Stability of car following with human memory effects and automatic headway compensation

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This paper addresses the study of some appropriate control strategies in order to guarantee the exponential stability of a class of deterministic microscopic car-following models including human drivers' memory effects and automated headway controllers. More precisely, the delayed action/decision of human drivers is represented using distributed delays with a gap and the considered automated controller is of proportional derivative type. The analysis is performed in both delay parameter and controller gain parameter spaces, and appropriate algorithms are proposed. Surprisingly, large delays and/or gains improve stability for the corresponding closed-loop schemes. Finally, some illustrative examples as well as various interpretations of the results complete the presentation.

Keywords: traffic; car-following model; memory; distributed delay; discrete delay; spacing control

1. Introduction and problem statement

According to the National Highway Traffic Safety Administration (NHTSA), motor vehicle crashes were the leading cause of death in the USA in 2002 for ages between 3 and 33 (Subramanian 2005). This tableau is one of the many reasons for studying traffic behaviour (Helbing 2001). Interest in the problem has increased in parallel with growing concerns about the undesirable impacts of traffic flow on the environment and energy consumption (Bose & Ioannou 2003). Numerous mathematical models have been developed by using macroscopic and microscopic approaches in order to investigate the dynamics of vehicles and to understand traffic flow (Chandler et al. 1958; Bando et al. 1998; Rothery 1998; Davis 2003; Treiber & Helbing 2003; Treiber et al. 2006). Based on the degree of detail and the physics to be captured, these approaches incorporate various parameters defining the traffic flow, and the behaviours of drivers and vehicles. Among them are considerations of single and multiple lanes, on–off ramps, lane changes, traffic lights and their synchronization, roundabouts, human driving and vehicle dynamics (Helbing 2001; Sipahi & Niculescu 2010).

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One contribution of 10 to a Theme Issue ‘Traffic jams: dynamics and control’.
Deriving appropriate mathematical models of traffic is a major undertaking, and it falls into a very broad field encompassing transportation research, physics, mathematics and engineering. In this paper, we will focus on a particular subclass of problems widely investigated in the literature (e.g. Bando et al. 1998; Stankovic et al. 2000; Helbing 2001; Orosz & Stépán 2004; Orosz et al. 2004, 2005, 2009; Zhou & Peng 2005; Treiber et al. 2006). Our starting point is a type of Pipes model (Pipes 1953) represented by a set of differential equations depicting a deterministic microscopic follow-the-leader type behaviour in which drivers cruise at a constant velocity in a single lane without changing lanes (figure 1). With this model, we wish to reveal the intrinsic features of driving, where humans make delayed decisions and an automated controller works in parallel to compensate the inter-vehicle spacings. Research along these lines has direct connections to developing traffic control strategies by constructing decentralized (Stankovic et al. 2000), adaptive (Swaroop et al. 2001), nonlinear spacing controllers (Yanakiev & Kanellakopoulos 1998; Santhanakrishnan & Rajamani 2003; Zhou & Peng 2005), gain scheduling techniques (Yanakiev & Kanellakopoulos 1998) and collision avoidance among automated heavy-duty vehicles (Cook 2007).

The common objectives in the cited contributions are: (i) to analytically investigate how the headway dynamics (spacing between consecutive vehicles) propagates upstream of the traffic flow (Swaroop & Hedrick 1996; Bose & Ioannou 2003; Seiler et al. 2004); (ii) to propose analysis tools to reveal how headway dynamics behaves under perturbations (Bose & Ioannou 2003; Cook 2007); and finally, (iii) to design appropriate controllers to prevent amplification of such perturbations (Yanakiev & Kanellakopoulos 1998; Stankovic et al. 2000; Swaroop et al. 2001; Santhanakrishnan & Rajamani 2003; Zhou & Peng 2005). These efforts fall within the analysis of the so-called car string stability, which aims to investigate the headway propagation upstream of the traffic flow. For an appropriate definition of such a stability notion and related problems, we also refer to Yanakiev & Kanellakopoulos (1998), Swaroop et al. (2001) and Bose & Ioannou (2003), to cite only a few.

Roughly speaking, the car string stability indicates attenuation of the periodic perturbations (excitations) arising in the acceleration, velocity and position differences between consecutive vehicles, whereas exponential stability refers to the exponential decay of the response of these physical entities in time.
against impulsive perturbations. While the explanations here are kept at a simple level, it is important to note that there exist various complex instability mechanisms in traffic flow as investigated through linear and nonlinear analysis (Bando et al. 1998; Orosz et al. 2005, 2009; Treiber et al. 2006; Helbing & Moussaïd 2009). Despite the simplicity of the mathematical models, assessment of these mechanisms is not a trivial task, as discussed in the cited studies.

In this paper, we study the exponential stability mechanisms of individual vehicles following each other. Our focus is mainly on the effects of delayed reactions of human drivers (Chandler et al. 1958; Bando et al. 1998; Green 2000; Davis 2003; Murray 2003), as well as on the effects of automated headway controllers, on the overall exponential stability of each controlled vehicle, where exponential stability notions follow from Stépán (1989). Although this study alone does not address the instability mechanisms of entire traffic flow dynamics, it attempts to capture new information that can be useful in designing automated vehicle controllers, which properly and collaboratively function with human drivers.

Delay in closed-loop dynamics is a well-known source of poor performance, weak robustness and instability (Bellman & Cooke 1963; Niculescu 2001; Gu et al. 2003). Furthermore, the presence of multiple constant delays leads to unexpected behaviours, as shown by the stability crossing boundaries in the delay parameter space, such as stability rays, delay-ratio sensitivity, delay interference, bounded and unbounded stability regions (Michiels & Niculescu 2007). Capturing these stability phenomena is far from trivial, and one of the main research interests in such cases is to find appropriate algorithms for characterizing them globally. In this context, some interesting ideas proposed in the literature include appropriate frequency-sweeping tests (Chen & Latchman 1995; Gu et al. 2003, 2005; Sipahi & Delice 2009), which are also exploited here to assess the exponential stability with respect to delays, drivers’ measures of aggressiveness and gains of the automated controllers.

In order to obtain tractable results, we will follow the lines of Bando et al. (1998), Konishi et al. (2000), Helbing (2001), Bose & Ioannou (2003), Treiber et al. (2006) and Sipahi & Niculescu (2008), and assume that the delays are time-invariant. The bottleneck in analysing the exponential stability in this paper is then due to our consideration of multiple delays, each one of which is likely to play a different or counter-intuitive role in stability.

(a) Memory effects of human drivers

Modelling delayed reactions of drivers using time delays is a challenge. Some of the existing models consider the selective memory of the drivers, which mathematically corresponds to using constant ‘discrete’ delays representing arrival of information from a ‘particular’ point of time in the past (Bando et al. 1998; Treiber et al. 2006). Such discrete-delay models, however, may not capture the possible ‘memory’ effects that arise from the use of continuously received information distributed over the past. The presence of human drivers naturally suggests that such memory effects should be taken into consideration.

In the present work, motivated by the above reasoning, we consider that the delayed actions/decisions of human drivers are represented by distributed delays. The physical basis of the model is the fact that the drivers perform their decisions
by taking into account what they continuously observe within an appropriate memory window, during which some information is retained and used in the decision-making process. It is known that distributed delays facilitate stability (Atay 2003), and one of the interests in this paper will be to investigate whether distributed delays improve the stability of vehicles driven by humans.

(b) Automated proportional derivative controller strategy

Automated inter-vehicle spacing control strategy is not new and has been widely studied. In Swaroop et al. (2001), Swaroop & Hedrick (1996) and Zhou & Peng (2005), we see that the position and velocity of the vehicles are used in constructing sliding mode controllers; in Cook (2007) and Zhang et al. (1999) the same quantities are combined to develop proportional derivative (PD) type controllers; in Naranjo et al. (2003), PD type controllers are deployed with adaptive fuzzy controllers; and in Seiler et al. (2004) lead controllers that are similar to PD controllers are used for spacing control. In our case study, we believe that the choice of PD controllers makes sense since these controllers offer sufficient speed, reactiveness and prediction, which are crucial in our control problem, where human drivers perform delayed and relatively slow reactions.

(c) Specific objectives and approach

This work is devoted to analysing the stability of a vehicle in traffic in the case when the driver has reaction delays and the vehicle is equipped with an automated controller that assists the driver in controlling the headway between the vehicle and the preceding vehicle. The idea that the driver delay involves distributed functions for describing a driver’s short-term memory has already been discussed in Sipahi et al. (2007), and the novelty here is that we explore how an automated PD controller in parallel with the driver works to maintain a certain headway between consecutive vehicles (see §2 for further discussions). It is assumed that the automated controller acts much faster than the drivers, i.e. the controller does not have delays in its construction. Additional scenarios including consideration of delays in the automated controller or in the engine power control (Yildiz et al. 2010) are left for future work.

Among the problems of interest to be considered are to understand how the added automated controller would improve the overall behaviour of the human–vehicle system. Furthermore, it is of interest to see whether a stability improvement can be rendered in the collaboration strategy between the human and the automated controller. Particular attention will be paid to the dynamic behaviour in the presence of large delay values. More precisely, we address two independent problems: (i) find the way the controller gains affect the closed-loop stability in the presence of known (average) driver memory effects, and (ii) characterize the way the memory window changes the stability properties of the controlled vehicle. To the best of our knowledge, a complete analytical study in this context has not been pursued in the literature, and we form our main objective along this line (§3). Illustrative examples in §4 complete the work.

Notation. The notation is standard. Sets of real and positive real numbers are denoted by \( \mathbb{R} \) and \( \mathbb{R}_+ \), respectively, and \( \mathbb{C} \), \( \mathbb{C}_- \) (\( \mathbb{C}_+ \)) represent the entire complex plane and left (right) half of the complex plane, respectively. The imaginary axis is denoted by \( j\mathbb{R} \), where \( j = \sqrt{-1} \), and we use \( s \) for the Laplace variable.

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2. Mathematical modelling

Some parts of the mathematical modelling and the pertaining discussions in this section are borrowed from Bando et al. (1998) and Bose & Ioannou (2003). Mathematical models that consider cars following each other can be expressed as nonlinear systems of the form

$$\ddot{x}_k = \Gamma_k([x_{k+1}(t) - x_k(t)] - [\dot{x}_{k+1}(t) - \dot{x}_k(t)], \dot{x}_k(t)), \quad (2.1)$$

where $x_k(t)$ is the position of the $k$th vehicle, $\Gamma_k$ is a nonlinear function that is continuous with respect to its arguments, and $\tau_{k,1}$ and $\tau_{k,2}$ are the positive driver reaction delays. The terms with delays on the right-hand side of model (2.1) emanate from the human driver, while the non-delayed term $\dot{x}_k(t)$ appears due to air-drag forces.

Simplified forms of model (2.1) have been studied extensively in the literature (e.g. Bando et al. 1998; Bose & Ioannou 2003; Helbing 2001; and references therein). Most of these works start by linearizing model (2.1) around an equilibrium configuration of the traffic flow. Assuming that the objective of each vehicle is to follow the vehicle ahead of it, this equilibrium is defined when all the vehicles are travelling at a constant velocity $V$. The linearization of model (2.1) at $\dot{x}_k(t) = V + \dot{y}_k(t)$ can be carried out to analyse the dynamics of the small perturbations, $y(t)$ (Pipes 1953; Chandler et al. 1958; Rothery 1998; Helbing 2001; Bose & Ioannou 2003; Sipahi et al. 2007). Most such dynamics are simplified forms of model (2.1) describing the acceleration perturbations as a function of velocity and position perturbations (see the cited references). We can represent such dynamics with the linear delay differential equation

$$\ddot{y}_k(t) = \alpha_k H_k(t - \tau_{k,1}) + \beta_k \dot{H}_k(t - \tau_{k,2}) - b_k \dot{y}_k(t), \quad (2.2)$$

where $H_k(t) = y_{k+1}(t) - y_k(t)$ is the headway perturbation between vehicles $k$ and $(k+1)$, the terms with $y_n(t)$ are the non-homogeneous parts of equation (2.2) and thus $k = 1, \ldots, n - 1$, with $n$ being the number of vehicles. Furthermore, the last term on the right-hand side of equation (2.2) represents the force perturbations due to drag forces, with $b_k$ being the drag term at equilibrium, and the gains $\alpha_k > 0$ and $\beta_k > 0$ can be seen as a measure of the driver’s aggressiveness per unit vehicle mass. Equation (2.2) formulates the driver $k$ attempting to vanish the perturbed velocity and position errors by penalizing these errors using the gains $\alpha_k$ and $\beta_k$ with the control objective of capturing the constant velocity $V$ (Sipahi et al. 2007; Sipahi & Niculescu 2008, 2010).

We note that the dynamics in equation (2.2) can also be extended to multiple vehicle-following strategies (Sipahi & Niculescu 2010). In view of the complexity of the problem and the many possibilities that can be considered, we shall restrict ourselves to a simplified version of equation (2.2) with $\beta_k = 0$. With the framework developed in §3, it will be apparent that consideration of $\beta_k \neq 0$ can be done in a straightforward manner. Other aspects not considered in this paper include constant time–headway driving strategies, where drivers allow larger headway at higher cruising speeds (Bando et al. 1998; Bose & Ioannou 2003; Orosz & Stépán 2004; Orosz et al. 2004; Sipahi et al. 2009). While the simplicity and the reliability of the mathematical model we consider here under these conditions...
is appealing, combining the memory effects and an automated controller with equation (2.2) requires us to develop new mathematical tools for the exponential stability analysis of the controlled vehicle.

Let us now discuss the modelling of memory effects and how an automated controller is considered in our control problem. As mentioned earlier, the memory effects of human drivers can be modelled using particular distribution functions. One of the simplest distribution functions is the uniform distribution (figure 2), which is a good fit for modelling the short-term memory of drivers,

\[ f_k(\tau) = \begin{cases} 
1/\delta_k, & h_k \leq \tau \leq h_k + \delta_k, \\
0, & \text{otherwise},
\end{cases} \tag{2.3} \]

where \( h_k \) is the memory dead-time, \( \delta_k \) is the memory window and the average of the memory window is denoted by

\[ \tau_{\text{avg},k} = h_k + \frac{\delta_k}{2}. \tag{2.4} \]

The delay \( h_k \) is also known as the ‘gap’, which is the length of time after which the memory with size \( \delta_k \) becomes effective (Sipahi et al. 2007). We wish to state that other forms of distributions, such as \( \gamma \)-distributions, can also be adapted to the stability analysis. However, at this point, there is no clear evidence as to which distribution better represents the memory effects. As a starting point, we believe that a distribution of the form (2.3) conveniently represents either the short-term memory effects or averaging of some distributions around \( 1/\delta_k \).

In light of the above discussions, we express the acceleration perturbations in the \( k \)th vehicle following the \((k + 1)\)th vehicle as

\[ \ddot{y}_k(t) = \alpha_k \int_0^\infty f_k(\tau)H_k(t - \tau) \, d\tau + K_{p,k} H_k(t) + K_{d,k} \dot{H}_k(t) - b_k \dot{y}_k(t), \tag{2.5} \]

where \( K_{p,k} \) and \( K_{d,k} \) are constant values denoting the proportional gain and derivative gain of the automated controller per unit mass of the vehicle, respectively. The driver’s decision is defined by the integral term in equation (2.5), which accounts for the total weighted headway perturbation errors using the kernel \( f_k(\tau) \) describing the human memory.
3. Stability analysis

In this section, we develop tools to analyse the asymptotic stability of the controlled vehicle dynamics, leaving string stability analysis to a future study. We note that, for the linear control problem considered here, exponential stability and asymptotic stability concepts are equivalent (Michiels & Niculescu 2007). We start by identifying the characteristic equation first. Next, a frequency-sweeping method is adapted to capture the stability properties in both the controller gain and delay parameter spaces.

(a) The characteristic equation

The traffic flow scenario considered here is a linear spatial configuration of the vehicles (figure 1). Therefore, we can focus on the asymptotic stability of each one of the vehicles separately. To keep the notation simpler, in the remainder of the text and without any loss of generality, we set \( k = 1 \) and denote \( f_k(t) = f(t), d_k = d, K_{p,1} = K_p, K_{d,1} = K_d, \tau_{avg,k} = \tau_{avg}, b_k = b, h_k = h \) and \( \alpha_1 = \alpha \) in equation (2.5). The transfer function between two consecutive vehicles can then be expressed by taking the Laplace transform of equation (2.5),

\[
G(s) = \frac{Y_1(s)}{Y_2(s)} = \frac{K_p + K_ds + \alpha F(s)}{s^2 + (K_d + b)s + K_p + \alpha F(s)},
\]

where \( Y_1(s) \) and \( Y_2(s) \) are the Laplace transforms of the perturbations \( y_1(t) \) and \( y_2(t) \), respectively (see also figure 3), and \( F(s) \) is the Laplace transform of the kernel \( f(t) \) given by

\[
F(s) = e^{-sh} \frac{1 - e^{-s\delta}}{s\delta}.
\]

The characteristic equation of the system (2.5) is the denominator of \( G(s) \),

\[
\Phi = s^2 + (K_d + b)s + K_p + \alpha e^{-sh} \frac{1 - e^{-s\delta}}{s\delta} = 0,
\]

from which the stability properties of the input–output system in equation (3.1) can be studied. Notice that, as \( \delta \to 0^+ \), the uniform distribution approaches a Dirac delta function, i.e. \( F(s) = 1 \), and equation (3.3) only carries the patterns of \( h \), that is, the ‘discrete’ delay-type behaviour (selective memory). In such cases, the stability analysis follows from earlier work (Michiels & Niculescu 2007). In the case when \( \delta \neq 0 \), however, there are additional problems. First of all, the characteristic equation includes two independent delay parameters \( h \) and \( \delta \). Secondly, the delay \( \delta \) appears also as a coefficient of the characteristic equation, which is non-standard.

There exist several studies addressing the stability in the delay parameter space using frequency domain approaches (e.g. Gu et al. (2005), Sipahi & Delice (2009) and Stépán (1989); see also Michiels & Niculescu (2007) for an overview). However, the existing work falls outside the coverage of equation (3.3). Some studies focusing on quasi-polynomials similar to what we observe in equation (3.3) can be found in the works of Bélair & Mackey (1989) and Beretta & Kuang (2002); however, these works do not exactly consider the structure in equation (3.3). The recent work by Sipahi et al. (2007) among others considers the analysis of the
stability of a dynamical system with two independent delays, but that study was
developed for a particular form of characteristic equation, which does not match
with equation (3.3).

In this paper, the stability analysis of equation (3.3) is addressed via
two approaches:

— assuming that the delays as well as the driver aggressiveness measures are
given, we will reveal the stability features of the system defined by equation
(3.1) in the corresponding controller gain space \((K_p, K_d)\), and

— assuming that the controller gains \(K_d\) and \(K_p\) as well as the driver
aggressiveness measures are given, we will reveal the stability features of
the system defined by equation (3.1) in the corresponding delay parameter
space \((h, \delta)\).

It is worth mentioning that the two approaches lead to complementary insights
into understanding the dynamics of the original closed-loop system as a function
of the parameter changes (see also Datko (1978), Michiels & Niculescu (2007)
and Stépán (1989) for further comments).

The stability of equation (3.3) for a given set of parameters holds if and only if
the characteristic roots of equation (3.3) all lie in \(\mathbb{C}_-\). The continuity of these roots
with respect to the parameters can be shown to hold (Datko 1978). We suppress
these technicalities here in order not to disrupt the flow of the discussions. With
the continuity argument at hand, when the delays \((h, \delta)\) or the gains \((K_p, K_d)\) are
varied, a loss or acquisition of exponential stability of the closed-loop system is
associated with the characteristic roots located on the imaginary axis \(j\mathbb{R}\), that
is, with the existence of at least one critical characteristic root located on the
imaginary axis. The set of parameters for which such critical roots exist will
divide the parameter space into several regions, each region being characterized
by the same number of unstable characteristic roots (see Michiels & Niculescu
(2007) and Stépán (1989) on the so-called \(D\)- or \(\tau\)-decomposition methods). In
the case of a two-dimensional parameter space, the boundaries of such regions

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Figure 3. Block diagram of the controller vehicle dynamics represented by the delay
integro-differential equation (2.5).
are called *stability crossing curves*, as suggested by Stépán (1989) and Gu *et al.* (2005). The points on these curves are the parameters (either delay or controller gain parameters) that engender \( s = j\omega \) solutions in the characteristic equation (3.3), where \( \omega \geq 0 \) without loss of generality. In other words, in order to capture the aforementioned stability crossing curves, it is necessary to solve

\[
-\omega^2 + j\omega(K_d + b) + K_p + \alpha e^{-j\omega h} \frac{1 - e^{-j\omega\delta}}{j\omega\delta} = 0,
\]

with respect to the parameter space of interest.

Finally, with the help of these curves and developing sensitivity analysis, we can determine the parameter space that renders equation (3.3) asymptotically stable. The analysis requires explicit computation of the crossing directions on the stability crossing curves (Gu *et al.* 2005; Sipahi & Olgac 2005), which may be towards either \( \mathbb{C}_+ \) or \( \mathbb{C}_- \), for the critical roots located on the imaginary axis and subject to variations with respect to the corresponding set of parameters. It is worth mentioning that the particular structure of the problem we are dealing with suggests treating the gain parameter and the delay parameter spaces independently, and to compute the stability crossing curves in each case.

**(b) Extraction of stability switching curves in gain space**

A parameter-sweeping approach is adapted to compute the regions in the gain space \((K_p, K_d)\), where the dynamics is characterized as stable or unstable. For this, the boundaries separating these regions need to be captured. The effort to achieve this requires explicit computations, as we demonstrate next. Assume that the pair \((h, \delta)\) is given, and \( \omega \) is a sweep parameter. Collect only \( K_p \) and \( K_d + b \) terms in equation (3.4) on one side of the equation, and let the other side of the equation be denoted by the complex function \( L = L(h, \delta, \omega) \), which can be computed for each sweep parameter \( \omega \). With this manipulation, it is easy to express the controller gains as a function of \( \omega \),

\[
K_p = \Re(L(h, \delta, \omega)) \tag{3.5}
\]

and

\[
K_d = \frac{\Im(L(h, \delta, \omega))}{\omega} - b. \tag{3.6}
\]

**Corollary 3.1 (\( \omega \neq 0 \) case).** The parameter \( K_d + b \) found from equation (3.6) is upper bounded by \( \alpha \tau_{\text{avg}} \) for all \( h, \delta \) and \( \omega > 0 \), where \( \tau_{\text{avg}} = \tau_{\text{avg}, k} \) is given in equation (2.4).

**Proof.** With some algebraic manipulations, one can show from equation (3.6) that

\[
\Im(L(h, \delta, \omega)) = \frac{\alpha}{\omega \delta} \{ \cos(\omega h)[1 - \cos(\omega \delta)] + \sin(\omega \delta) \sin(\omega h) \}. \tag{3.7}
\]

We then have

\[
K_d + b = \frac{2\alpha}{\omega^2 \delta} \sin(\omega \tau_{\text{avg}}) \sin \left( \frac{\omega \delta}{2} \right), \tag{3.8}
\]

which is upper bounded by \( \alpha \tau_{\text{avg}} \) as \( \omega \to 0^+ \).
The proof of corollary 3.1 provides information about the boundaries that separate the stability regions in the controller gain parameter space. Except when \(\omega = 0\), we know that the points on the stability boundaries cannot exceed the upper bound \(\alpha \tau_{\text{avg}}\) in \(K_d + b\). A similar analysis can also be done on the controller gain \(K_p\), which is found as

\[
K_p = \Im(A(h, \delta, \omega)) = \omega^2 - \frac{2\alpha}{\omega \delta} \cos(\omega \tau_{\text{avg}}) \sin\left(\frac{\omega \delta}{2}\right). \tag{3.9}
\]

Notice that \(K_p\) does not have an upper bound, as it goes to infinity when \(\omega \to \infty\). In what follows, we investigate the particular case when \(\omega \to 0^+\). Once this is established, the results are combined to construct a stability theorem.

**Corollary 3.2 \((\omega \to 0^+)\) case.** For a given \((h, \delta) \in \mathbb{R}_+^2\), the system defined by (3.3) has an invariant root at \(s = 0\) for \(K_p = -\alpha\) and has two roots at \(s = 0\) for \(K_p = -\alpha\) and \(K_d + b = \alpha \tau_{\text{avg}}\).

**Proof.** When \(K_p \to -\alpha\), it is easy to see from equation (3.3) that \(s \to 0\) is always a solution, making \(s = 0\) an invariant root. From the same equation, one can verify that the limit

\[
\left.\frac{d\Phi}{ds}\right|_{s \to 0^+} = 0
\]

holds only when \(K_p = -\alpha\) and \(K_d + b = \alpha \tau_{\text{avg}} - b\).

**Corollary 3.3 (Sensitivity of \(\omega \to 0^+)\).** The sensitivity of the invariant root \(s = 0\) with respect to \(K_p\) is instability (or stability) favouring when \(K_d + b < \alpha \tau_{\text{avg}}\) (or \(K_d + b > \alpha \tau_{\text{avg}}\)).

**Proof.** By using the implicit function theorem on \(\Phi\) in a small neighbourhood around the origin of the complex plane, it follows that

\[
\left.\frac{ds}{dK_p}\right|_{s \to 0^+} = -\frac{\partial \Phi/\partial K_p}{\partial \Phi/\partial s}\left|_{s \to 0^+}\right. = \frac{1}{\alpha \tau_{\text{avg}} - (K_d + b)},
\]

from which we conclude that the root at \(s = 0\) tends to move either to \(\mathbb{C}_+\) when \(K_d + b < \alpha \tau_{\text{avg}}\) or to \(\mathbb{C}_-\) when \(K_d + b > \alpha \tau_{\text{avg}}\).

A rule of thumb arising from the above corollary is that the root at \(s = 0\) is stability favouring when the coefficient \(K_d + b\) is larger than the multiplication of the driver aggressiveness \(\alpha\) and the average delay \(\tau_{\text{avg}} = h + \delta/2\). Note that this result does not conclude on stability, which we prove next.

**Theorem 3.4 (Stability theorem).** Given \(K_d + b = \alpha \tau_{\text{avg}}\), the controlled vehicle dynamics is exponentially stable as \(K_p \to -\alpha + \epsilon, \epsilon > 0, |\epsilon| \ll 1\).

**Proof.** Owing to the boundedness of the last term in equation (3.4) with respect to \(\omega\), one can see that, for sufficiently large \(K_p^* > 0\) and \(K_d^* + b^* > 0\) in equation (3.4), it is possible to enforce equation (3.3) not to have \(\omega \geq 0\) solutions, no matter what \(\delta\) and \(h\) are. This indicates that the controlled vehicle dynamics is stable for large gains independent of the delays. Furthermore, the point \(\Lambda^* = (K_p^*, K_d^* + b^*)\) can be chosen such that \(K_p^*>-\alpha\) and \(K_d^* + b^* > \alpha \tau_{\text{avg}}\). Since \(\alpha \tau_{\text{avg}}\) is the upper bound of \(K_d + b\) for all \(\omega > 0\) and since \((K_d + b) \to \infty\) can occur only when \(K_p \to -\alpha\), it is easy to see that there exists a continuous path that connects...
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stability (necessary and sufficient conditions)

0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0

$K_d + b$

$K_p$

Figure 4. The necessary and sufficient stability conditions determine the shaded region for the controlled vehicle dynamics. Representative numerical values are taken as $h=0.1$, $\delta=0.2$ and $\alpha=2$.

point $A^*$ to point $B^* = (-\alpha, \alpha \tau_{\text{avg}})$ in the $(K_p, K_d + b)$ parameter space without piercing through any stability crossing curves. From the continuity argument with respect to the parameters $K_p$ and $K_d + b$ (see §3a), we conclude that the point $C^* = B^* + (\epsilon, 0)$, $\epsilon > 0$, $|\epsilon| \ll 1$, also belongs to the stability region.

We now summarize the obtained results in figure 4. In this figure, the shaded region corresponds to the necessary and sufficient stability conditions of the controlled vehicle dynamics for representative values of $h$, $\delta$ and $\alpha$.

(c) Sufficient stability conditions in controller gain parameter space

Recall that we consider only the cases with $\alpha > 0$. Furthermore, the delay-independent stability concept studied below is defined in the sense of Michiels & Niculescu (2007), i.e. the delay system maintains its exponential stability for all finite delays. We then have the following properties.

Lemma 3.5 (Delay-independent stability, sufficient condition, $h \neq 0$, $\delta \neq 0$). The input–output system defined in equation (3.1) is stable independent of delays if the following conditions hold simultaneously:

(i) $K_d + b > 0$, $K_p + \alpha > 0$,
(ii) $K_p^2 - \alpha^2 \geq 0$,
(iii) $[4K_p - (K_d + b)^2](K_d + b)^2 > 4\alpha^2$ whenever $(K_d + b)^2 - 2K_p < 0$.

Proof. Conditions in (i) guarantee the stability of the delay-free system ($\tau = 0$). For $\tau \neq 0$, if this system does not exhibit any $s = j\omega$ solutions, it is guaranteed
that the system maintains its stability. Note that the last term in equation (3.4) is bounded

$$\frac{2a^2[1 - \cos(\delta\omega)]}{(\delta\omega)^2} \leq a^2, \quad \forall (\omega, \delta) \in \mathbb{R}.$$ 

It is therefore sufficient that \((K_p - \omega^2)^2 + \omega^2(K_d + b)^2 - a^2 = 0\) for all \(\omega \geq 0\) such that there does not exist any \(\omega\) satisfying equation (3.4). This condition can be analysed in two parts: when \(\omega = 0\) and when \(\omega \neq 0\). It is easy to show that \(\omega = 0\) is not a solution of equation (2.5) for finite delays, yielding condition (ii) of the lemma; and for all \(\omega \geq 0\), we have either condition (ii) or (iii) in view of the plot of \((K_p - \omega^2)^2 + \omega^2(K_d + b)^2 - a^2 = 0\) in figure 5.

Let us now depict the sufficient delay-independent stability region in the parameter space \((K_p, K_d + b)\). For this, the inequalities listed in lemma 3.5 are plotted and the stability region is identified as the shaded grey region shown in figure 6, where the boundary of condition (iii) is tangent to the boundary of \(K_p = \alpha\) at \((K_p, K_d + b) = (\alpha, \sqrt{2a\alpha})\). With this key observation, the conditions of the lemma can be seen to be equivalent to the following three conditions: (i) \(K_d + b > 0\), (ii) \(K_p \geq \alpha > 0\), (iii) \([4K_p - (K_d + b)^2](K_d + b)^2 > 4\alpha^2\), whenever \((K_d + b) < \sqrt{2a\alpha}\), all of which can be further simplified to obtain practical control design rules as stated in the following corollary.

**Corollary 3.6.** The input–output system defined in equation (3.1) is stable independent of delays if the following conditions hold simultaneously:

(i) \(K_p \geq \alpha > 0\),

(ii) \(K_d + b > \sqrt{2K_p}\).

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Figure 5. The parameter \(\omega^2\) is strictly non-negative. In this case, the function \((K_p - \omega^2)^2 + \omega^2(K_d + b)^2 - a^2 = 0\) can exhibit its minimum at either point \(Q_1\), \(Q_2\) or \(Q_3\) depending on whether \((K_d + b)^2 - 2K_p\) is positive, zero or negative. The minima at \(Q_1\) and \(Q_2\) to be greater than \(a^2\) lead to condition (ii), while the minimum at \(Q_3\) to be greater than \(a^2\) gives rise to condition (iii) of lemma 3.5.
Stability of car following

\[ K_p = \text{stability (sufficient conditions)} \]

4.0 3.5 3.0 2.5 2.0 1.5 1.0 0.5 0 0.2 0 0 0 4 0 0 8 0 100

\( K_d + b = \sqrt{2K_p} \)

Figure 6. The sufficient delay-independent stability conditions from lemma 3.5 depict the grey-shaded region. The dashed lines and the thick curve correspond to the boundaries defined by the inequalities in conditions (i), (ii) and (iii) of the lemma. The curve labelled \( K_d + b = \sqrt{2K_p} \) is shown to visualize the boundary of condition (ii) of corollary 3.6. The remaining thin curves represent the necessary and sufficient conditions shown in figure 4 given here for comparison purposes. All the curves correspond to the \( \alpha = 2 \) case.

When \( \delta \to 0^+ \), the closed-loop system reduces to a single delay problem, and lemma 3.5 leads to an explicit necessary and sufficient delay-independent stability condition. For \( \delta \neq 0 \), however, lemma 3.5 captures only the sufficient conditions of the asymptotic stability of the system in equation (3.1). While an analytical solution with both necessary and sufficient conditions does not exist due to the difficulties in managing the trigonometric functions, the frequency-sweeping method in §3b can be used to obtain these conditions numerically. A comparison is given in figure 6, where the necessary and sufficient conditions from figure 4 are also displayed. As expected, the boundaries defining the shaded sufficient delay-independent stability conditions do not intersect with the curves related to both necessary and sufficient conditions, and the stability region associated with sufficient conditions is smaller than that with necessary and sufficient conditions.

We conclude this subsection with the following corollary, which is immediate in the absence of the controller.

**Corollary 3.7.** The input–output system defined in equation (3.1) cannot be stable independent of delays if the controller gains \( K_p \) and \( K_d \) vanish.

**Proof.** The proof follows from the fact that condition (iii) of lemma 3.5 cannot be satisfied for vanishing controller gains.

\( (d) \) Extraction of stability switching curves in delay space

In this subsection, we will develop frequency-sweeping tools to reveal the stability crossing curves in the delay parameter space \((h, \delta)\), given \( K_p \) and \( K_d + b \). While the stability analysis in the controller gain parameter space
requires the solution of explicit functions (see §3c), the stability analysis in delay parameter space needs further effort since the equations arising are in implicit forms.

To address the solutions of $h$ and $\delta$ from the characteristic equation, we start by developing the magnitude conditions in equation (3.4). With some algebraic manipulations, and noting that $|e^{-j\omega h}| = 1$ and $|1 - e^{-j\omega \delta}|^2 = 2[1 - \cos(\delta \omega)]$, we arrive at the following expression:

$$\left(K_p - \omega^2\right)^2 + \omega^2(K_d + b)^2 = \frac{2\alpha^2[1 - \cos(\delta \omega)]}{(\omega \delta)^2}. \quad (3.10)$$

The right-hand side of equation (3.10) is a numerically known entity for a given $\theta = \delta \omega$. This observation leads us to sweep $\theta \in \mathbb{R}_+$ and solve for $\omega$ as a function of the remaining parameters. Owing to the specific form of equation (3.10), the four $\omega$ roots can be analytically solved. Among these roots, those that are positive real, $\omega \in \mathbb{R}_+$, are selected. One can then compute $\delta \in \mathbb{R}_+$ via the relation $\delta = \theta / \omega$. The solution technique can also be displayed using geometry (figure 7), where $\varphi_1(\omega) = (K_p - \omega^2)^2 + \omega^2(K_d + b)^2$ and $\varphi_2(\omega, \delta) = 2\alpha^2[1 - \cos(\delta \omega)]/((\omega \delta)^2$, and we are looking for the common solutions, that is, the intersection points (such as A and B) between the curves $\varphi_1$ and $\varphi_2$.

Once $\delta$ is found, the detection of $h$ follows from the phase conditions obtained from equation (3.4) using equations (3.8) and (3.9):

$$h = -\frac{\delta}{2} + \frac{1}{\omega} \left[ \arctan \left( \frac{\omega(K_d + b)}{\omega^2 - K_p^2} \right) \pm 2\pi \ell \right], \quad \ell = 0, 1, \ldots. \quad (3.11)$$
Notice from the formulation of $h$ in equation (3.11) that there exist infinitely many solutions of $h$ for a given $(\delta, \omega)$ pair. These solutions are periodic with periodicity $2\pi/\omega$. This property can also be seen directly from equation (3.4) by inspecting the exponential function carrying the $h$ term. A similar property for $\delta$, however, does not exist, since $\delta$ appears as a coefficient in equation (3.4) as well.

Combining the results from above, we can present a practical sweeping procedure as follows. Sweep $\theta > 0$ in a range with sufficiently large upper bound with sufficiently small steps, and perform the following for each $\theta$ value starting from $\theta = 0$.

Step 1. Increase $\theta$ by a small step size.

Step 2. Compute the right-hand side of equation (3.10) and solve $\omega$ from this equation. For each $\omega \in \mathbb{R}_+$ that exists, perform steps 3 and 4, otherwise go to step 1.

Step 3. Find $\delta \in \mathbb{R}_+$ via the formula $\delta = \theta/\omega$.

Step 4. Using $\delta$ and $\omega$, find $h \geq 0$ from (3.11) for $\ell = 0, 1, \ldots, \bar{\ell}$, where $\bar{\ell}$ is sufficiently large to capture all the stability crossing curves in a predetermined range of $(h, \delta)$ (see also the examples in §4).

Step 5. Go back to step 1.

(e) Identification of stability regions

Identification of stability regions in the controller gain parameter space is as follows. The identification starts with the knowledge in the stability theorem (theorem 3.4), which indicates that the point $K_d + b = \alpha \tau_{\text{avg}}$ and $K_p \rightarrow -\alpha + \epsilon$, $\epsilon > 0$, $|\epsilon| \ll 1$, is a stability point. All the regions $S_1$ that are connected to this region will also lead to stability in the control problem. At this stage, the sensitivity analysis performed earlier for the $\omega = 0$ root can be carried out to identify another region $S_2$ in $(K_p, K_d + b)$ in which any control gain combination leads to one unstable root. For $\omega \neq 0$, there exists another boundary defined by equations (3.8) and (3.9). This boundary separates the two regions $S_1$ and $S_3$, where in region $S_3$ there exist two unstable roots in $\mathbb{C}_+$. Depending on the delay values, there may appear other regions in the controller gain parameter space in which a larger number of unstable roots exist.

Identification of stability regions in the delay parameter space starts with determining the stability of the delay-free control problem, that is, when $\delta = h = 0$. When delays vanish, the stability of the control problem requires us to inspect the two roots of the quadratic polynomial $s^2 + (K_d + b)s + K_p + \alpha = 0$ found from equation (3.3). These roots have negative real parts if and only if $K_d + b > 0$ and $K_p + \alpha > 0$. Obviously, it makes sense to choose $K_p$ and $K_d$ such that these inequalities are satisfied, and the control problem without delays is asymptotically stable. Because of the continuity properties discussed in the previous section, the stability regions will form as the regions enclosed by the $\delta = 0$ axis, the $h = 0$ axis and the stability crossing curves that can be computed by the procedure developed in §3d. As per the delay decomposition theorem and the continuity arguments, any region forming around the origin of the delay space will carry the same stability properties that the origin carries. With this starting point, the sensitivity of the $\mp j\omega$ characteristic roots corresponding to the stability

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crossing boundaries can be computed in order to determine whether these roots move to $\mathbb{C}_+$ or to $\mathbb{C}_-$. In this way, the number of roots moving to $\mathbb{C}_+$ or to $\mathbb{C}_-$ can be counted in each region in the delay parameter space, and the control system can be declared to be exponentially stable in regions that do not render any roots in $\mathbb{C}_+$ (Stépán 1989).

4. Case studies

In this section, we present three case studies that consider different conditions. Of particular interest is to analyse stability in both delay and controller gain parameter spaces. With these analyses, we will reveal how stability is affected by the change of controller gains and damping in the controlled vehicle dynamics, and how different driver aggressiveness coefficients are related to controller gains.

(a) Stability in gain parameter space

The first investigation is performed in the gain parameter space $(K_p, K_d + b)$. We choose $\alpha = 2$, $h = 0.1$ and $\delta = 0.2$. Following the procedure developed in §3, we find the stability boundary and the stability region as shown in figure 8a, where the vertical boundary represents the combination of gains that give rise to characteristic roots residing at $s = 0$, the specific point $B^*$ represents a particular gain combination that renders double roots at $s = 0$, and the remaining boundary corresponds to gain pairs leading to $s = j\omega$, $\omega > 0$, roots in the characteristic equation. Figure 8a can also be seen as the zoomed-out version of figure 4.

Using the root-sensitivity expressions, the number of unstable roots can be identified in figure 8a. The stable region, that is, the region that leads to zero unstable roots, is shaded and marked. We clearly visualize in this figure the analytical results we derived in the previous section. Particularly, we observe that sufficiently large $K_p$ and $K_d + b$ values make the vehicle dynamics stable, and these large gains can be connected to point $B^*$ with a continuous path without intersecting the stability crossing curves. In figure 8a, it is also noticeable that relatively larger $K_p$ gains are needed to stabilize the flow dynamics compared with the magnitudes of the damping terms $K_d + b$.

Moreover, a simple stability criterion can be formulated from corollary 3.6. Let $K_p = 2.5 > \alpha = 2$, then $K_d + b > \sqrt{5}$ guarantees delay-independent stability of the closed-loop system.

We next study how driver aggressiveness $\alpha$ affects the stability regions in the controller gain parameter space. For this objective, we vary $\alpha$ and find the stability boundaries for each $\alpha$ value chosen. The results are presented in figure 8b, where the stable region is not shaded to avoid confusion. In this figure, we see that increase of driver aggressiveness $\alpha$ shrinks some parts of the stability regions while it enlarges some other parts of the stability regions in the controller gain parameter space.

(b) Stability in delay parameter space

In this case study, the interest is on the asymptotic stability region in the space of delays $h$ and $\delta$, while keeping the controller gains fixed. The gains are chosen as $K_p = 1$ and $K_d + b = 0.55$, and driver aggressiveness is taken as $\alpha = 2$. 

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Figure 8. (a) The stability region (shaded) in the controller gain parameter space with $\alpha = 2$, $h = 0.1$ and $\delta = 0.2$. (b) Effect of $\alpha$ on stability regions. Given $h = 0.1$ and $\delta = 0.2$, the stability region changes as a function of driver aggressiveness coefficient $\alpha$, where $\alpha = 1, 1.5$ and 2.

Figure 9. (a) There are two separate stability regions (shaded) of the controlled vehicle dynamics in the delay parameter space. The numerical values are $\alpha = 2$, $K_p = 1$ and $K_d + b = 0.55$. (b) Effect of $K_p$ on stability regions. Increasing $K_p$ controller gain enlarges the stability regions. The remaining parameters are taken as $\alpha = 2$ and $K_d + b = 0.55$.

The computation of stability crossing curves follows the algorithmic construct presented in the previous section, leading to figure 9a. In order to keep the presentation concise, only the points that separate stability from instability are displayed.

Interestingly, there are two separate stability regions (shaded) in figure 9a. The first region is attached to the origin of the delay space. The zoomed inset shows this stability region in the same figure. The second stability region is the one detached from the origin of the delay space and it is much larger when compared with the first one. Moreover, there is an interesting phenomenon we observe here; we see that larger dead-time $h$ can be compensated for by increasing the memory window appropriately. In other words, detrimental effects that are brought by a large discrete delay $h$ can be eliminated by the uniform distributed delay with a large memory size $\delta$. The numerical example demonstrates a trade-off between $h$ and $\delta$ in this second stability region.
Figure 10. (a) The stability map of the controlled vehicle dynamics in the delay parameter space. The stability regions are shaded. The parameters are taken as $\alpha = 2$, $K_d + b = 0.55$ and $K_p = 3$. (b) The effects of different damping terms on stability are shown. For the particular choices of parameters, increased damping terms enlarge the stability regions. The parameters are taken as $\alpha = 2$ and $K_p = 3$, with different damping effects represented by $K_d + b = 0.25$, 0.55, 0.75 and 1.

(c) Effects of $K_p$ and $K_d$ in delay parameter space

We next investigate how the stability regions in figure 9a are affected when the proportional controller $K_p$ is varied. We choose several $K_p$ values and generate the stability boundaries (figure 9b). In order to avoid any confusion, stability regions are not shaded in this figure, but the stability regions follow the same pattern as those in figure 9a. In figure 9b, we see that increasing $K_p$ enlarges the stability regions that are not connected to the origin. The stability region around the origin is also enhanced; however, the effects of different $K_p$ are negligible in this region, where the stability boundary varies less than 2 per cent along the $\delta$ axis.

We note that the geometry of the stability boundaries is not limited to those shown in figure 9a,b. When we increase $K_p$ to $K_p = 3$ without changing the remaining parameters, the stability regions are found as in figure 10a. Increasing $K_p$ in this case bends the boundaries in figure 9b towards the negative $\delta$ axis. This ultimately enlarges the stability regions significantly, allowing the delay-independent stability region found for $\delta > 2.5$.

We finally investigate the effects of damping by choosing several values for $K_d + b$. In figure 10b, these effects are shown, where we suppress the shading of the stability regions as they are similar to those in figure 10a. We observe that increased damping enlarges the stability regions, offering more choices for stability. This can be seen as a way of increasing the robustness of the system against possible uncertainties in parameters $h$ and $\delta$ describing the memory window.

5. Conclusions

Asymptotic stability of a deterministic linear time-invariant car-following model is studied with the effects of both human drivers’ memories and automated spacing controllers. The memory effects are modelled with uniform distributions with discrete dead-time, while the spacing controller is a PD controller free
of delays. Under these circumstances, we investigate the asymptotic stability of the controlled vehicle, where vehicles follow each other without changing lanes. The problem is investigated and the results are presented under two categories: one in the controller parameter space associated with the automated spacing controller and the other in the delay parameter space associated with drivers’ memories. Appropriate frequency-sweeping techniques are developed to reveal the results and exploited to prove the stability as well as the inherent features of the stability crossing curves in the associated parameter spaces. Case studies are provided to demonstrate the effects of damping, driver aggressiveness, controller gains and delays to asymptotic stability of the controlled vehicle. The highlights of the results are that large controller gains and increased damping have stability favouring effects, while driver aggressiveness can either favour or disfavour stability. We also reveal the controller design rules by which the vehicle dynamics can be rendered delay-independent stable no matter what the delay effects are from the human drivers, and we find that detrimental effects of large dead-times in the memory can be compensated for by appropriately enlarging the memory size, that is, the spread of the distributed delays.

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