Bose–Einstein condensation and superfluidity of trapped polaritons in graphene and quantum wells embedded in a microcavity

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The theory for spontaneous coherence of short-lived quasiparticles in two-dimensional excitonic systems is reviewed, in particular, quantum wells (QWs) and graphene layers (GLs) embedded in microcavities. Experiments with polaritons in an optical microcavity have already shown evidence of Bose–Einstein condensation (BEC) in the lowest quantum state in a harmonic trap. The theory of BEC and superfluidity of the microcavity excitonic polaritons in a harmonic potential trap is presented. Along the way, we determine a general method for defining the superfluid fraction in a two-dimensional trap, within the angular momentum representation. We discuss BEC of magnetoexcitonic polaritons (magnetopolaritons) in a QW and GL embedded in an optical microcavity in high magnetic field. It is shown that Rabi splitting in graphene is tunable by the external magnetic field $B$, while in a QW the Rabi splitting does not depend on the magnetic field in the strong $B$ limit.

Keywords: polaritons; Bose–Einstein condensation; superfluidity

1. Introduction

(a) Bose–Einstein condensation

When a system of bosons is cooled to low temperatures, a substantial fraction of the particles spontaneously occupy the single lowest energy quantum state. This phenomenon is known as Bose–Einstein condensation (BEC). The existence of

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BEC was first proposed by Einstein (1925) after he had generalized the work of Bose (1924) for statistical calculation of photons for an alternative derivation of the radiation law earlier found by Planck, on the statistics of monoatomic ideal gases. Then, in the middle of the twentieth century, variations of the theory of BEC were applied to explain the unusual properties of superfluid helium-4 (Griffin 1993; Pitaevskii & Stringari 2003). However, superfluid helium is a system of strongly interacting particles, while BEC was predicted by Einstein for weakly interacting bosons. In 1995 such a BEC of weakly interacting particles was produced in a gas of rubidium (Anderson et al. 1995; Ensher et al. 1996) and sodium (Ketterle & Druten 1996; Ketterle & Miesner 1997) atoms. Cornell, Ketterle and Wieman shared the 2001 Nobel Prize in Physics ‘for the achievement of BEC in dilute gases of alkali atoms’, which recognized the enormous technical challenges that had to be overcome in achieving the nanokelvin temperatures needed to create this atomic BEC. The experimental and theoretical achievements in the studies of the phenomenon of BEC of dilute supercold alkali gases in traps are reviewed from a theoretical perspective in Daflovo et al. (1999).

Since the two-dimensional thermal de Broglie wavelength is proportional to the inverse square root of the mass of a particle, if one makes a high-density gas of small mass Bose quasiparticles, then BEC can occur at much higher temperatures than for regular relatively heavy alkali atoms. The very light bounded quasiparticles can be made through the absorption of a photon by a semiconductor which creates an electron in an excited state while leaving behind a positively charged ‘hole’. This electron–hole pair can be bound into an ‘atomic’ state, just like the proton and electron of the hydrogen atom, but the mass of the new quasiparticle called an exciton is much smaller than the mass of a regular atom. Therefore, such excitons are expected to undergo BEC and be superfluid at temperatures \( T \sim 4 \text{K} \) much higher than alkali atoms (Moskalenko & Snoke 2000) at achievable exciton densities. Another quasiparticle, resulting from strong coupling of electromagnetic waves with excitons, is the polariton. The BEC of trapped microcavity excitonic polaritons resulting from coupling of photons with excitons presents another interesting phenomenon seen in semiconductor structures (Snoke 2002, 2008a; Littlewood 2007). In addition, extremely comprehensive research activity, both experimental and theoretical, is going on today in the study of properties of graphene (Novoselov et al. 2004; Zhang et al. 2005a; Castro Neto et al. 2009). Recent research of the collective properties of Bose quasiparticles such as excitons, biexcitons and polaritons in various graphene-based structures in strong magnetic fields predicts the existence of BEC and superfluidity of these quasiparticles (Berman et al. 2008a,b,c, 2009b).

\( (b) \) Bose–Einstein condensation and superfluidity of dipole excitons in coupled quantum wells

Predicted in Lozovik & Yudson (1975, 1978), superfluidity of the system of spatially separated electrons and holes in BCS-like pairing regime or in BEC of dipole excitons as shown in figure 1 induced general interest in this system. The possibility of BEC and superfluidity in the system would manifest itself in coupled quantum wells as persistent electric currents in each well and also through coherent optical effects (Zhu et al. 1995; Lozovik & Ovchinnikov 2002) and Josephson-like phenomena (Lozovik & Poushnov 1997). Observation of coherent
properties has been claimed in this type of system for driven non-equilibrium creation of excitons (Butov 2004; Yang et al. 2006) and for permanent excitons in strong magnetic fields (Eisenstein & MacDonald 2004). A dramatic ring effect in this type of system (Snoke 2002) has been explained as a classical effect of charge separation (Rapaport et al. 2004), but progress has been made on equilibrium trapping of this type of exciton in a harmonic potential (Vörös et al. 2006, 2009). In strong magnetic fields ($B > 7$ T), two-dimensional excitons survive in a substantially wider temperature range, as the exciton binding energies increase with magnetic field (Lerner & Lozovik 1981; Kallin & Halperin 1984; Paquet et al. 1985; Yoshioka & MacDonald 1990; Dzyubenko & Lozovik 1991; Lozovik & Ruvinsky 1997; Lozovik et al. 1999). The influence of disorder on BEC and superfluidity of excitons and magnetoexcitons has been investigated in Berman et al. (2004, 2007, 2006).

(c) Microcavity polaritons

Another type of quasiparticles which can undergo BEC are polaritons in an optical microcavity in which a quantum well (QW) is embedded. A microcavity simply consists of two mirrors opposite each other, as in a laser cavity, which are built into a heterostructure (Snoke 2002, 2008a; Littlewood 2007). The two mirrors are typically fabricated as sets of dielectric layers forming a Bragg reflector (Snoke 2002). A cavity is made using two sets of Bragg mirrors, made from layers with alternating dielectric constant, as shown in figure 2a. Photons are free to move in the two directions parallel to the mirrors, but confined to specific modes in the direction perpendicular to the mirrors. The QW is placed in between these two mirrors. If the thicknesses of the layers are designed correctly, the exciton energy can be made exactly resonant with the photon energy of the lowest confined mode. As shown in figure 2b, when the exciton and cavity photon energies are resonant, the dipole coupling of the photon and exciton leads to an anticrossing and formation of two new branches of excitations known as the upper and lower polaritons, which are a superposition of the exciton and photon (Hopfield 1958; Kavokin & Malpuech 2003; Snoke 2008a). The effective mass of the lower polariton (LP) is given by the curvature of the band at the zero wave vector in the plane of the QW.

Polaritons are extremely light relative to atoms and excitons, owing to the effective mass of the two-dimensional cavity photon which is defined by the cavity width (for discussion and references see below). For typical experimental
structures, the effective mass of polaritons can be less than $10^{-4}$ times the vacuum electron mass (Snoke 2008a). Hence, BEC and superfluidity in the system of polaritons are expected to occur at much higher temperatures than in the systems of atoms or even excitons (Snoke 2006, 2008b; Littlewood 2007). The theory of BEC and superfluidity of microcavity polaritons in an in-plane harmonic potential trap was developed in Berman et al. (2008d).

(d) **Bose–Einstein condensation and superfluidity of electron–hole pairs and magnetoexcitons in graphene layers**

Recent advances in fabrication techniques have made it possible to produce graphene, which is a two-dimensional honeycomb lattice of carbon atoms forming the basic planar structure in graphite (Novoselov et al. 2004; Zhang et al. 2005a; Castro Neto et al. 2009). Graphene has stimulated considerable theoretical interest as a semimetal the electron effective mass of which may be described by an unusual massless Dirac fermion band structure. Several novel many-body effects in graphene have been investigated (Das Sarma et al. 2007). In recent experiments, the integer quantum Hall effect (IQHE) has been reported (Novoselov et al. 2005; Zhang et al. 2005b, 2006). Quantum Hall ferromagnetism in graphene has been investigated from a theoretical point of view (Nomura & MacDonald 2006). Graphene has a number of interesting properties as a result of its unusual band structure which is linear near two inequivalent points ($K$ and $K'$) in the Brillouin zone. The single-electron quantum states near $K$ and $K'$ are described by a Dirac-type equation with zero mass, where the wave functions are spinors because of the two-point basis of the honeycomb lattice. In the presence of a magnetic field, the specific
structure of graphene results in the unusual features of both the Shubnikov–de Haas oscillations (Mikitik & Sharlai 1999) as well as the step pattern of the IQHE (Zheng & Ando 2002). Both these effects have been recently observed experimentally and reported in Novoselov et al. (2005) and Zhang et al. (2005b). The spectrum of plasmon excitations in a single graphene layer (GL) embedded in a material with effective dielectric constant $\varepsilon_b$ in the absence of an external magnetic field was calculated in Hwang & Das Sarma (2007). The collective magnetoplasmon excitations have been considered in GL structures (Berman et al. 2008e) and quantum dots in graphene (Berman et al. 2009a). The electron–hole pair condensation in graphene-based bilayers has been studied in Lozovik & Sokolik (2009), Min et al. (2008), Bistritzer & MacDonald (2008) and Kharitonov & Efetov (2008).

The collective properties of Bose quasiparticles such as excitons, biexcitons and polaritons in various graphene-based structures in high magnetic field are very interesting in relation to BEC and superfluidity, since the random field influence in graphene is potentially much weaker than in a QW, particularly, owing to the absence of backscattering (for smooth potentials). Let us mention that if the interaction of bosons with the random field is stronger, then the BEC critical temperature is lower (Berman et al. 2004).

The energy spectrum and wave function of a single magnetoexciton in graphene have been obtained by Iyengar et al. (2007). The collective properties of different quasiparticles in various graphene-based structures in high magnetic field have been studied theoretically. BEC and superfluidity of two-dimensional spatially indirect magnetoexcitons in two-layer graphene has been predicted (Berman et al. 2008a). The superfluid density and the temperature of the Kosterlitz–Thouless phase transition were shown to be increasing functions of the excitonic density but decreasing functions of magnetic field and the interlayer separation (Berman et al. 2008a). The instability of the ground state of the interacting two-dimensional indirect magnetoexcitons in a slab of superlattice with alternating electron and hole GLs was established (Berman et al. 2008b,c). The stable system of indirect two-dimensional magnetobiexcitons, consisting of a pair of indirect excitons with opposite dipole moments, was considered in graphene superlattices (Berman et al. 2008b). The superfluid density and the temperature of the Kosterlitz–Thouless phase transition for magnetobiexcitons in graphene superlattices were predicted (Berman et al. 2008b). The essential property of magnetoexcitonic systems based on graphene (in contrast, for example, to a QW) is a stronger influence of magnetic field and a weaker influence of disorder. Observation of BEC and superfluidity of two-dimensional quasiparticles in graphene in high magnetic field would be interesting confirmation of the phenomena we have described.

The paper is organized in the following way. In §2, we describe the experimental studies of BEC of microcavity polaritons in a trap. In §3, we present the theory of BEC and superfluidity of two-dimensional polaritons in an in-plane harmonic potential. In §4, the theoretical analysis of trapped QWs and graphene polaritons in a microcavity in a high magnetic field is given. In §5, we calculate the Rabi splitting constant in graphene and a QW in a high magnetic field. In §6, we analyse BEC of trapped graphene and QW magnetopolaritons in a microcavity. Finally, the discussion of the results and conclusions follow in §7.
2. Experimental evidence of Bose–Einstein condensation of microcavity polaritons in a trap

Several early experiments (Deng et al. 2002, 2003, 2006; Kasprzak et al. 2006) demonstrated spontaneous coherence in exciton–polariton gases in various two-dimensional semiconductor microcavity structures. In each of these experiments, a laser was focused on the sample to generate high-density polaritons at the laser spot. The coherent effects were seen only at the same place where the laser excited the samples and only during the time when the laser was on. Although it was argued with reasonable justification (Snoke 2006) that the coherent state created in these experiments had many of the characteristics of a BEC, a basic question remained: because the coherent emission occurs only in the region excited by the laser, is it possible that the coherent effects are essentially the same as a nonlinear amplification of the laser itself? Besides, because the polaritons were not created in a confining geometry in those experiments, the ground state of the system was poorly defined: the polaritons could freely diffuse away from the excitation region or fall into local minima created by disorder.

The demonstration of a spatial trap for the polaritons in the plane of their motion was reported in Balili et al. (2006, 2007, 2009). This trap can be well approximated by a harmonic potential at its minimum, allowing confinement of the polaritons at low temperature. Confinement of the particles in a macroscopic trap is known (Nozières 1995) to make BEC allowable in two dimensions with a condensate of finite size, similar to a condensate in a three-dimensional trap (Pitaevskii & Stringari 2003). The trap also produces an evaporative cooling effect for the polaritons. Moreover, the polaritons can be generated with a laser that is focused far from the centre of the trap and their accumulation can be observed in the bottom of the trap where there is no laser excitation. In a trap, the polaritons exhibit the effects associated with spontaneous Bose coherence (Balili et al. 2007). The studied sample consisted of three sets of four GaAs/AlAs QWs embedded in a GaAs/AlGaAs microcavity and a trap is induced by a mechanical stress (figure 3).

The resonant exciton–photon coupling leads to two polariton branches in the spectrum. The LP branch has a minimum at $k = 0$ with a very small effective mass, in the range $10^{-5}$–$10^{-4}$ of the vacuum electron mass, depending on the details of the structure. These quasiparticles act as a weakly interacting gas of bosons in two dimensions. Since the thermal de Broglie wavelength in two dimensions varies inversely with mass, the extremely light mass of these bosonic particles means that the critical temperature for superfluidity can in principle be 100 K or above for experimentally achievable number densities.

In a translationally invariant two-dimensional system, without a trap, superfluidity occurs via a Kosterlitz–Thouless superfluid (KTS) transition. Experiments on untrapped systems (Deng et al. 2002, 2003, 2006; Kasprzak et al. 2006; Baumberg et al. 2008; Love et al. 2008) have shown promising indications of the onset of spontaneous coherence effects. In principle, superfluidity in a finite two-dimensional system can be viewed as a type of BEC, with coherence length of the order of the size of the cloud of particles, what is sometimes called a ‘quasicondensate’ (Malpeuch et al. 2003). In the above experiments, however, it was argued that condensation occurred in a confining potential either in a local disorder minimum or created by band-gap shift owing to laser heating.

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If there is a confining potential, it is possible to have a true BEC quantum phase transition in two dimensions (Bagnato & Kleppner 1991; Nozières 1995). Recently, a macroscopic confining potential trap was created in a two-dimensional exciton–polariton system, in which the exciton energy was shifted using a stress-induced inhomogeneous band-gap shift (Balili et al. 2006), and evidence for BEC of polaritons has been observed in this system (Balili et al. 2007). In these experiments, the trap is about 30 μm across compared with a typical interparticle distance of 0.3 μm, and the spring constant is low enough that the spacing between the quantized states $\hbar \omega_0$ in the harmonic potential is small compared with $k_B T$, so that the states may be treated as continuum states. The diffusion length of the polaritons is comparable to the trap size, so that we may consider them to be in equilibrium spatially.

The polaritons have been created in this type of harmonic potential trap analogous to atoms in optical traps (Balili et al. 2007). The trap can be loaded by creating polaritons 50 μm from its centre that are allowed to drift into the trap. When the density of polaritons exceeds a critical threshold, a number of signatures of BEC are observed: spectral and spatial narrowing, a peak at zero momentum...
in the momentum distribution, first-order coherence and spontaneous linear polarization of the light emission. The polaritons, which are eigenstates of the light–matter system in a microcavity, remain in the strong coupling regime while going through this dynamical phase transition (Balili et al. 2009). The effects indicating BEC of trapped polaritons include both a real-space and momentum-space narrowing, first-order coherence, and onset of linear polarization above a particle density threshold (Balili et al. 2009).

3. Theory of Bose–Einstein condensation and superfluidity of two-dimensional polaritons in an in-plane harmonic potential

The properties of polaritons have been studied in several theoretical works. The theory of polariton dynamics owing to polariton–polariton interaction has been developed in Ciuti et al. (1998, 2003), Tassone & Yamamoto (1999) and Porras et al. (2002). The crossover between lasing and polariton coherence has been studied in Eastham & Littlewood (2001) and Keeling et al. (2004). Polariton superfluidity (Carusotto & Ciuti 2004) as well as spontaneous linear polarization of light emission (Laussy et al. 2006) have been predicted. In these previous studies, the coherent polaritonic phases were analysed in the two-dimensional infinite system. In such system a critical threshold to form a highly degenerate state can be achieved and a ‘quasicondensate’ is possible (Malpeuch et al. 2003).

The Hamiltonian of the polaritons is

\[ \hat{H}_{\text{tot}} = \hat{H}_{\text{exc}} + \hat{H}_{\text{ph}} + \hat{H}_{\text{exc-ph}}, \]

where \( \hat{H}_{\text{exc}} \) is an excitonic Hamiltonian, \( \hat{H}_{\text{ph}} \) is a photonic Hamiltonian and \( \hat{H}_{\text{exc-ph}} \) is a Hamiltonian of exciton–photon interaction. Analogous to the case of Bose atoms in a trap (Fernández & Mullin 2002; Pitaevskii & Stringari 2003), in the case of a slowly varying external potential, we can make the quasiclassical approximation, assuming that the effective exciton mass is not a function of \( r \). Then the Hamiltonian of the two-dimensional excitons can be written as

\[ \hat{H}_{\text{exc}} = \sum_{\mathbf{P}} (\varepsilon_{\text{ex}}(P) + V(r)) \hat{b}_{\mathbf{P}} \hat{b}_{\mathbf{P}} + \frac{1}{2A} \sum_{\mathbf{P}, \mathbf{P}' \mathbf{q}} U_{\mathbf{q}} \hat{b}_{\mathbf{P}+\mathbf{q}} \hat{b}_{\mathbf{P}'-\mathbf{q}} \hat{b}_{\mathbf{P}} \hat{b}_{\mathbf{P}'}, \]

where \( \hat{b}_{\mathbf{P}} \) and \( \hat{b}_{\mathbf{P}}^\dagger \) are excitonic creation and annihilation operators obeying Bose commutation relations, \( A \) is the macroscopic quantization area, \( V(r) \) is the potential of the external field and \( U_{\mathbf{q}} \) is the Fourier transform of the exciton–exciton interaction potential. In equation (3.2) \( \varepsilon_{\text{ex}}(P) = E_{\text{band}} - E_{\text{binding}} + \varepsilon_0(P) \) is a spectrum of a single exciton in a QW with a semiconductor energy gap \( E_{\text{band}} \) and the binding energy of a two-dimensional exciton \( E_{\text{binding}} = 2\mu_{\text{eh}} \epsilon^4 / (\hbar^2 \epsilon) \), where \( \mu_{\text{eh}} = m_e m_h / (m_e + m_h) \) is the reduced mass (\( m_e \) and \( m_h \) are the masses of electron and hole, respectively), \( \epsilon \) is the dielectric constant, \( e \) is the charge.

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where an electron and $e_0(p) = P^2/2M$; $M = m_e + m_h$ is the mass of an exciton. As discussed in Ciuti et al. (1998) and Ben-Tabou de-Leon & Laikhtman (2001), in the low-density limit, the excitons can be treated as pure bosons, with an interaction potential that includes the effects of the underlying fermion nature of the electrons and holes. For small wave vectors ($q \ll a_{2D}^{-1}$, where $a_{2D} = \hbar^2/2\mu_e e^2$ is the effective two-dimensional Bohr radius of excitons) the exciton–photon pair repulsion is determined by the contact potential $U_q \simeq U_0 = 3c^2 a_{2D}/\epsilon$. We also assume that the exciton gas is very dilute and the average distance between excitons $r_s \sim (\pi n)^{-1/2} \gg a_{2D}$. A much smaller contribution to the exciton–exciton interaction is also given by the saturation effects (Rochat et al. 2000), which are neglected here.

The spatial dependence of the external field $V(r)$ comes about owing to the shifting of the exciton energy with inhomogeneous stress (Negoiţă et al. 1999); the photon states in the cavity are assumed to be unaffected by stress. The minimum of the exciton energy can be approximated as $V(r) = 1/2\gamma r^2$.

The Hamiltonian of the non-interacting photons in a semiconductor microcavity is $\hat{H}_{\text{ph}} = \sum_P \epsilon_{\text{ph}}(P) \hat{a}_P^\dagger \hat{a}_P$, where $\hat{a}_P$ and $\hat{a}_P^\dagger$ are photonic creation and annihilation Bose operators and $\epsilon_{\text{ph}}(P) = (c/n)\sqrt{P^2 + \hbar^2(2\pi)^2/\lambda^2}$ is the cavity photon spectrum. Here $c$ is the speed of light, $\lambda$ is the wavelength of the cavity mode and $n = \sqrt{\epsilon}$ is the effective refractive index. The Hamiltonian of the resonant exciton–photon coupling in the cavity has the form $\hat{H}_{\text{exc–ph}} = \hbar \Omega_R \sum_P \hat{\gamma}_P^\dagger \hat{\gamma}_P + \text{h.c.}$, where the exciton–photon coupling energy represented by the Rabi constant $\hbar \Omega_R$ depends only on the overlap between the exciton and photon wave functions and the semiconductor interband dipole moment (Pau et al. 1995; Savona et al. 1999). For a GaAs cavity, this coupling energy was estimated in Balili et al. (2006) as $\hbar \Omega_R \approx 15\text{ meV}$. We neglect anharmonic terms for exciton–photon coupling since $k_B T \ll \hbar \Omega_R$ in the experiments (Balili et al. 2006).

The quadratic part of the total Hamiltonian $\hat{H}_{\text{tot}}$ (without the second term on the right-hand side of equation (3.2)) can be diagonalized by applying unitary transformations and has the form (Ciuti et al. 2003)

$$\hat{H}_0 = \sum_P \epsilon_{\text{LP}}(P) \hat{\gamma}_P^\dagger \hat{\gamma}_P + \sum_P \epsilon_{\text{UP}}(P) \hat{a}_P^\dagger \hat{a}_P, \quad (3.3)$$

where $\hat{\gamma}_P$ and $\hat{a}_P$ are the Bose creation operators for the lower and upper polaritons, respectively, defined as

$$\hat{\gamma}_P = C_P \hat{a}_P + X_P \hat{b}_P \quad \text{and} \quad \hat{a}_P = X_P \hat{a}_P - C_P \hat{b}_P, \quad (3.4)$$

where $|X_P|^2$ and $|C_P|^2 = 1 - |X_P|^2$ represent the exciton and cavity photon fractions as that in the LP (Ciuti et al. 2003).

We define the small parameters $\alpha \equiv (1/2)(M^{-1} + c\lambda \hbar^{-1}n^{-1}P^2/|\hbar \Omega_R| \ll 1$ and $\beta \equiv \gamma r^2/|\hbar \Omega_R| \ll 1$. In the limit that these are small, which occurs when the temperature is low enough so that the characteristic momentum and cloud size are small, the resulting effective Hamiltonian for the LPs in the parabolic trap

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within the effective mass approximation has the form (we count energy from \((c/n^2\pi^2\hbar^2\lambda - |\hbar\Omega_R|)\))

\[
\hat{H}_{\text{eff}} = \int \text{d}r \hat{\psi}^\dagger(r) \left(-\frac{\Delta}{2M_{\text{eff}}} + V_{\text{eff}}(r)\right) \hat{\psi}(r) + \frac{U^{(0)}_{\text{eff}}}{2} \int \text{d}r \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r) \hat{\psi}(r), \tag{3.5}
\]

where \(\hat{\psi}^\dagger(r)\) and \(\hat{\psi}(r)\) are real space Bose field operators of creation and annihilation of LPs, respectively, and the effective mass of a polariton is given by

\[
M^{-1}_{\text{eff}} = \frac{1}{2} \left( M^{-1} + \frac{c}{2\pi^2 n} \right). \tag{3.6}
\]

In equation (3.5) the effective external potential is given by \(V_{\text{eff}}(r) = (1/2)\gamma_{\text{eff}} r^2\) with \(\gamma_{\text{eff}} = (1/2)\gamma\), and the effective polariton–polariton pair repulsion potential is given by the hard-core contact potential \(U_{\text{eff}}(r - r') = U^{(0)}_{\text{eff}} \delta(r - r') = 3e^2 a_{2D}/(2\epsilon)\delta(r - r')\). In other words, the linear expansion of the polaritonic Hamiltonian with respect to small \(\alpha\) and \(\beta\) results in \(V_{\text{eff}}(r) = (1/2) V(r)\) and \(U_{\text{eff}}^{(0)} = (1/4) U\).

Thus, in the real space the effective Hamiltonian for trapped polaritons maps to the Hamiltonian of weakly interacting dilute two-dimensional Bose gas in confinement (Fernández & Mullin 2002). Although BEC cannot happen in a two-dimensional homogeneous ideal gas at non-zero temperature, in a harmonic trap BEC can occur in two dimensions below a critical temperature given by (Bagnato & Kleppner 1991)

\[
T_c^0 = k_B^{-1}\hbar \pi^{-1} \sqrt{6}\gamma_{\text{eff}} N M_{\text{eff}} s, \tag{3.7}
\]

where \(k_B\) is the Boltzmann constant, \(s\) is the spin degeneracy (\(s = 2\) for bright excitons in GaAs QWs), and \(N\) is the total number of polaritons in the trap. This expression for the temperature of BEC can be also used to estimate \(T_c^0\) for weakly interacting polaritons.

In the following, we assume thermal equilibrium for the polaritons. This is reasonable as long as the time for the polariton–polariton scattering is short compared with their lifetime, in which case the polariton gas will establish a well-defined temperature, which can be substantially higher than the lattice temperature. Although the polariton lifetime is typically short, approximately 2 ps, the polariton–polariton scattering time can be even shorter at high polariton density, and recent experimental work (Kasprzak et al. 2006), using the trick of detuning the structure to make the polaritons slightly more excitonic, has shown that the polaritons can indeed have a well-defined temperature. At temperatures \(T \gg k_B^{-1}\hbar \sqrt{\gamma_{\text{eff}}/M_{\text{eff}}}\) (for \(\gamma = 100\text{ eV cm}^{-2}\), this implies \(T \gg 0.292\text{ K}\)) one can ignore the effects of the discontinuous jumps between the quantized states in the harmonic potential. Neglecting the anomalous averages \(\langle \hat{\psi} \hat{\psi} \rangle\) and \(\langle \hat{\psi}^\dagger \hat{\psi}^\dagger \rangle\) via the Popov approximation (Giorgini et al. 1997; Fernández & Mullin 2002), implying the system to be very dilute (i.e. \(n(0)a_{2D}^2 \ll 1\), where \(n(0)\) is the number density of polaritons at the centre of the trap),
the self-consistent equation for the non-condensate density $n'(r)$ can be written as (Berman et al. 2004)

$$n'(r) = -\frac{sM_{\text{eff}} k_B T}{2\pi \hbar^2} \log \left( 1 - \exp \left[ -\frac{\hbar}{2k_B T} \sqrt{\frac{\gamma_{\text{eff}}}{M_{\text{eff}}}} \tilde{g} \left( r + \frac{\sqrt{2M_{\text{eff}} U(0)_{\text{eff}} n_0(r)}}{\hbar \gamma_{\text{eff}}} \right)^2 \right. \right. $$

$$\left. \left. - \frac{4 \left| U(0)_{\text{eff}} \right|^2 M_{\text{eff}} n_0(r)^2}{\hbar^2 \gamma_{\text{eff}}} \right]^{1/2} \right),$$

(3.7)

where $n_0(r)$ is the density of the condensate and $\mu = 2 U(0)_{\text{eff}} n(r) - U(0)_{\text{eff}} n_0(r)$ is the chemical potential of the system in the Popov approximation, which is valid in a wide temperature range (Griffin 1993).

Analogously to Fernández & Mullin (2002), the total number of polaritons in condensate $N_0$ is given by $N_0 = N - N'$, where $N' = 2\pi \int_0^\infty n'(r) r \, dr$ is the total number of non-condensate particles and $N = 2\pi \int_0^\infty n(r) r \, dr$. Writing $n_0(r) = n(r) - n'(r)$, and solving the self-consistent equation (3.7) with respect to the non-condensate density $n'(r)$, we obtain the dependence of the fraction $N_0(T)/N$ of the total number of condensate particles on the temperature $T$. Figure 4 shows this dependence for the experimental parameters provided in Berman et al. (2008d).

For small quasimomenta $P \ll \sqrt{2M_{\text{eff}} U(0)_{\text{eff}} n_0}$ and low temperatures the energy spectrum of quasiparticles $\epsilon(P)$ is given by $\epsilon(P, r) \approx c_s(r) P$ (Fernández & Mullin 2002), where $c_s(r) = \sqrt{U(0)_{\text{eff}} n_0(r, T)/M_{\text{eff}}}$ is the sound velocity in the Popov...
approximation (Griffin 1993). Since the spectrum of quasiparticles is a linear sound spectrum satisfying the Landau criterion of superfluidity (Abrikosov et al. 1963), there can occur the superfluidity of cavity polaritons in a trap. At low temperatures there are normal and superfluid components of the trapped polariton gas. We define the total number of particles in the superfluid component as \( N_s \equiv N - N_n \), where \( N_n \) is the total number of particles in the normal component. We define \( N_n \) analogously to the procedure applied for definition of the density of the normal component in the infinite system, \( n_n \) (Abrikosov et al. 1963), using the isotropy of the trapped polaritonic gas instead of the translational symmetry for an infinite system. Let us suppose that a gas of quasiparticles rotates in the liquid in the plane perpendicular to the axis of the trap with some small macroscopic angular velocity \( \omega \). In this case, the distribution function of the gas of quasiparticles can be obtained from the distribution function of a gas at rest by substituting for the energy spectrum of quasiparticles \( \epsilon(P) - L\omega \), where \( L = r \times P \) is the angular momentum of the particle; for the two-dimensional case both \( \omega \) and \( L \) are normal to the trap. Assuming \( P r / \hbar \gg 1 \) we apply the quasiclassical approximation for the angular momentum: \( L \approx P r \) and \( \frac{3}{2} (L, r) = r - \frac{1}{2} c_s(r) L \). The total angular momentum in a trap per unit of volume \( L_{\text{tot}}(r) \) is given by

\[
L_{\text{tot}}(r) = s \int \frac{d^2 L}{(2\pi \hbar r)^2} L n_B(\epsilon(r, L) - L\omega),
\]

where we assume that at low temperatures the quasiparticles are non-interacting, and they are described by the Bose–Einstein distribution function \( n_B(\epsilon) = (\exp[\epsilon/(k_B T)] - 1)^{-1} \). For small angular velocities, \( n_B(\epsilon - L\omega) \) can be expanded with respect to \( L\omega \). As a result we get

\[
L_{\text{tot}}(r) = -s \int \frac{d^2 L}{(2\pi \hbar r)^2} L(L\omega) \frac{\partial n_B(\epsilon)}{\partial \epsilon}.
\]

Assuming that only quasiparticles contribute to the total angular momentum, we define the density of the normal component \( n_n(r) \) by \( L_{\text{tot}}(r) = n_n(r)L_0 \), where \( L_0 = M_{\text{eff}} r\omega \) is the angular momentum of a single quasiparticle. For the total number of particles in the normal component we then obtain (Berman et al. 2008d)

\[
N_n = 2\pi \int_0^\infty n_n(r) r dr = s \int_0^\infty \frac{3\zeta(3) k_B^3 T^3}{\hbar^2 c_s^4 M_{\text{eff}}} r dr,
\]

where \( \zeta(z) \) is the Riemann zeta function. The radial dependence is entirely through \( c_s \), which depends on the density of the condensate \( n_0(r) = n - n'(r) \) (the density of non-condensate polaritons \( n'(r) \) can be obtained from equation (3.7)). The dependence of the fraction \( N_s(T)/N \) of the total number of polaritons in the superfluid component \( N_s(T) = N - N_n(T) \) on the temperature \( T \) is presented in figure 5. The superfluid fraction depends only weakly on the spring constant \( \gamma \), and in the limit \( \gamma \to 0 \) approaches the superfluid density for a two-dimensional translationally invariant system (Berman et al. 2004). For a discussion of the specific features of the superfluid density of polaritons see, for example, Lozovik & Semenov (2007).
Figure 5. Superfluid fraction $N_s/N$ as a function of temperature for the same three trap spring constants as figure 4. The dashed-dotted line shows the superfluid fraction in the limit $\gamma \to 0$, namely the translationally invariant two-dimensional case.

4. Trapped quantum well and graphene polaritons in a microcavity in a high magnetic field

Trapped polaritons in an optical microcavity with an embedded single GL embedded into an optical microcavity in a high magnetic field were considered in Berman et al. (2009b). When an undoped electron system in graphene in a magnetic field without an external electric field is in the ground state, half of the zeroth Landau level is filled with electrons, all Landau levels above the zeroth one are empty, and all levels below the zeroth one are filled with electrons. We suggest using the gate voltage to control the chemical potential in graphene in two ways: to shift the chemical potential above the zeroth level so that it is between the zeroth and first Landau levels (case 1) or to shift the chemical potential below the zeroth level so that it is between the first negative and zeroth Landau levels (case 2). Therefore, in the first case magnetoexcitons formed in graphene by the electron on the first Landau level and the hole on the zeroth Landau level or in the second case the electron on the zeroth Landau level and the hole on the first negative Landau level. Note with an appropriate gate potential we can also use any other neighbouring Landau levels $n$ and $n+1$. In both cases, all Landau levels below the chemical potential are completely filled and all Landau levels above the chemical potential are completely empty. Based on the selection rules for optical transitions between the Landau levels in single-layer graphene (Gusynin et al. 2007), in the first case, there are allowed transitions between the zeroth and the first Landau levels, while in the second case there are allowed transitions between the first negative and zeroth Landau levels.

For the relatively high dielectric constant of the microcavity, $\epsilon \gg e^2/(\hbar c)$, the magnetoexciton energy in graphene can be calculated by applying perturbation theory with respect to the strength of the Coulomb electron–hole attraction analogously to that done in Lerner & Lozovik (1981) for...
two-dimensional QWs in a high magnetic field with non-zero electron \( m_e \) and hole \( m_h \) masses. This approach allows us to obtain the spectrum of an isolated magnetoexciton with the electron on the Landau level 1 and the hole on the Landau level 0 in a single GL, and it will be exactly the same as for the magnetoexciton with the electron on the Landau level 0 and the hole on the Landau level \(-1\). The characteristic Coulomb electron–hole attraction for the single GL is \( e^2/(\epsilon r_B) \). The energy difference between the first and zeroth Landau levels in graphene is \( \hbar v_F/r_B \). For graphene, the perturbative approach with respect to the strength of the Coulomb electron–hole attraction is valid when \( e^2/(\epsilon r_B) \ll \hbar v_F/r_B \) (Lerner & Lozovik 1981). This condition can be fulfilled with all magnetic fields \( B \) if the dielectric constant of the surrounding media satisfies the condition \( e^2/(\epsilon \hbar v_F) \ll 1 \). Therefore, we claim that the energy difference between the first and zeroth Landau levels is always greater than the characteristic Coulomb attraction between the electron and the hole in the single GL at any \( B \) if \( \epsilon \gg e^2/(\hbar v_F) \approx 2 \). Thus, applying perturbation theory with respect to weak Coulomb electron–hole attraction in graphene embedded in a GaAs microcavity (\( \epsilon = 12.9 \)) is more accurate than for graphene embedded in an SiO\(_2\) microcavity (\( \epsilon = 4.5 \)). However, the magnetoexcitons in graphene exist in a high magnetic field. Therefore, we restrict ourselves by consideration of high magnetic fields. This condition for perturbation theory in graphene is different from the two-dimensional QW in GaAs, since in the latter case the energy difference between the neighbouring Landau levels is \( \hbar \omega_c \), where \( \omega_c = eB/(c\mu_{ch}) \) is the cyclotron frequency, \( \mu_{ch} = m_em_h/(m_e + m_h) \), and \( m_e \) and \( m_h \) are the effective masses of the electron and the hole, respectively (Lerner & Lozovik 1981). Therefore, for the QW in GaAs, the binding energy of the magnetoexciton is much smaller than the energy difference between two neighbouring Landau levels only in the limit of high magnetic field \( B \gg e^3c\mu_{ch}^2/(2\hbar^3) \), and perturbation theory with respect to weak electron–hole attraction can be applied only for high magnetic field.

The projection of the Hamiltonian of the trapped electrons and holes in the presence of microcavity photons for the QW and GL in a magnetic field onto the lowest Landau level results in the effective Hamiltonian which is similar to the Hamiltonian (3.1). The first term of this effective Hamiltonian is provided by equation (3.2), where \( \epsilon_0(P) = P^2/(2m_B) \) as well as the third term of equation (3.1) as renormalized by magnetic field Rabi splitting constant \( \Omega_R \) only for GL (Berman et al. 2009b). So the trapped electron–hole QW and GL systems with photons embedded in a microcavity in a high magnetic field are described by the effective Hamiltonian with a renormalized mass and Rabi splitting constant for the GL where the term related to the vector potential is missing (Berman et al. 2009b). The magnetic field in the effective Hamiltonian enters in the renormalized mass of the magnetoexciton \( m_B \) instead of exciton mass \( M \). The magnetic field \( B \) manifests itself in the effective Hamiltonian only through the effective magnetic mass of a magnetoexciton \( m_B \) in the expression for \( \epsilon_0(P) \) in the first term of \( \hat{H}_{ex} \) provided by equation (3.2). Besides, for the GL the magnetic field enters the Rabi splitting constant, while for the QW the Rabi splitting constant does not depend on magnetic field. There are two differences in the expressions for the effective Hamiltonian for the QW and GL. The first difference is that \( m_B \) for the GL is four times less than that for the QW owing to the four-component spinor structure of the wave function of the relative motion for the isolated
non-interacting electron–hole pair in a magnetic field (Berman et al. 2009b). The second difference is that for the GL the Rabi splitting constant $\Omega_R$ depends on the magnetic field, which is not the case for the QW in the limit of high magnetic field (Berman et al. 2009b).

Polaritons are linear superpositions of excitons and photons. In high magnetic fields, when magnetoexcitons may exist, the polaritons become linear superpositions of magnetoexcitons and photons. Let us call the superpositions of magnetoexcitons and photons as magnetopolaritons. It is obvious that magnetopolaritons in graphene are two-dimensional, since graphene is a two-dimensional structure. The Hamiltonian of magnetopolaritons in a strong magnetic field is given by equation (3.5). It can be shown that the interaction between two direct two-dimensional magnetoexcitons in graphene with the electron on the Landau level 1 and the hole on the Landau level 0 can be neglected in a strong magnetic field, by analogy to what is described in Lerner & Lozovik (1981) for two-dimensional magnetoexcitons in a QW. Thus, the Hamiltonian does not include the term corresponding to the interaction between two direct magnetoexcitons in a single GL. So in a high magnetic field there is BEC of the ideal magnetoexcitonic gas in graphene. Therefore, in a single GL in a high magnetic field we assume the second term in equation (3.5) vanishes.

The binding energy $E_B^{(b)}$ and effective magnetic mass $m_B$ of a magnetoexciton in graphene obtained using the first-order perturbation with respect to the electron–hole Coulomb attraction similarly to the case of a single QW (Lerner & Lozovik 1981) are given by

$$E_B^{(b)} = \sqrt{\frac{\pi}{2} \frac{e^2}{\epsilon R B}} \quad \text{and} \quad m_B = \frac{2^{7/2} e^2}{\sqrt{\pi} c^2 \hbar B}. \quad (4.1)$$

We obtain the effective Hamiltonian of polaritons by applying the standard procedure (Hopfield 1958; Agranovich 1968, 2009; Ciuti et al. 2003), when we diagonalize the Hamiltonian $\hat{H}_{\text{tot}}$ (3.1) by using Bogoliubov transformations. If we measure the energy relative to the $P=0$ lower magnetopolariton energy $(c/n)\hbar \pi L_C^{-1} - |\hbar \Omega_R|$, we obtain the resulting effective Hamiltonian for trapped magnetopolaritons in graphene in a magnetic field. At small momenta $\alpha \ll 1$ ($L_C = \hbar \pi c/n (E_{band} - E_B^{(b)})^{-1}$) and weak confinement $\hat{\beta} \ll 1$, this effective Hamiltonian is

$$\hat{H}_{\text{eff}} = \sum_P \left( \frac{P^2}{2M_{\text{eff}}(B)} + \frac{1}{2} V(r) \right) \hat{p}_P \hat{p}_P, \quad (4.2)$$

where the effective magnetic mass of a magnetopolariton is given by

$$M_{\text{eff}}(B) = 2 \left( m_B^{-1} + \frac{c L_C(B)}{n \hbar \pi} \right)^{-1}. \quad (4.3)$$

According to equations (3.6) and (4.1), the effective magnetopolariton mass $M_{\text{eff}}$ increases with the increment of the magnetic field as $B^{1/2}$. Let us emphasize that the resulting effective Hamiltonian for magnetopolaritons in graphene in a magnetic field for the parabolic trap is given by equation (4.2) for both physical realizations of confinement represented by cases 1 and 2.

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5. The Rabi splitting constant in graphene and a quantum well in high magnetic field

Neglecting anharmonic terms for the magnetoexciton–photon coupling, the Rabi splitting constant \( \Omega_R \) can be estimated quasiclassically as

\[
|\hbar \Omega_R| = |\langle f | \hat{H}_{\text{int}} | i \rangle|,
\]

where \( \hat{H}_{\text{int}} \) is the Hamiltonian of the electron–photon interaction. For graphene this interaction is determined by Dirac electron Hamiltonian as

\[
\hat{H}_{\text{int}} = -\frac{v_F e}{c} \hat{\sigma} \cdot \mathbf{A}_{\text{ph}0} = \frac{v_F e}{i \omega} \hat{\sigma} \cdot E_{\text{ph}0},
\]

where \( \hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y) \), \( \hat{\sigma}_x \) and \( \hat{\sigma}_y \) are Pauli matrices, \( \mathbf{A}_{\text{ph}0} \) is the vector potential corresponding to a single cavity photon, and \( E_{\text{ph}0} = (8 \pi \hbar \omega / (\epsilon W))^{1/2} \) is the magnitude of electric field corresponding to a single cavity photon of frequency \( \omega \) in the volume of microcavity \( W \), while for the QW this interaction is

\[
\hat{H}_{\text{int}} = \mathbf{d}_{12} \cdot \mathbf{E}_{\text{ph}0},
\]

where

\[
\mathbf{d}_{12} = e \sum_i \mathbf{r}_i
\]

is the dipole momentum of transition and the sum is taken over the coordinate vectors related to the positions of all the electrons in the system.

In equation (5.1) the initial \( |i\rangle \) and final \( |f\rangle \) electron states are different for graphene and a QW. For the case of graphene these electron states are defined as

\[
|i\rangle = \prod_k \hat{c}_{0,k}^\dagger |0\rangle_0 |0\rangle_1
\]

and

\[
|f\rangle = \hat{b}_{1,0}^\dagger |i\rangle.
\]

In equation (5.5), \( \hat{c}_{n,k}^\dagger \) is the Fermi creation operator of the electron with the \( y \) component of the wavevector \( k \) on the Landau level \( n \), \( |0\rangle_n \) denotes the wave function of the vacuum on the Landau level \( n \), \( \prod_k \hat{c}_{0,k}^\dagger |0\rangle_0 \) corresponds to the completely filled zeroth Landau level and \( \hat{b}_{n,n'}^\dagger \) is the Bose creation operator of the magnetoexciton with the electron on the Landau level \( n \) and the hole on the Landau level \( n' \). We consider magnetoexcitons with magnetic momenta equal to zero, for which the Bose condensate in the system of non-interacting particles is the exact solution of the problem (Lerner & Lozovik 1981). Following Lerner & Lozovik (1981), \( \hat{b}_{n,n'}^\dagger \) for this case is defined as

\[
\hat{b}_{n,n'}^\dagger = \frac{1}{\sqrt{N_d}} \sum_k \hat{h}_{n',k}^\dagger \hat{c}_{n,-k}^\dagger,
\]

where \( \hat{h}_{n',k} \) is the Fermi creation operator of the hole with the \( y \) component of the wavevector \( k \) on the Landau level \( n' \), \( N_d = S/(2 \pi r_0^2) \) is the macroscopic degeneracy of Landau levels, and \( S \) is the area of the system.
Let us use the Landau gauge for the wave function of the single electron \( \psi_{n,k}(x,y) \) with the \( y \) component of the wavevector \( k \) on the Landau level \( n \). In the Landau gauge with the vector potential \( \mathbf{A} = (0, Bx, 0) \), the two-component eigenfunction \( \psi_{n,k}(\mathbf{r}) \) is given in Zheng & Ando (2002). The corresponding eigenenergies depend on the quantum number \( n \) only and are given by

\[
\epsilon_n = \frac{\hbar v_F}{r_B} \sqrt{2n}. \quad (5.7)
\]

Substituting the two-component single electron eigenfunction \( \psi_{n,k}(\mathbf{r}) \) from Zheng & Ando (2002) into equation (5.5) and using the electron–photon interaction \( \hat{H}_{\text{int}} \) of equation (5.2), we finally obtain from equation (5.1):

\[
|\hbar \Omega_R| = \frac{|e v_F| |E_{\text{ph}0}|}{\sqrt{2} \omega}. \quad (5.8)
\]

In equation (5.8) the energy of the photon absorbed at the creation of the magnetoexciton (at \( E^{(b)}_B \ll \epsilon_1 - \epsilon_0 \)) is given by

\[
\hbar \omega = \epsilon_1 - \epsilon_0 = \sqrt{2} \frac{\hbar v_F}{r_B}. \quad (5.9)
\]

Substituting the photon energy from equation (5.9) into equation (5.8), we obtain the Rabi splitting corresponding to the creation of a magnetoexciton with the electron on the Landau level 1 and the hole on the Landau level 0 in graphene:

\[
\hbar \Omega_R = 2e \left( \frac{\pi \hbar v_F r_B}{\sqrt{2} \epsilon W} \right)^{1/2}. \quad (5.10)
\]

As follows from equation (5.10), the Rabi splitting in graphene is related to the creation of the magnetoexciton, which decreases when the magnetic field increases and is proportional to \( B^{-1/4} \). Therefore, the Rabi splitting in graphene can be controlled by the external magnetic field. Note that in a semiconductor QW contrary to graphene the Rabi splitting does not depend on the magnetic field (at high \( B \)).

Substituting equation (5.3) and the initial \( |i\rangle \) and final \( |f\rangle \) electron states from Lerner & Lozovik (1981) into equation (5.1) after the integration we obtain the Rabi splitting constant \( \Omega_R \) for a QW

\[
\hbar \Omega_R = d_{12} E_{\text{ph}0}, \quad (5.11)
\]

where \( d_{12} \) is the matrix term of a magnetoexciton generation transition in a QW represented as

\[
d_{12} = e \left| \left\langle f \left| \sum_i r_i \right| i \right\rangle \right|. \quad (5.12)
\]
Similar calculations for the transition dipole moment and the photon energy corresponding to the formation of magnetoexciton with the electron and hole on zeroth Landau level in the QW give

\[
d_{12} = \frac{e \rho B}{2\sqrt{2}} \quad \text{and} \quad \hbar \omega = \epsilon_1 - \epsilon_0 = \hbar \omega_c = \frac{\hbar e B}{c\mu_{eh}},
\]

(5.13)

Substituting the transition dipole moment and the photon energy given by equation (5.13) into equation (5.1), we obtain the Rabi splitting for the QW

\[
\hbar \Omega_R = 2e\hbar \left( \frac{\pi}{\epsilon \mu_{eh} W} \right)^{1/2}.
\]

(5.14)

Thus, as follows from equation (5.14), the Rabi splitting in a QW does not depend on the magnetic field in the limit of high magnetic field. Therefore, only in graphene can the Rabi splitting be controlled by the external magnetic field in the limit of high magnetic field.

It is easy to show that the Rabi splitting related to the creation of the magnetoexciton with the electron on the Landau level 0 and the hole on the Landau level \(-1\) will be exactly the same as for the magnetoexciton with the electron on the Landau level 1 and the hole on the Landau level 0. Let us mention that dipole optical transitions from the Landau level \(-1\) to the Landau level 0, as well as from the Landau level 0 to the Landau level 1, are allowed by the selection rules for optical transitions in single-layer graphene (Gusynin et al. 2007).

### 6. Bose–Einstein condensation of trapped microcavity magnetopolaritons in graphene and quantum well

Although BEC cannot take place in a two-dimensional homogeneous ideal gas at non-zero temperature, as discussed in Bagnato & Kleppner (1991), in a harmonic trap BEC can occur in two dimensions below a critical temperature \(T_0^c\). Below, we estimate this temperature. In a harmonic trap at a temperature \(T\) below a critical temperature \(T_0^c\) \((T < T_0^c)\), the number \(N_0(T, B)\) of non-interacting magnetopolaritons in the condensate is given by (Bagnato & Kleppner 1991)

\[
N_0(T, B) = N - \frac{\Gamma(2)\zeta(2)(g_s^{(e)} g_v^{(e)} + g_s^{(h)} g_v^{(h)}) M_{eff}(B)}{\hbar^2 \gamma_{eff}} (k_B T)^2
\]

\[
= N - \frac{\pi(g_s^{(e)} g_v^{(e)} + g_s^{(h)} g_v^{(h)}) M_{eff}(B)}{3\hbar^2 \gamma} (k_B T)^2,
\]

(6.1)

where \(N\) is the total number of magnetopolaritons, \(g_s^{(e),(h)}\) and \(g_v^{(e),(h)}\) are the spin and graphene valley degeneracies for an electron and a hole, respectively, \(k_B\) is the Boltzmann constant, \(\Gamma(x)\) is the gamma function and \(\zeta(x)\) is the Riemann zeta function.
Applying the condition $N_0 = 0$ to equation (6.1), and assuming that the magnetopolariton effective mass is given by equation (3.6), we obtain BEC critical temperature $T_c^{(0)}$ for the ideal gas of magnetopolaritons in a single GL in a magnetic field:

$$T_c^{(0)}(B) = \frac{1}{k_B} \left( \frac{3\hbar^2 \gamma N}{\pi (g_s^{(e)} g_s^{(h)} + g_s^{(h)} g_s^{(h)} + g_s^{(h)} g_s^{(h)}) M_{\text{eff}}(B)} \right)^{1/2}. \quad (6.2)$$

At temperatures above $T_c^{(0)}$, BEC of magnetopolaritons in a single GL does not occur. In our calculations, we used $g_s^{(e)} = g_s^{(h)} = 1$ and $g_s^{(h)} = g_s^{(h)} = 2$. The functional relations between the spring constant $\gamma$ and the magnetic field $B$ corresponding to different constant values of $T_c^{(0)}/\sqrt{N}$ are presented in figure 6. According to equation (6.2), BEC critical temperature $T_c^{(0)}$ decreases with the magnetic field as $B^{-1/4}$ and increases with the spring constant as $\gamma^{1/2}$. If we take into account the virtual transitions between Landau levels (which have the order $1/\epsilon$), then magnetopolariton–magnetopolariton interaction occurs and results in the superfluidity in the magnetopolariton system.

7. Discussion and conclusions

In conclusion, the theoretical and experimental statuses of BEC and superfluidity of trapped QW polaritons in a microcavity are presented. Besides, we report BEC of trapped magnetoexciton polaritons in GL and QW embedded in an...
optical microcavity in a high magnetic field. In both the cases, the polaritons are considered in a harmonic potential trap. The effective Hamiltonian of QW and GL polaritons in a microcavity in a high magnetic field and the BEC temperature as functions of magnetic field are obtained. It is shown that the effective magnetic mass of a magnetoexciton polariton depends on magnetic field. It is shown that Rabi splitting in a GL can be controlled by the external magnetic field, while in a QW the Rabi splitting does not depend on the magnetic field when it is strong.

At low temperature, the Hamiltonian of two-dimensional exciton polaritons in a slowly varying external parabolic potential acting on the exciton energy and bringing it into resonance with a cavity photon mode corresponds directly to the case of a weakly interacting Bose gas with an effective mass and effective pairwise interaction in a harmonic potential trap. The condensate fraction and the superfluid component are decreasing functions of temperature, as expected, and increasing functions of the curvature of the parabolic potential. The mixing with the photon states leads to a smaller lifetime for high-energy states, that is, an evaporative cooling effect, but does not fundamentally prevent condensation. Since harmonic potential traps are now possible for microcavity polaritons, it should be possible to compare these calculations with experimental results for the critical density and spatial profile of the polariton condensate.

In our calculations, we have assumed that the system under consideration is in thermal equilibrium. This assumption is valid if the relaxation time is less than the quasiparticle lifetime. Although the magnetopolariton lifetime is short, thermal equilibrium can be achieved within the regime of a strong pump. Porras et al. (2002) claimed that the time scale for polariton–exciton scattering can be small enough to satisfy this condition for the existence of a thermalized distribution of polaritons in the lowest \( k \)-states in a QW. We expect a similar characteristic time for magnetopolariton–magnetoexciton scattering in graphene. However, the consideration of pump and decay in a steady state may lead to results which are different from those presented in this paper. The consideration of the influence of decay on BEC may be the subject of further studies of a trapped gas.

Even though the estimated BEC critical temperatures for polariton condensates are relatively high, the current experiments are still performed at cryogenic temperatures. The reason for this is that the transition under consideration is determined by the Rabi splitting (Littlewood 2007). The Rabi splitting is limited by fundamental properties of the material. In the GaAs QWs used in Balili et al. (2009) the Rabi splitting is 13 meV (~150 K) which is about twice as large as in CdTe used in Kasprzak et al. (2006). Let us mention that since in graphene Rabi splitting depends on magnetic field (Berman et al. 2009b), the transition temperature for the occurrence of polariton BEC can be tuned by altering the magnetic field, which does not happen for QW-based polaritons.

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