anti-phase solutions near supercritical Hopf bifurcation. The numbers in parentheses indicate the total multiplicity $E(b)$ of eigenvalues with strictly positive real parts, at Hopf bifurcation $\lambda = 2a$. Note that straight lines through $b = -1/\pi \in \mathbb{C}$ touch the boundary of $\Lambda_a$ in the points $b_0$ and $b_0$ (see §6). $a = 0.1$.

**Theorem 3.1.** Consider the coupled oscillator systems (2.1) and (2.2) with diffusive coupling constant

$$0 < a < \frac{1}{\pi}$$

(3.4)

in the supercritical case $\text{Re} \gamma < 0$. Then, there exists a strictly decreasing real analytic function $b_* = b_*(a) > a$ with limits $b_*(0) = \infty$ and $b_*(1/\pi) = 1/\pi$ such that, for real controls,

$$a < b < b_*(a),$$

(3.5)

the anti-phase periodic orbits $z_-(t) \not\equiv 0$, $z_+ \equiv 0$ of equations (2.10) and (2.11) are stabilized non-invasively by a delayed coupling (3.2) with half-period delay

$$\tau = \frac{1}{2} p_-, (3.6)$$

for small amplitudes $r_- = |z_-(t)|$, and for parameters $\lambda$ near the anti-phase Hopf bifurcation at $\lambda = 2a$.

**Theorem 3.2.** Consider the subcritical case $\text{Re} \gamma > 0$ of theorem 3.1, again for $0 < a < 1/\pi$. Then, there exists a continuous, strictly increasing function $\beta = \beta(a)$ with limits $\beta(0) > 0$ and $\beta(1/\pi) = \infty$ such that the following holds for

$$|\text{Im} \gamma| > \beta(a) \text{ Re} \gamma.$$  

(3.7)

For each pair of $a, \gamma$ satisfying equation (3.7), there exists an open region of controls $b \in \mathbb{C} \setminus \mathbb{R}$, depending on $a$ and $\gamma$, for which equations (3.2) achieve non-invasive delayed-feedback stabilization locally near Hopf bifurcation at $\lambda = 2a$, as asserted in theorem 3.1.

See figure 2 for a sketch of the stabilization regime $b \in \mathbb{C} \setminus \mathbb{R}$ to which theorem 3.2 applies. The shaded region $\Lambda_a$ indicates the region of those strictly complex controls $b$ for which stabilization is possible, locally near anti-phase Hopf
bifurcation, provided that $|\text{Im} \gamma|/|\text{Re} \gamma|$ is large enough. For decreasing values $|\text{Im} \gamma|/|\text{Re} \gamma| \downarrow \beta(a)$, the regime of stabilizing $b$ shrinks to two complex-conjugate boundary points $b_0(a), \overline{b}_0(a)$. See §5 for details on $b_0(a)$ and the end of §6 for $b_0(a), \beta(a)$. See §7 for numerical examples.

4. Beyond odd-number limitation for planar Hopf bifurcation

As a preparation for the proof of theorems 3.1 and 3.2 in §6, we revisit delayed-feedback stabilization of planar Hopf bifurcation in a single system,

$$\dot{z} = f(z) + b \cdot (z(t - \tau) - z),$$

with the same normal-form nonlinearity $f(z) = (\lambda + i + \gamma |z|^2)z \in \mathbb{C}$ as in equation (2.2). See also Fiedler et al. (2007), Just et al. (2007) and Schöll & Schuster (2008) for our previous analysis of this case.

Of course, we keep in mind that equation (4.1) also describes stabilization within the invariant subspace $\mathcal{Z} = \{(z_1, z_2)|z_2 = 0\}$ of the in-phase solutions $z_1(t) \equiv z_2(t)$, introduced in equations (2.5) and (2.7), under the naive delayed-feedback control scheme

$$\dot{z}_1 = f(z_1) + a (z_2 - z_1) + b \cdot (z_1(t - \tau) - z_1),$$

and

$$\dot{z}_2 = f(z_2) + a (z_1 - z_2) + b \cdot (z_2(t - \tau) - z_2).$$

We comment on this case in the discussion (§8).

**Theorem 4.1.** Consider the planar Hopf normal-form system (4.1) with subcritical Hopf bifurcation in the absence of control $b = 0$, i.e. with

$$\text{Re} \gamma > 0.$$  

Then, there exist complex control gains $b$ such that the bifurcating periodic orbits $z(t) = r \exp(2\pi i t / p), r^2 = -\lambda/\text{Re} \gamma, p = 2\pi/(1 - \lambda \text{Im} \gamma/\text{Re} \gamma)$ are stabilized non-invasively by a delayed feedback (4.1) with delay equal to the period

$$\tau = p.$$  

This holds for small amplitudes $r$ and for parameters $\lambda$ near Hopf bifurcation at $\lambda = 0$.

In the absence of control, $b = 0$, the bifurcating periodic solutions $z(t)$ are one-dimensionally unstable. Their stabilization by non-invasive delayed feedback therefore refutes the so-called ‘odd-number limitation’ of Pyragas control. See Fiedler et al. (2007) and Just et al. (2007) for a detailed discussion of this aspect, including a rigorous mathematical analysis illustrated by numerical examples.

To prepare for our proof of theorems 3.1 and 3.2 in §§3 and 4, we now sketch a proof of theorem 4.1, in the same spirit. For brevity, we only consider the hard-spring case,

$$\text{Im} \gamma > 0,$$

where period $p = 2\pi/(1 + r^2 \text{Im} \gamma)$ decreases with amplitude $r$, leaving the opposite soft-spring case $\text{Im} \gamma < 0$ to the reader.

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Figure 3. Subcritical Hopf bifurcation in the parameter plane \((\lambda, \tau)\) with fixed control \(b\) for: (a) soft springs \(\text{Im}\, \gamma < 0\) and (b) hard springs \(\text{Im}\, \gamma > 0\). Solid black lines display the Hopf bifurcation curve \((\lambda(\omega), \tau(\omega))\) emanating from \(\lambda = 0\) and non-invasive delay \(\tau = 2\pi\). Hopf curves are oriented with increasing \(\omega\). The dashed black lines correspond to the period \(\tau = p(\lambda)\) of the Pyragas curve of bifurcating periodic solutions. Strict unstable dimensions \(E(b)\) of the trivial equilibrium \(z \equiv 0\) are indicated in parentheses. (a) Soft spring: \(\text{Re}\, \gamma = 1, \text{Im}\, \gamma = -10, b = 0.3 e^{i\pi/4}\); (b) hard spring: \(\text{Re}\, \gamma = 1, \text{Im}\, \gamma = 10, b = 0.1 e^{-i3\pi/4}\).

The mechanical terminology ‘soft’ and ‘hard’ spring arises from the pendulum equation \(\ddot{x} + D x = 0\) with nonlinear spring constant \(D = D(x)\). For ‘soft’ springs \(D(x)\), where \(D(x)\) decreases with increasing \(|x|\), the minimal period increases with amplitude. Examples are mathematical pendula \(D(x) = \sin x\), or rubber balloons. For ‘hard’ springs, \(D(x)\) increases with \(|x|\) and the period decreases with amplitude.

The basic idea of the proof is easily sketched in the two parameter diagrams of figure 3. There are two ingredients. First, we linearize at the equilibrium \(z \equiv 0\) and study the strict unstable dimensions. Let \(E(b)\) denote the total number of eigenvalues \(\eta\) with \(\text{Re}\, \eta > 0\), counting real multiplicities. Even for fixed non-zero \(b\), these numbers still depend on \((\lambda, \tau)\), as indicated in figure 3 in parentheses. Second, we evaluate the dashed period curve, 

\[
\tau = p = p(\lambda) = \frac{2\pi}{1 - \lambda \text{Im}\, \gamma / \text{Re}\, \gamma},
\]

of non-invasive control in figure 3, as it emanates from the Hopf point \(\lambda = 0, \tau = 2\pi\) to the subcritical side \(\lambda < 0\).

With these two ingredients, the proof works as follows. Suppose we can choose \(b\) such that the dashed period curve enters a region with \(E(b) = 2\) at \(\lambda = 0, \tau = 2\pi\), transversely to the Hopf curve and pointing away from the \(E(b) = 0\) region. Then, the subcritical Hopf bifurcation along the \(\lambda\)-axis \(\tau = 0\) (alias \(b = 0\)) has become supercritical for the chosen parameters along the dashed curve. Hence, the bifurcating unstable orbits, for \(\tau = 0,\) alias \(b = 0\), have become stable along the dashed curve of non-invasive delayed-feedback control, by standard exchange of stability at Hopf bifurcation.

Let us implement the above idea for our specific case (4.1). Linearization at \(z \equiv 0\) yields the characteristic equation 

\[
0 = \chi(\eta) = \lambda + i + b(e^{-\tau\eta} - 1) - \eta
\]

for the eigenvalues \(\eta\).
First, consider the starting point $\lambda = 0, \tau = 2\pi$ with purely imaginary eigenvalue $\eta = i$. We determine the strict unstable dimension $E(b)$ there, aiming for $E(b) = 0$ to ensure that the Hopf eigenvalue $\eta = i$ actually effects a change from $E = 0$ to $E = 2$. Indeed, any $E > 0$ at that starting point, i.e. the presence of any non-Hopf eigenvalues with a strictly positive real part, would be inherited as an instability of any bifurcating periodic orbit, obstructing stabilization. Note here that the Hopf eigenvalue pair $\eta = \pm i$ itself does not yet contribute to the strict unstable dimension $E(b)$ at $\lambda = 0, \tau = 2\pi$.

At $\lambda = 0, \tau = 2\pi$, the characteristic equation for $\eta = \omega i = (1 + \omega)i$ reads

$$b = b(\omega) = \frac{i\omega}{e^{-2\pi i\omega} - 1} = -\frac{\omega}{2} (\cot(\pi \omega) + i). \quad (4.8)$$

Note $b(0) = -1/(2\pi)$ and the singularities of $b(\omega)$ at integer $\omega \in \mathbb{Z}\backslash\{0\}$. In particular, $\eta = 0$ is never an eigenvalue at $\lambda = 0, \tau = 2\pi$, and $E = E(b)$ can only change by Hopf bifurcation there.

As $E(0) = 0$ at $b = 0$, by planarity and the eliminated trivial Hopf eigenvalue $\eta = i$, the strict unstable dimensions $E(b)$ are as indicated in figure 4. Indeed, complex analytic maps, such as $\omega \mapsto b(\omega)$, preserve orientation. Instability $E(b)$ tracks eigenvalues $\Re \eta > 0$ to the right of the imaginary axis $\eta = i\omega = i(1 + \omega)$. Therefore, $E(b)$ is larger by two on the right-hand side of any of the oriented curves $\omega \mapsto b(\omega)$, when compared with the left-hand side. Elsewhere, $E(b)$ does not change. Therefore, all unstable dimensions $E(b)$ in figure 4 follow from $E(0) = 0$ and the indicated orientations of the solid Hopf curves $\omega \mapsto b(\omega)$ for real $\omega$.

We calculate the tangent $\lambda = \hat{\lambda}, \tau = 2\pi + \hat{\tau}$ to the dashed periodic curve $\tau = p(\lambda)$ shown in figure 3. Expanding the explicit representation (4.6), we immediately obtain

$$\hat{\tau} = 2\pi \frac{\Im \gamma}{\Re \gamma} \hat{\lambda}. \quad (4.9)$$
To compute the tangent $\lambda = \tilde{\lambda}, \tau = 2\pi + \tilde{\tau}$ to the solid Hopf curve $\lambda = \lambda(\omega)$, $\tau = \tau(\omega)$ of solutions $\eta = i\tilde{\omega} = i(1 + \omega)$ at $\omega = 0$, we linearize the characteristic equation (4.7), for fixed $b$, keeping in mind that $\tilde{\lambda}$ and $\tilde{\tau}$ are of the order $\omega$. Thus, we obtain $0 = \tilde{\lambda} + b(-2\pi i\omega - i\tilde{\tau}) - i\omega$ and the tangent of the Hopf curve

$$\tilde{\lambda} = \frac{\text{Im } b}{\text{Re } b} \omega \quad (4.10a)$$

and

$$\tilde{\tau} = -\frac{1 + 2\pi \text{Re } b}{\text{Re } b} \omega. \quad (4.10b)$$

To achieve the geometric hard-spring situation shown in figure 3a, and hence prove theorem 4.1 in the hard-spring case, we recall that both $\text{Re } \gamma$ and $\text{Im } \gamma$ are positive; see equations (4.3) and (4.5). Hence, equation (4.9) makes the slope of the dashed periodic curves positive in figure 3. By equations (4.10), the slope of the solid Hopf curve, by contrast, is given by

$$\tilde{\tau} = -\frac{1 + 2\pi \text{Re } b}{\text{Im } b} \tilde{\lambda}. \quad (4.11)$$

We now determine the strict unstable dimensions $E(b)$ resulting from different sides of the Hopf curve $\eta = i\tilde{\omega}$ of the characteristic equation (4.7), for fixed $b$ and in the $(\lambda, \tau)$ plane. It is advisable to proceed here with analytic care. Let $(\tilde{\lambda}, \tilde{\tau}, \tilde{\eta}) \in \mathbb{R}^2 \times \mathbb{C}$ denote infinitesimal variations of $(\lambda, \tau, \eta)$. By multi-variate linearization of equation (4.7) at $(\lambda, \tau, \eta)$, we obtain the equivalent system

$$\varphi(\tilde{\lambda}, \tilde{\tau}) := \tilde{\lambda} - \eta b e^{-\tau \eta} \tilde{\tau} = \tilde{\xi} \quad (4.12a)$$

and

$$\psi(\tilde{\eta}) := (1 + \tau b e^{-\tau \eta}) \tilde{\eta} = \tilde{\zeta}, \quad (4.12b)$$

with real linear maps $\varphi : \mathbb{R}^2 \to \mathbb{C}$ and $\psi : \mathbb{C} \to \mathbb{C}$. In other words,

$$\tilde{\eta} \mapsto (\tilde{\lambda}, \tilde{\tau}) = (\varphi^{-1} \circ \psi)(\tilde{\eta}), \quad (4.13)$$

if we eliminate the dummy variable $\tilde{\xi} \in \mathbb{C}$.

The map $\psi$ preserves real orientation, being just a multiplication by $1 + \tau b e^{-\tau \eta} \in \mathbb{C}$, for non-zero $1 + \tau b e^{-\tau \eta}$. The map $\varphi$, however, when viewed as a linear map $\varphi : \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$ by $\text{Re } \varphi$ and $\text{Im } \varphi$ possesses determinant

$$\det \varphi = -\text{Im}(\eta b e^{-\tau \eta})$$

$$= -\bar{\omega}(\text{Re } b \cos(\bar{\omega} \tau) + \text{Im } (b \sin(\bar{\omega} \tau))) \quad (4.14)$$

$$= -\text{Re } b$$

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at the point of interest $\eta = i\tilde{\omega}, \tilde{\omega} = 1, \lambda = 0, \tau = 2\pi$. In particular, $\varphi$ preserves orientation for $\text{Re} \ b < 0$ as chosen in figure 3b. Therefore $\varphi^{-1} \circ \psi$ also preserves orientation in this case.

By equations (4.10) and $\text{Im} \ b < 0$, the Hopf curve is oriented to the upper right at $\lambda = 0, \tau = 2\pi$, as indicated in figure 3b. Because $\varphi^{-1} \circ \psi$ from equation (4.13) preserves orientation, $E(b) = 2$ again holds to the right of the oriented Hopf curve $\omega \mapsto (\lambda(\omega), \tau(\omega))$, and the strict unstable dimensions of figure 3b follow.

The stabilizing condition for the dashed curve $\tau = p(\lambda)$ of the Pyragas periodic orbits (Fiedler et al. 2007) to enter the $E(b) = 2$ region when emanating from $\lambda = 0, \tau = 2\pi$ to the lower left therefore reads

\[
\frac{2\pi \text{Im} \gamma}{\text{Re} \gamma} = \frac{\tilde{\tau}}{\tilde{\lambda}} > \frac{1 + 2\pi \text{Re} b}{\text{Im} b}. \tag{4.15}
\]

Clearly, this can always be achieved in any subcritical, soft-spring case of positive $\text{Re} \gamma$ and $\text{Im} \gamma$, if we choose $1 + 2\pi \text{Re} b > 0$ small enough with $\text{Re} b \gtrsim -1/2\pi$, and $\text{Im} b < 0$ is also negative, such that $b$ resides in the lower left part of the shaded region indicated in figure 4.

We have thus achieved supercritical Hopf bifurcation along the dashed line of non-invasive delayed feedback in figure 3, and hence local stability of the bifurcating branch. This proves theorem 4.1.

5. Characteristic equations for two coupled oscillators

In this section, we return to our control system (3.2) of coupled oscillators, linearized at $z_1 \equiv z_2 \equiv 0$. In terms of the coordinates $z_{\pm} = (z_1 \pm z_2)/2$ from equation (2.5), the linearization reads

\[
\dot{z}_+ = (\lambda + i)z_+ + b \cdot (z_+(t - \tau) - z_-) \tag{5.1a}
\]

and

\[
\dot{z}_- = (\lambda - 2a + i)z_- - b \cdot (z_- (t - \tau) + z_+). \tag{5.1b}
\]

Note how the linearization decouples into $Z_{\pm}$ components $z_{\pm}$, just as in the case $b = 0$ of absent control; see equations (2.5)–(2.11). In the previous section, non-invasive delay stabilization of local subcritical Hopf bifurcation was achieved once that Hopf point itself was stabilized. Analogously, we now study stability at the Hopf point $\lambda = 2a$ itself, before addressing the bifurcating anti-phase orbits in the following section.

The exponential ansatz $z_{\pm} = \exp(\eta t)$ yields the following two characteristic equations:

\[
0 = \chi_+(\eta) = \lambda + i + b(e^{-\tau \eta} - 1) - \eta \tag{5.2a}
\]

and

\[
0 = \chi_-(\eta) = \lambda - 2a + i - b(e^{-\tau \eta} + 1) - \eta. \tag{5.2b}
\]
Because the linearization equations \((5.1)\) decouple in \(z\), each of equations \((5.2)\) contributes its own independent set of eigenvalues to the total spectrum; the strict unstable dimensions \(\text{Re } \eta > 0\) of \(\chi_+\) and \(\chi_-\) therefore add up to the total unstable dimension \(E(b) = E_+(b) + E_-(b)\) of the trivial equilibrium, see equation \((5.8)\).

Our analysis of equations \((5.2)\) does not aim at solving these equations for their complex roots \(\eta\), for example, by tools such as the Lambert function. Instead, we are interested in the stability boundary in the space of the five parameters \(a\), \(\text{Re } \gamma\), \(\text{Im } \gamma\), \(\text{Re } b\) and \(\text{Im } b\). For this purpose, we will insert \(\eta = i \tilde{\omega}\) in equation \((5.7)\) below and study the resulting Hopf curves in the complex \(b\) plane, with parameters \(a\), \(\text{Re } \gamma\) and \(\text{Im } \gamma\).

For \(b = 0\), we find a Hopf bifurcation in \(Z_+ = \{(z_+, z_-) | z_+ = 0\}\) at \(\lambda = 2a\) and \(\eta = i\), i.e. for period \(p_- = 2\pi\). Therefore, \(\tau = p_- / 2 = \pi\) at the Hopf bifurcation, and equations \((5.2)\) become

\[
0 = \chi_+(\eta) = 2a + i + b(e^{-\pi \eta} - 1) - \eta \tag{5.3a}
\]

and

\[
0 = \chi_-(\eta) = i - b(e^{-\pi \eta} + 1) - \eta. \tag{5.3b}
\]

First, consider \(b = 0\). The complex notation that we have employed then provides a single eigenvalue \(\eta = 2a + i\) with positive real part in \(Z_+ = \{(z_+, z_-) | z_- = 0\}\) from equation \((5.3a)\). In \(Z_-,\) the characteristic equation \((5.3b)\) provides the simple Hopf eigenvalue \(\eta = i\), as expected. Let \(E(b)\) again denote the total number of eigenvalues \(\eta\) with \(\text{Re } \eta > 0\), adding both \(Z_+\) and \(Z_-\) and counting real multiplicities. Then, we have just proved

\[
E(0) = 2, \tag{5.4}
\]

at \(\lambda = 2a\), \(\tau = \pi\), for this strict unstable (or expanding) dimension \(E(b)\).

Could the unstable dimension \(E(b)\) change as \(b\) varies? To achieve our goal,

\[
E(b) = 0, \tag{5.5}
\]

of Hopf stabilization at \(\lambda = 2a\), \(\tau = \pi\), it should change, somehow.

We first note \(E(b) = 2\) for small \(|b|\). Indeed, the delay exponential \(\exp(-\pi \eta)\) then just generates a plethora of countably infinitely many discrete eigenvalues \(\eta\), in each of the characteristic equations \((5.3)\), all with large negative real parts.

Can \(E(b)\) change by an eigenvalue \(\eta\) crossing zero as \(b\) varies? Inserting \(\eta = 0\) in equations \((5.3)\) shows that this cannot happen via \(\chi_+\). In \(\chi_-\), however, \(\eta = 0\) is a solution if and only if

\[
b = \frac{i}{2}. \tag{5.6}
\]

It remains to study the changes of \(E(b)\) by a purely imaginary Hopf eigenvalue

\[
\eta = i \tilde{\omega}. \tag{5.7}
\]

Let \(E_\pm(b)\) count the solutions \(\eta\) with \(\text{Re } \eta > 0\) of \(\chi_\pm(\eta) = 0\), with algebraic multiplicity, so that

\[
E(b) = E_+(b) + E_-(b). \tag{5.8}
\]
Figure 5. Oriented Hopf curves $b = b_+ (\omega)$ (solid) and $b = b_- (\omega)$ (dashed) for $\omega \in \mathbb{R}$ in control system (3.2), linearized at $z_1 \equiv z_2 \equiv 0$. Orientation arrows indicate increasing $\omega$. Note the symmetry with respect to the real axis, owing to complex conjugation $b_\pm (-\omega) = \overline{b_\pm (\omega)}$. Unstable dimensions are indicated by $(E(b))$. Note $E(b) = 0$ inside the shaded loop $\Lambda_a$. $a = 0.1$.

As $E_\pm \geq 0$ and $E_+(0) = 2, E_-(0) = 0$, we study the changes of $E_+(b)$ via $\eta = i\tilde{\omega}$, first, hoping for a region of $b \in \mathbb{C}$ where $E(b) = 0$. Solving equation (5.3a) with

$$\eta = i\tilde{\omega} = i(1 + 2\omega),\quad (5.9)$$

we obtain the Hopf curves

$$b = b_+(\omega) = 2 \frac{a - i\omega}{1 + \exp(-2\pi i\omega)} = a + \omega \tan(\pi \omega) + i(-\omega + a \tan(\pi \omega)),\quad (5.10)$$

with singularities at odd integers $2\omega$. See the solid lines of figure 5 for a sketch of these Hopf curves when $a = 0.1$.

Solving equation (5.3b) with $\eta = i\tilde{\omega} = i(1 + 2\omega)$, we obtain the Hopf curves

$$b = b_-(\omega) = 2 \frac{i\omega}{\exp(-2\pi i\omega) - 1} = -\omega(cot(\pi \omega) + i),\quad (5.11)$$

with singularities at integer $\omega \neq 0$. See the dashed lines of figure 5. Note how these dashed lines correspond to the solid lines of figure 4 because equation (5.11) corresponds to equation (4.8) in the sense that $b_- (\omega) = 2b(\omega)$.

To determine the changes of the real unstable dimensions $E(b) = E_+(b) + E_-(b)$ along the curves $\omega \mapsto b_\pm (\omega)$, we observe that the zeros of $\chi_\pm$ are given as complex analytic functions. An elementary calculation shows that the complex derivative $b'_-(\omega)$ never vanishes and hence implicit differentiation applies to the Hopf curve. The complex derivative $b'_+(\omega)$ vanishes if and only if

$$a = \frac{1}{\pi} \quad \text{and} \quad \omega = 0.\quad (5.12)$$
This is precisely where the shaded loop $\Lambda_a$ of figure 5 is formed. For

$$0 < a < \frac{1}{\pi},$$  \hfill (5.13)

this loop stretches over the real interval

$$a < b < b_*(a),$$  \hfill (5.14)

where $b_*(a) = b_+(\omega_*(a)) > 0$ is given by equation (5.10) evaluated at the first positive solution $\omega_* = \omega_*(a) > 0$ of the transcendental equation

$$0 = \text{Im}(b_+(\omega_*)) = -\omega_* + a \tan(\pi \omega_*).$$  \hfill (5.15)

To determine $E(b)$ in figure 5, we proceed as for figure 4. Complex analytical maps such as $\omega \mapsto b_\pm(\omega)$, with non-vanishing derivatives, preserve orientation. Instability $E(b) = E_+(b) + E_-(b)$ tracks eigenvalues $\text{Re} \eta > 0$, i.e. $\eta$ to the right of the imaginary axis $\eta = i\tilde{\omega} = i(1 + 2\omega)$. (The only zero eigenvalue $\eta = 0$ at $\omega = -1/2$ of $b_-(1/2) = i/2$ noted in equation (5.6) makes no exception here.) Therefore, $E(b)$ is larger by two on the right-hand side of any of the oriented curves $\omega \mapsto b = b_\pm(\omega)$, when compared with the left-hand side. Elsewhere, $E(b)$ does not change. Starting from $E(0) = 2$, as noted in equation (5.4), it is therefore elementary to derive all strict real unstable dimensions $E(b)$ of the Hopf bifurcation point $\lambda = 2a$, as given in figure 5.

In particular, $E(b) = 0$ if and only if $b$ is inside the shaded loop $\Lambda_a$ of figure 5, and that the stabilizing loop exists if and only if $0 < a < 1/\pi$.

### 6. Proof of stabilization theorems

Based on the analysis of the strict unstable dimension $E(b)$ at the anti-phase Hopf bifurcation $\lambda = 2a$, $\tau = \pi$, $\eta = i$ as given in the previous section, we now proceed to prove local non-invasive delayed-feedback stabilization of the bifurcating anti-phase periodic solutions, as claimed in theorems 3.1 and 3.2 for the supercritical and subcritical cases, respectively. Both proofs are based on the strategy of §4. The stabilization region $E(b) = 0$ of figure 4 for the complex control $b \in \mathbb{C}$ now has to be replaced by the shaded-loop region $\Lambda_a$ of $E(b) = 0$ derived in figure 2 and, in further detail, in figure 5. We recall that the loop $\Lambda_a$ is bounded by the section $|\omega| \leq \omega_*$ of the curve

$$b_+(\omega) = a + \omega \tan(\pi \omega) + i(-\omega + a \tan(\pi \omega)),$$  \hfill (6.1)

where $\omega_* = \omega_*(a)$ is the first positive solution of

$$0 = \omega_* - a \tan(\pi \omega_*).$$  \hfill (6.2)

See equations (5.10) and (5.15). We denote $b_*(a) := b_+(\omega_*(a)) > 0$, as above.

Analogously to §4, figure 3, we now seek the geometric situation of figure 6 for the tangents and unstable dimensions of the solid oriented Hopf curve $(\lambda(\omega), \tau(\omega))$ and the dashed curve $\tau = p_-(\lambda)/2$ of bifurcating periodic solutions. We calculate their tangents at $\lambda = 2a$, $\tau = \pi$, $\eta = i$ next; see also equation (4.9) versus equations (4.10).
Linearizing the explicit representation

$$\tau = \frac{1}{2} p_-(\lambda) = \frac{\pi}{1 - (\lambda - 2a)\text{Im} \gamma / \text{Re} \gamma}$$

(6.3)

of equations (2.10), (2.11) and (3.6) at $\lambda = 2a$, $\tau = \pi$ with $\lambda = 2a + \hat{\lambda}$, $\tau = \pi + \hat{\tau}$, we obtain

$$\hat{\tau} = \pi \frac{\text{Im} \gamma}{\text{Re} \gamma} \frac{\hat{\lambda}}{\lambda}$$

(6.4)

for the tangent to the anti-phase periodics, in analogy to equation (4.9). Analogously to equations (4.10), we also linearize the characteristic equation (5.2b), for fixed $b$, and obtain $0 = \chi_-(\eta) = \hat{\lambda} - b(2\pi i\omega + i\hat{\tau}) - 2i\omega$ with $\eta = i(1 + 2\omega)$, $\lambda = 2a + \hat{\lambda}$, $\tau = \pi + \hat{\tau}$. This yields the tangent to the anti-phase Hopf curve

$$\hat{\lambda} = \frac{\text{Im} b}{\text{Re} b} 2\omega$$

(6.5a)

and

$$\hat{\tau} = -\frac{1 + \pi \text{Re} b}{\text{Re} b} 2\omega.$$  

(6.5b)

We now address the supercritical case $\text{Re} \gamma < 0$ of theorem 3.1. In §5, we have seen how the stabilizing loop $\Lambda_\sigma$ arises for $0 < a < 1/\pi$; see assumption (3.4). We claim that all real controls $b$ in $\Lambda_\sigma$ stabilize the local anti-phase branch of periodic solutions, i.e. all $a < b < b_*(a)$; see equation (3.5) and the explicit representation of $b_*(a)$ in equations (5.14) and (5.15), as well as equations (6.1) and (6.2). Analyticity, strict monotonicity and the claimed limits of $b_*(a)$ are obvious.

To prove stabilization, note that the slope of the anti-phase Hopf curve (6.5) is vertical in the $(\lambda, \tau)$ plane, for $\text{Im} b = 0$. Moreover, the orientation is downwards, decreasing $\hat{\tau}$, as $\text{Re} b > 0$ in the loop $\Lambda_\sigma$. It remains to prove $E = 2$

---

**Figure 6.** Anti-phase Hopf bifurcation in the parameter plane $(\lambda, \tau)$ for fixed control $b$. Solid line, Hopf bifurcation curve $(\lambda(\omega), \tau(\omega))$ through $\lambda = 0$ and non-invasive delay $\tau = \pi$. Dashed line, $\tau = p_-(\lambda)/2$, Pyragas curve of bifurcating periodic solutions. (a) Supercritical case $\text{Re} \gamma < 0$ and $a < b < b_*(a) := b_+(\omega_*(a))$; (b) subcritical soft-spring case $\text{Re} \gamma > 0$, $\text{Im} \gamma < 0$; and (c) subcritical hard-spring case $\text{Re} \gamma > 0$, $\text{Im} \gamma > 0$. (a) $\text{Re} \gamma = -1$, $\text{Im} \gamma = -1$, $b = 1.0$ ($b_* \approx 1.92$); (b) $\text{Re} \gamma = +1$, $\text{Im} \gamma = -10$, $b = 0.24 e^{i\pi/8}$; (c) $\text{Re} \gamma = +1$, $\text{Im} \gamma = +10$, $b = 0.24 e^{-i\pi/8}$; $a = 0.1$ in all plots.
along any periodic curve emanating to the supercritical right of the \( \tau \)-axis, and stabilization of the bifurcating anti-phase periodic orbits will follow, as claimed in theorem 3.1.

To determine the regions \( E = 2 \), we proceed as in §4, equations (4.13)–(4.15), but with modified real linear maps

\[
\varphi(\tilde{\lambda}, \tilde{\tau}) = \tilde{\lambda} + \eta b e^{-\tau \eta} \tilde{\tau} = \tilde{\zeta}
\]

and

\[
\psi(\tilde{\eta}) = (1 - \tau b e^{-\tau \eta}) \tilde{\eta} = \tilde{\zeta},
\]

which arise from the modified characteristic equation (5.2) for \( \chi_- (\eta) = 0 \), replacing equation (4.7). Again,

\[
(\tilde{\lambda}, \tilde{\tau}) = (\varphi^{-1} \circ \psi)(\tilde{\eta}),
\]

where \( \psi \) preserves real orientation. At the point of interest \( \eta = i\tilde{\omega}, \tilde{\omega} = 1, \lambda = 0 \) and \( \tau = \pi \) this time, the linear map \( \varphi : \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2 \) again possesses determinant

\[
\det \varphi = +\text{Im}(\eta b e^{-\tau \eta})
\]

\[
= +\tilde{\omega}(\text{Re}(b) \cos(\tilde{\omega} \tau) + \text{Im}(b) \sin(\tilde{\omega} \tau))
\]

\[
= -\text{Re} b
\]

because \( \tilde{\omega} \tau = \pi \), this time. As \( \text{Re} b > 0 \) in the loop \( \Lambda_a \) of linear in-phase stabilization of figure 4, the map \( \varphi \) reverses orientation, this time, and so does the composition \( \varphi^{-1} \circ \psi \) of equation (6.7). Therefore, the region \( E = 2 \) now appears to the left of the Hopf curve in the \((\lambda, \tau)\) plane. As the Hopf curve is oriented vertically downwards, the \( E = 2 \) region contains the tangent of any supercritical Pyragas curve \( \tau = p_-(\lambda)/2 \) emanating to the right of the \( \tau \)-axis; see figure 6a. This proves theorem 3.1.

Finally, we settle the subcritical case \( \text{Re} \gamma > 0 \) of theorem 3.2. Fix \( 0 < a < 1/\pi \) and consider strictly complex \( b \) in the stability loop \( \Lambda_a \) of figure 5, first in the soft-spring case

\[
\text{Im} \gamma < 0.
\]

We determine those \( \text{Im} \gamma \) next, for which such choices of \( b \) are able to stabilize the local bifurcating anti-phase branch, non-invasively. From the orientation analysis of equations (6.6)–(6.8), we again conclude \( E = 2 \) to the left of the oriented Hopf curve in the \((\lambda, \tau)\) plane. Analogously to equation (4.15), it is therefore immediate that we encounter the stabilizing soft-spring geometric situation of figure 6b, if and only if the Hopf slope \( \tilde{\tau}/\tilde{\lambda} \) of equations (6.5) exceeds the slope \( \hat{\tau}/\hat{\lambda} \) of the periodics (6.4), i.e.

\[
0 > -\pi \frac{\text{Re} b + 1/\pi}{\text{Im} b} > \pi \frac{\text{Im} \gamma}{\text{Re} \gamma}.
\]

In particular, \( \text{Im} b \) is required to be positive for the proper orientation of the Hopf curve. The least restrictive choice is given by \( \Lambda_a \ni b \to b_0(a) \) defined such that the minimum
\[
\beta(a) = \min \left\{ \frac{\Re b + 1/\pi}{\Im b} \mid b \in \Lambda_a, \ \Im b > 0 \right\}
\] (6.11)

over the closure of the upper half loop \( \Lambda_a \cap \{ \Im b > 0 \} \) is attained, at complex \( b = b_0(a) \). Note that \( b_0(a) \) is the tangent point where straight lines through \( b = -1/\pi \in \mathbb{C} \) touch the boundary of the upper half of the stabilizing loop \( \Lambda_a \) (figure 2). This allows us to stabilize the local anti-phase branch for all subcritical soft spring \( \gamma \) such that

\[
0 < \beta(a) \Re \gamma < |\Im \gamma|.
\] (6.12)

It is completely analogous to consider stabilization in the subcritical hard-spring case \( \Re \gamma > 0, 0 < a < 1/\pi \), where

\[
\Im \gamma > 0.
\] (6.13)

We then arrive at the stabilizing geometric situation of figure 6b, for slopes

\[
0 < -\pi \frac{\Re b + 1/\pi}{\Im b} < \pi \frac{\Im \gamma}{\Re \gamma}
\] (6.14)

and suitable \( b \) in the lower-half loop \( \Lambda_a \cap \{ \Im b < 0 \} \), again if and only if equation (6.12) holds. Indeed, the symmetry \( b_+(-\omega) = b_-(\omega) \) of the loop \( \Lambda_a \) implies

\[
\beta(a) = \min \left\{ \frac{\Re b + 1/\pi}{\Im b} \mid b \in \Lambda_a, \ \Im b > 0 \right\}
\] (6.15a)

\[
= \min \left\{ -\frac{\Re b + 1/\pi}{\Im b} \mid b \in \Lambda_a, \ \Im b < 0 \right\}.
\] (6.15b)

Equation (6.12) therefore allows us to locally stabilize the supercritical anti-phase branch for both the soft- and hard-spring cases, as was claimed in equation (3.7).

We conclude with deriving the claimed monotonicity and continuity of the minimum function \( \beta(a) \) (figure 7). It is sufficient to show that the loops \( \Lambda_a \) strictly shrink with increasing \( a \). Then, the maximum slopes \( 1/\beta(a) \) of straight lines through \( b = -1/\pi \in \mathbb{C} \) and points of \( \Lambda_a \cap \{ \Im b > 0 \} \) likewise decrease.

To show that the loops \( \Lambda_a \) strictly shrink with increasing \( a \), we consider the map

\[
(\omega, a) \mapsto (\Re b_+(\omega), \Im b_+(\omega)),
\] (6.16)

which defines the boundary of the loop \( \Lambda_a \) for \( 0 \leq \omega \leq \omega_+(a) < 1/2 \); see equations (6.1) and (6.2). It is easy to check that the map (6.16) is an orientation preserving local diffeomorphism on \( 0 \leq a < 1/\pi \), \( |\omega| < 1/2 \). With the given counterclockwise orientation of each loop boundary, by \( \omega \), this shows that the loops \( \Lambda_a \) shrink to \( b = 1/\pi \) for \( a \nearrow 1/\pi \). This completes the proof of theorems 3.1 and 3.2.
Figure 7. $\beta(a)$ is plotted (see equations (6.15)). The inset shows the function in a wider range. Note that $\beta$ is monotonically increasing and $\beta(0) > 0$.

7. Numerical illustrations

In this section, we present some numerical results. Figure 8 explains the stabilization of the anti-phase orbit for the subcritical case by looking at the rotating waves (circular orbits) present in the system. The radii of rotating waves within the anti-phase manifold are plotted. The target orbit is stabilized through a transcritical bifurcation with a delay-induced rotating wave. For feedback strength a little above the stabilization, the trivial equilibrium loses its stability in a subcritical Hopf bifurcation. In the limit $\lambda \to 2a$, the Hopf bifurcation and the transcritical bifurcation occur at the same coupling strengths $b_0$, resulting in an instant exchange of stability.

Figure 9 displays exemplary time series for the stabilized anti-phase circular orbit in the subcritical case. When the target circular orbit is stabilized, the control signal vanishes, demonstrating the non-invasiveness of the method.

Figure 10 illustrates how the stabilization fails if condition (3.7) $|\text{Im}\gamma| > \beta(a)\text{Re}\gamma$ is not satisfied. In this case, we have chosen $\gamma = 1 - 4i$. As $\beta(0.1) \approx 4.37$, the choice of $\gamma$ slightly violates the inequality and leads to amplitude death instead of stabilized anti-phase orbits.

8. Discussion

The in-phase orbits are stable and unstable for the super- and subcritical cases, respectively. As the dynamics in the flow-invariant in-phase plane is equivalent to the dynamics of a single planar system, the stabilization of the in-phase orbit in the subcritical case reduces to the stabilization of an orbit born in a planar subcritical Hopf bifurcation by delayed feedback, which has previously been demonstrated (Fiedler et al. 2007).
Figure 8. Stabilization of the anti-phase branch in the subcritical case. The radii of circular orbits versus the feedback gain $b_0$ within the anti-phase manifold are plotted; solid and dashed lines correspond to dynamically stable and unstable circular orbits, respectively. For increasing feedback strengths, $b_0$, a pair of stable and unstable orbits is born in a saddle–node bifurcation ($b_0 \approx 0.03$). The stable sibling then stabilizes the target orbit in a transcritical bifurcation at $b_0 \approx 0.1$ and subsequently, having lost its stability, destabilizes the trivial equilibrium $r = 0$ in a subcritical Hopf bifurcation ($b_0 \approx 0.17$). Note that the control is non-invasive on the target orbit, i.e. its radius does not change. With further increase of feedback strengths, there is a cascade of saddle–node bifurcations generating new feedback-induced circular orbits. One of these bifurcations is shown ($b_0 \approx 0.14$). Unstable dimensions are indicated in parentheses, for the trivial equilibrium $z \equiv 0$, and in square brackets, for the bifurcating periodic orbits. $a = 0.1$, $\lambda = 2a - 0.01$, $\gamma = 1 - 10i$, $\tau = p_- / 2 = \pi / (1 - (\lambda - 2a) \Im \gamma / \Re \gamma)$, $b = b_0 e^{i\pi/8}$.

Figure 9. Stabilization of the anti-phase branch in the subcritical case. (a) Time series of Re $z_+$ (black) and Re $z_-$ (grey) and (b) time series of coupling forces $F_1 := b \cdot (z_1(t - \tau) + z_2)$ (solid) and $F_2 := b \cdot (z_2(t - \tau) + z_1)$ (dotted) acting on the systems. The system starts away from the anti-phase plane $z_+ = 0$. After a short time, the in-phase component $z_+$ decays and the system goes to the anti-phase plane. After a longer transient, the system approaches the stabilized anti-phase orbit. Once the anti-phase orbit is reached, the control forces vanish. $a = 0.1$, $\lambda = 2a - 0.01$, $\gamma = 1 - 10i$, $\tau = p_- / 2 = \pi / (1 - (\lambda - 2a) \Im \gamma / \Re \gamma)$, $b = 0.24 e^{i\pi/8}$.
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Figure 10. The time series of \( \text{Re}\ z_+ \) (black) and \( \text{Re}\ z_- \) (grey) are plotted for the same parameters as in figure 9 except \( \gamma = 1 - 4i \). This \( \gamma \) violates condition (3.7) and fails to stabilize the anti-phase circular orbit. Instead, the feedback system dies out to its trivial equilibrium \( z_+ = z_- = 0 \).

Now, we are also able to stabilize unstable anti-phase orbits in certain combinations of super-/subcritical and soft-/hard-spring cases. Stabilization is achieved locally near Hopf bifurcation. In the supercritical case of two unstable Floquet multipliers, real feedback gains were sufficient, both for soft and for hard springs. In the subcritical case of three unstable Floquet multipliers, by contrast, stabilization could only be achieved by complex feedback gains and for sufficiently nonlinear (soft or hard) springs. The crucial limitation \( |\text{Im}\ \gamma| > \beta(a)\text{Re}\ \gamma \) was derived in equation (3.7). Here, \( |\text{Im}\ \gamma| \) measures the dependence of the minimal period of the individual oscillator upon amplitude, \( \text{Re}\ \gamma > 0 \) measures the subcriticality of the Hopf bifurcation and \( a \) is the strength of the diffusive coupling. The coupling strength was limited to \( 0 < a < 1/\pi \), where \( \pi \) corresponds to the normalized half period at anti-phase Hopf bifurcation.

9. Conclusion

In conclusion, we have studied two identical, diffusively coupled Hopf normal-form oscillators of both super- and subcritical type. By introducing a delay coupling of half the minimal period, we are able to non-invasively stabilize anti-phase orbits of the coupled systems, which are inherently unstable. We prove stabilization theorems for the hard-spring case and sketch the similar proof for the soft-spring case.

10. Outlook

Building on this work, it will be interesting to apply our method to stabilize anti-phase orbits in physical and biological systems such as coupled lasers and coupled neurons. From a mathematical point of view, generalizations to \( n \) oscillators and thus \( n \)-fold symmetries may be an interesting direction to pursue (D’Huys et al. 2008; Choe et al. submitted). For the Pyragas control scheme of planar in-phase circular orbits, it has been shown (Fiedler 2008) that...
only orbits whose real Floquet multipliers $\mu$ obey $\mu < \exp(9)$ can be stabilized. Similar Floquet constraints may apply for our control scheme. Although we believe our non-invasive stabilization strategy to be well adapted to anti-phase periodic oscillations, only the derivation and comparison of such fundamental constraints will settle our quest for efficient non-invasive feedback stabilization of spatio-temporal patterns.

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