Long-range interactions in turbulence and the energy decay problem

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We discuss the long-range interactions that arise in homogeneous turbulence as a consequence of the Biot–Savart law. We note that, somewhat surprisingly, these long-range correlations are very weak in decaying, isotropic turbulence, and we argue that this should also be true for magnetohydrodynamic, rotating and stratified turbulence. If this is indeed the case, it is possible to make explicit predictions for the rate of decay of energy in these anisotropic systems, and it turns out that these predictions are consistent with the available numerical and experimental evidence.

Keywords: rotating turbulence; stratified turbulence; Loitsyansky; magnetohydrodynamics

1. Introduction

In 1941, when Kolmogorov was first formulating his most important contributions to turbulence, it was generally assumed that all two-point velocity correlations, such as \( \langle u_i(x)u_j(x+r) \rangle = \langle u_iu'_j \rangle \), or the triple correlation \( \langle u_iu_ju'_k \rangle \), decayed rapidly with separation \( r = |r| \). In particular, it was usually assumed that \( \langle u_iu'_j \rangle_\infty \) and \( \langle u_iu_ju'_k \rangle_\infty \) are transcendentally small. (Here, the subscript \( \infty \) indicates \( |r| \to \infty \).)

If this is true for freely decaying, isotropic turbulence, then the Karman–Howarth equation in the form

\[
\frac{\partial}{\partial t} \langle u \cdot u' \rangle = \frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r} \left( r^4 u^3 K \right) + 2\nu \nabla^2 \langle u \cdot u' \rangle, \tag{1.1}
\]

where \( u^3 K(r) = \langle u_x^2(x)u_x(x+r\hat{e}_x) \rangle \), may be integrated to yield

\[
I = -\int r^2 \langle u \cdot u' \rangle dr = \text{const.} \tag{1.2}
\]

Kolmogorov was quick to realize the importance of this would-be invariant. In particular, the large scales in isotropic turbulence are large-similar when normalized by the integral scales \( \ell \) and \( u \), where \( u \) is defined by \( u^2 = \frac{1}{3} \langle u^2 \rangle \), and so equation (1.2) demands

\[
I = cu^2\ell^5 = \text{const.}, \tag{1.3}
\]

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where \( c \) is a constant of order unity. This may be combined with the empirical, but well-established, relationship

\[
\frac{du^2}{dt} = -\alpha \frac{u^3}{\ell}, \quad \alpha = \text{const.,}
\]

(1.4)
to yield Kolmogorov’s celebrated decay laws [1]:

\[
\frac{u^2}{u_0^2} = \left[ 1 + \frac{7\alpha}{10} \left( \frac{u_0 t}{\ell_0} \right) \right]^{-10/7} \\
and \quad \frac{\ell}{\ell_0} = \left[ 1 + \frac{7\alpha}{10} \left( \frac{u_0 t}{\ell_0} \right) \right]^{2/7}.
\]

(1.5)

Here, the dimensionless constant \( \alpha \) is of the order of unity and \( u_0 \) and \( \ell_0 \) are the initial values of \( u \) and \( \ell \). These decay laws were subsequently brought into question by Batchelor & Proudman [2], who pointed out that it was not reasonable to assume that \( \langle u_i u_j \rangle_\infty \) or \( \langle u_i u_j u_k \rangle_\infty \) are transcendentally small. The key point is the following. A fluctuation in \( u \) at position \( x \) sends out pressure waves, which travel infinitely fast in an incompressible fluid, and these fall off with distance from the source as \( p(x') = p' \sim r^{-3} \), where \( r = |r| = |x' - x| \). (The \( r^{-1} \) and \( r^{-2} \) contributions to the far-field pressure are zero because of incompressibility.) This, in turn, induces long-range pressure–velocity correlations of the form \( \langle u_i u_j p' \rangle_\infty \sim r^{-3} \). Moreover, the evolution equation for \( \langle u_i u_j u_k' \rangle \) turns out to be

\[
\rho \frac{\partial}{\partial t} \langle u_i u_j u_k' \rangle = -\frac{\partial}{\partial r_k} \langle u_i u_j p' \rangle + \cdots,
\]

and so we might expect these long-range pressure–velocity correlations to induce long-range triple correlations of the form

\[
\langle u_i u_j u_k' \rangle_\infty = d_{ijk} u^3 \left( \frac{r}{\ell} \right)^{-4} + \text{higher order terms},
\]

(1.6)

where the \( d_{ijk} \) are dimensionless prefactors. (This equation can also be obtained directly from the Biot–Savart law, as shown in [3, p. 639].) The Karman–Howarth equation then yields \( \langle u_i u_j' \rangle \sim r^{-6} \) for isotropic turbulence, and \( \langle u_i u_j' \rangle \sim r^{-5} \) for anisotropic, homogeneous turbulence. (Symmetry kills the \( r^{-5} \) term in the isotropic case.) This algebraic fall-off in \( \langle u_i u_j u_k' \rangle_\infty \), if correct, is enough to invalidate equations (1.2) and (1.3), since integration of equation (1.1) now yields

\[
\frac{dI}{dt} = 8\pi [u^3 r^4 K]_\infty = 8\pi d_{xxx} u^3 \ell^4 \neq 0.
\]

(1.7)

So it would seem that the findings of Batchelor & Proudman [2] undermine the entire basis for Kolmogorov’s decay laws.

It is worth pointing out, however, that there is no rigorous theory which can predict the magnitude of the prefactors \( d_{ijk} \) in equation (1.6). They could, for example, be zero. Nevertheless, following Batchelor & Proudman’s seminal paper it became widely assumed that the \( d_{ijk} \) are non-zero and of order unity, and by implication the decay laws (1.5) are incorrect. (See, for example, the discussion in [4,5].) Certainly the heuristic two-point closure models which emerged in the second half of the twentieth century all predict...
dI/dt \neq 0$, starting with Proudman & Reid’s [6] dissection of the quasi-normal model and carrying through to the commonly used eddy-damped quasi-normal Markovian (EDQNM) closure (again, see [5] for a discussion of EDQNM).

Curiously, though, recent numerical simulations performed in very large computational domains by Davidson et al. [7] and Ishida et al. [8] show that $I \approx \text{const.}$ once the turbulence has become fully developed. They also show that $u^2 \sim t^{-10/7}$ during the same period, which is consistent with the classical predictions and inconsistent with the closure models. The implication of these simulations is that either the $r^{-4}$ tail predicted by equation (1.6) is not in fact there, or it is present, but the corresponding prefactors, $d_{ijk}$, are very small. Ishida et al. [8] proposed the latter option. In either case, it seems that the algebraic tails predicted by Batchelor & Proudman can be ignored for most practical purposes. These simulations also suggest that the suppression of the long-range interactions is somehow connected to the morphology of the vorticity field of fully developed turbulence, since $I$ is found to be strongly time-dependent during the initial stages of the decay, when the vorticity field is more or less structureless, but $I$ is constant once the vorticity field has had a chance to develop its characteristic fully developed structure, consisting of a sponge or tangle of very thin, tube-like Burgers vortices. (We note in passing that, if the morphology of the vorticity field is indeed important, then two-point closure theories will not do a good job of predicting the long-range interactions, since they contain little or no information about geometry.) Interestingly, the importance of the detailed structure of the vorticity field for the suppression of long-range correlations was anticipated by Ruelle [9].

These observations have wide ramifications for other types of homogeneous turbulence, such as magnetohydrodynamic (MHD), rotating and stratified turbulence. In particular, provided the long-range interactions are equally weak in these more complex systems, we can predict the corresponding rate of decay of energy following a generalization of Kolmogorov’s argument. Indeed, in §§5–7 we shall make explicit predictions for the decay of energy in all three of these homogeneous turbulent systems, predictions which are consistent with the available experimental and numerical data.

The question remains, however, as to why the long-range correlations predicted by Batchelor & Proudman are so weak. We shall return to this issue in §3, but perhaps it is worth noting now that similar behaviour is seen in other physical systems with a large number of degrees of freedom and subject to random fluctuations. The most obvious example is Debye–Hückel screening in electrolytes and plasmas, where one might expect to see interactions between distant charge carriers induced by the non-local Coulomb force. In practice, however, there are no such interactions, with the long-range Coulomb forces suppressed by the clustering of oppositely signed charges, which leaves the plasma electrically neutral, at least in a coarse-grained sense. In such cases, the two-point correlations fall off exponentially with separation. We may think of this as a kind of energy minimization process, since the electrostatic energy associated with the far-field forces is released as these forces shut down. Yet other systems exhibit a form of partial screening, such as tangles of current loops, in the sense that the energy associated with the long-range Lorentz forces can be reduced through a clustering of oppositely signed current loops.
We shall return to these issues in §3. First, however, it is natural to enquire as to the physical basis for the conservation of $I$ (in those cases where the long-range interactions are absent). This brings us to the important contribution of Landau, who showed that equation (1.2) is related to the principle of conservation of angular momentum. Although Landau’s analysis is flawed, as pointed out in Davidson [10], it still captures that spirit of the idea and provides the key to extending the analysis to other homogeneous systems.

Throughout this paper, we restrict ourselves to turbulence where the energy spectrum at low $k$ takes the form $E(k \to 0) \sim k^4$. In such cases, the invariant is equation (1.2), or something close to equation (1.2). The other canonical case is $E(k \to 0) \sim k^2$ (so-called Saffman turbulence), in which the key invariant is Saffman’s integral,

$$L = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{r}.$$  

Decay laws for $E(k \to 0) \sim k^2$ turbulence are discussed in Davidson [11] and in §9.

2. Landau’s interpretation of the conservation of $I$

Let $H = \int \mathbf{x} \times \mathbf{u} dV$ be the angular momentum held in some volume $V$ of the turbulent flow. In order to apply the principle of angular momentum conservation to turbulence, Landau first considered the case of inhomogeneous turbulence evolving in a large, closed domain. In particular, the identity

$$(\mathbf{x} \times \mathbf{u}) \cdot (\mathbf{x}' \times \mathbf{u}') = 2\mathbf{x} \cdot \mathbf{x}' (\mathbf{u} \cdot \mathbf{u}') - u'_i x'_j \nabla \cdot [x_i x_j \mathbf{u}]$$

integrates to yield

$$H^2 = \left[ \int_V \mathbf{x} \times \mathbf{u} dV \right]^2 = \int_V \int_V 2\mathbf{x} \cdot \mathbf{x}' (\mathbf{u} \cdot \mathbf{u}') d\mathbf{x}' d\mathbf{x},$$

since $\mathbf{u} \cdot d\mathbf{S} = 0$ on the surface of $V$. This can be rewritten as

$$H^2 = -\int_V \int_V (\mathbf{x}' - \mathbf{x})^2 (\mathbf{u} \cdot \mathbf{u}') d\mathbf{x}' d\mathbf{x}, \quad (2.1)$$

since $\int \mathbf{u} dV = 0$, and on ensemble averaging, we obtain

$$\langle H^2 \rangle = \int_V \left[ -\int_{V^*} r^2 (\mathbf{u} \cdot \mathbf{u}') d\mathbf{r} \right] d\mathbf{x}, \quad (2.2)$$

where the shape of $V^*$ depends on the location of $\mathbf{x}$ within $V$. Landau now assumed that $\langle \mathbf{u} \cdot \mathbf{u}' \rangle$ falls off rapidly with separation, $r$, say as $\langle \mathbf{u} \cdot \mathbf{u}' \rangle_\infty \sim \exp(-r^2/\ell^2)$. Then for all points $\mathbf{x}$ which are remote from the boundary the inner integral in equation (2.2) can be replaced by an integral over all $r$. Since $V \gg \ell^3$, this is a good approximation for all points $\mathbf{x}$ in $V$, except those which lie within a distance $O(\ell)$ from the surface. It follows that, in the limit of $V \gg \ell^3$ [12],

$$\frac{\langle H^2 \rangle}{V} = -\int r^2 (\mathbf{u} \cdot \mathbf{u}') d\mathbf{r}. \quad (2.3)$$

So it appears that there is indeed a link between $I$ and $H$. 

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We have not yet used conservation of angular momentum to explain the invariance of $I$. To do this we need to consider the particular situation in which the closed domain is spherical, of radius $R$. In such a case, $H$ is conserved in each realization, in the sense that the viscous stresses on the surface of $V$ have a negligible influence on $H$ in the limit of $(R/\ell) \to \infty$. Equation (2.3) then suggests that the invariance of $I$ is indeed a consequence of angular momentum conservation provided, of course, that the long-range correlations can be ignored. Actually, it turns out that there are several technical difficulties associated with Landau’s analysis, as pointed out in Davidson [10], but nevertheless, it does capture the essential reason for the invariance of $I$.

3. Is there screening in turbulence?

Let us now return to the observation that the long-range correlations, as measured by the prefactors $d_{ijk}$, are very weak in fully developed, isotropic turbulence. This is a curious finding since $\langle u_i u_j u_k' \rangle \sim d_{ijk} r^{-4}$ seems to be an inevitable consequence of the long-range pressure forces, or equivalently the Biot–Savart law. Indeed, simple closure models, such as the quasi-normal closure, have $d_{ijk}$ as an order-one quantity [6], while later closures, such as EDQNM, predict a modest but finite time dependence for $I(t)$. For example, in Davidson [10] it is shown that EDQNM predicts $I \sim (u^2)^{-m}$, where

$$m = \frac{7\pi}{20 a_1 A \sqrt{2}} \int_0^\infty \frac{F(x)}{x} \left[ \int_0^\infty F(x) \, dx \right]^{-3/2} \int_1^\infty \frac{F^2(x) / x^2 \sqrt{\int_0^x y^2 F(y) \, dy}} {\sqrt{\int_0^x y^2 F(y) \, dy}} \, dx. \quad (3.1)$$

Here, $k_i$ is the wavenumber at which the energy spectrum $E(k)$ is a maximum, $F(k/k_i)$ is the normalized spectrum, $F = 24\pi^2 E(k)/Ik_i^4$, $a_1$ is a model constant of order unity, and $A \approx 1/3$. Equation (3.1) is cumbersome and inelegant, but the key point is that it tells us that, according to EDQNM, $m$ is an order-one quantity, since there are no small parameters in equation (3.1). Indeed, the standard version of EDQNM has $m \sim 0.12$.

Why, then, do the recent simulation show that the $d_{ijk}$ are very small in fully developed turbulence, contrary to intuition and to the predictions of the closures? This remains an open question, though there have been some tentative suggestions. Perhaps the most striking suggestion came from Ruelle [9], who speculated that the long-range correlations might vanish by analogy with Debye–Hückel screening in electrolytes and plasmas, whereby the long-range Coulomb forces are suppressed through a clustering of oppositely signed charges, leaving the plasma electrically neutral at each point. Ruelle noted that, in a similar way, current loops interacting via their induced magnetic fields can, in certain situations, exhibit a form of partial screening, whereby the magnetic energy associated with far-field interactions is reduced. There is a kinematic analogy between such current loops and vortex tubes in a turbulent flow, and it was this analogy that motivated Ruelle’s speculation about screening in turbulence. It turns out, however, that the analogy is imperfect, as we now show.

Consider, for example, two remote vortex loops interacting in an inviscid fluid via their induced velocity fields. Let loop 1 have a vorticity field $\omega_1$, which is confined to the region $V_1$, and loop 2 have a vorticity field $\omega_2$, confined to the
distinct region $V_2$. We now consider the influence of loop 2 on the angular impulse of loop 1,

$$M_1 = \frac{1}{3} \int_{V_1} \mathbf{x} \times (\mathbf{x} \times \mathbf{\omega}_1) dV,$$  \hspace{1cm} (3.2)

and the corresponding influence of loop 1 on $M_2$. (The angular impulse of a blob of vorticity, defined by equation (3.2), is equal to the angular momentum introduced into the flow by virtue of the presence of the blob.) In Davidson [10] it is shown that

$$\frac{dM_1}{dt} = -\int_{V_1} \mathbf{x} \times (\mathbf{\omega}_1 \times \mathbf{u}_2) dV \quad \text{and} \quad \frac{dM_2}{dt} = -\int_{V_2} \mathbf{x} \times (\mathbf{\omega}_2 \times \mathbf{u}_1) dV. \hspace{1cm} (3.3)$$

Let us now compare this with the equivalent problem in magnetostatics. Here, we have two current loops with current densities $\mathbf{j}_1$ and $\mathbf{j}_2$, exerting forces and torques on each other through their induced magnetic fields, $\mathbf{B}_1$ and $\mathbf{B}_2$, with $\nabla \times \mathbf{B} = \mu \mathbf{j}$. (Here, $\mu$ is the permeability of free space.) There is an obvious kinematic analogy, with $\mu \mathbf{j} \leftrightarrow \mathbf{\omega}$ and $\mathbf{B} \leftrightarrow \mathbf{u}$. However, when we turn to dynamics the analogy breaks down, since the counterpart of equation (3.3) is

$$T_1 = \int_{V_1} \mathbf{x} \times (\mathbf{j}_1 \times \mathbf{B}_2) dV \quad \text{and} \quad T_2 = \int_{V_2} \mathbf{x} \times (\mathbf{j}_2 \times \mathbf{B}_1) dV, \hspace{1cm} (3.4)$$

where $T_1$ and $T_2$ are the torques exerted on the two current loops. Evidently, the torques in the two problems are in opposite senses. This suggests that the behaviour of interacting current loops will be different to that of interacting vortex tubes, so it is not at all clear that Debye-like screening will occur in turbulence.

An alternative explanation for the observed weakness of the long-range correlations is given in Davidson et al. [7]. Here, it is noted that the pressure field is obtained by inverting $\nabla^2 p = -(\rho^+ - \rho^-)$, where $\rho^+ = S_{ij} S_{ij}$, $\rho^- = \omega^2/2$, and $S_{ij}$ is the rate-of-strain tensor. In homogeneous turbulence $\langle \rho^+ - \rho^- \rangle = 0$, and if the regions of intense $\rho^+$ and $\rho^-$ are closely correlated in space, which is analogous to the clustering of oppositely signed charges in Debye screening, then the far-field pressure forces will be weak. That is, we can minimize the long-range interactions if we can ensure $\rho^+ - \rho^- \approx 0$, at least in a coarse-grained sense. Interestingly, the small-scale vortex tubes (Burgers vortices), which dominate the vorticity field in fully developed turbulence, consist of a central core of enstrophy, $\omega^2/2$, embedded in an annulus of $S_{ij} S_{ij}$ of equal overall strength. Thus, as this tangle of fine-scale tubes is swept around by the chaotic velocity field, it maintains a close spatial correlation between $\rho^+$ and $\rho^-$. Yet another possible explanation for the weakness of the long-range correlations is the presence of thin but broad vortex sheets in the turbulence, which can act as barriers to the transfer of information by the pressure field [13]. However, to date neither of these possibilities have been explored, and so they remain mere conjecture. It would seem, therefore, that we have no convincing explanation for the weakness of the long-range interactions, and this remains an important, open question in turbulence.

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4. Extending Landau’s analysis to other types of homogeneous turbulence

As we have seen, the underlying assumption in Landau’s theory, that long-range interactions may be neglected, turns out to be a reasonable approximation in isotropic turbulence. If the long-range interactions are weak in other forms of homogeneous turbulence, such as MHD, strongly rotating or stratified turbulence, then it is possible to generalize Landau’s analysis to incorporate these more complex, anisotropic systems. It turns out that the only requirement is that there is no net torque associated with the body force (the buoyancy, Coriolis or Lorentz force) in at least one direction. In such a case, one can repeat the steps of §2, but focusing on the conserved component of angular momentum only. It turns out that the Lorentz, Coriolis and buoyancy forces do indeed satisfy this constraint, and so the theory of §2 is readily adapted to these anisotropic flows.

The way in which the analysis of §2 can be generalized to MHD, rotating and stratified turbulence is given in Davidson [3,10,14]. In the case of rotating or stratified turbulence, the component of angular momentum parallel to the rotation axis or gravitational acceleration is clearly conserved. In MHD turbulence, on the other hand, it turns out to be the component of $H$ parallel to the applied magnetic field which is conserved, as can be seen from the following argument. Consider a turbulent, conducting fluid evolving in a large, electrically insulated, spherical domain and subject to an imposed magnetic field $B_0$. Let $j$ be the current density induced in the fluid, which satisfies the boundary condition $j \cdot dS = 0$, and $b$ be the magnetic field associated with $j$ by virtue of the Biot–Savart law. Then the net torque exerted on the fluid by the Lorentz force is

$$T = \int_V \mathbf{x} \times (j \times B_0) \, dV + \int_V \mathbf{x} \times (j \times b) \, dV,$$

where $V$ is the closed spherical domain. However, a closed system of currents produces zero net torque when interacting with its self-field, $b$, and it follows that the second integral on the right is zero. The first integral, on the other hand, can be transformed to give

$$T = \frac{1}{2} \int_V (x \times j) \, dV \times B_0 = m \times B_0,$$

where $m$ is the dipole moment associated with $j$. Ignoring the viscous stresses on the surface of $V$, the angular momentum of the fluid evolves according to $\rho dH/dt = m \times B_0$, and so the component of $H$ parallel to $B_0$ is indeed conserved, as noted above.

We now denote the conserved component of $H$ in each of these systems (i.e. MHD, rotating or stratified turbulence) by $H_{//}$, and introduce the integral

$$I_{//} = - \int r_\perp^2 \langle u_\perp \cdot u'_\perp \rangle \, dr,$$

where $r_\perp$ and $u_\perp$ are the components of $r$ and $u$ in the transverse plane. Assuming that all two-point correlations decay rapidly with separation, we may repeat the analysis of §2, but applied to $H_{//}$ only, and this gives

$$I_{//} = - \int r_\perp^2 \langle u_\perp \cdot u'_\perp \rangle \, dr = \text{const.} \quad (4.2)$$

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An independent check on the validity of equation (4.2) can be made by integrating the appropriate form of the generalized, anisotropic Karman–Howarth equation, incorporating the appropriate body forces. It has been shown that, provided the two-point correlations decay sufficiently rapidly with separation, equation (4.2) does indeed hold for these various types of anisotropic, homogeneous turbulence [3, pp. 514, 542].

The importance of equation (4.2) is clear. If the large scales are self-similar, then equation (4.2) demands

$$u_\perp^2 \ell_\perp^4 \ell_// = \text{const.},$$  \hspace{1cm} (4.3)

where $\ell_\perp$ and $\ell_//$, are suitably defined integral scales. This imposes a powerful constraint on the evolution of the large scales. For example, constraint (4.3) can be used in the spirit of Kolmogorov [1] to estimate the rate of decay of energy in MHD, rotating and stratified turbulence, and the resulting predictions are close to the available numerical and experimental evidence, as discussed below.

Of course, we need to determine whether or not the long-range interactions are indeed weak in these anisotropic systems, just as they are in mature, isotropic turbulence. If they are not weak, then constraint (4.3) cannot be justified. It is reassuring in this respect to note that the Coriolis and buoyancy forces are less efficient at transmitting information over large distances than the pressure force [10,11]. That is, the Coriolis and buoyancy forces can generate internal waves (inertial waves and gravitational waves), but these waves are less efficient at propagating information than the pressure force, which acts instantaneously over large distances in an incompressible fluid. The situation is a little more complicated in MHD turbulence, where the Lorentz force is a non-local function of $u$, but it is shown in Davidson [14] that the resulting long-range interactions are no worse than in conventional turbulence. It seems reasonable, therefore, to assume that these anisotropic systems will also exhibit weak (as distinct from strong) long-range interactions, at least in the fully developed state.

In the remainder of this paper, we show how constraint (4.3) plays a crucial role in freely decaying, homogeneous turbulence. We focus on four particular cases: (i) MHD turbulence in which the magnetic Reynolds number is small, $R_m = \sigma \mu u \ell \ll 1$, $\sigma$ being the electrical conductivity; (ii) MHD turbulence at high $R_m$; (iii) rapidly rotating turbulence; and (iv) turbulence evolving in the presence of a uniform background stratification. In each case, we derive an energy decay law for the system and predict the rate of change of the integral scales. Our analysis of stratified and high-$R_m$ MHD turbulence may be found in Davidson [10], whereas our discussion of low-$R_m$ MHD turbulence is based on that given in Davidson [3] and Okamoto et al. [15].

5. Decay laws in magnetohydrodynamic turbulence

We start with MHD turbulence at low $R_m$, where the effect of $B_0$ is to introduce anisotropy into the turbulence, with $\ell_// > \ell_\perp$. The low-$R_m$ regime, which typifies most terrestrial MHD, is characterized by $b \ll B_0$, so that the energy associated
with the induced field, \( b \), may be ignored. The energy equation is then
\[
\frac{du^2}{dt} = -\frac{2}{3} \left[ \nu \omega^2 + \frac{\langle j^2 \rangle}{\rho \sigma} \right],
\]

where the two terms on the right are the viscous and Joule dissipation, respectively. The Joule dissipation may be estimated from the low-\( R_m \) form of Ohm’s Law, \( j = \sigma (-\nabla V + u \times B_0) \), where \( V \) is the electrostatic potential. The curl of this yields
\[
\nabla \times j = \sigma B_0 \cdot \nabla u,
\]

from which we obtain the estimate
\[
\frac{\langle j^2 \rangle}{\rho \sigma} = \frac{3\beta}{2} \left( \frac{\ell_\perp}{\ell_//} \right)^2 \frac{u^2}{\tau}.
\]

Here, \( \tau = (\sigma B_0^2 / \rho)^{-1} \) is the so-called Joule dissipation time and \( \beta \sim 1 \). (In fact, it may be shown that \( \beta = 2/3 \) in isotropic turbulence.) Using equation (5.3) to estimate the Joule dissipation in equation (5.1), and making the usual high-\( Re \) estimate of the viscous dissipation term, we obtain
\[
\frac{du^2}{dt} = -\alpha \frac{u^3}{\ell_\perp} - \beta \left( \frac{\ell_\perp}{\ell_//} \right)^2 \frac{u^2}{\tau}.
\]

In low-\( R_m \) turbulence the relative intensity of the imposed magnetic field is usually measured using the dimensionless interaction parameter, \( N_I = \ell_\perp / u \tau \). When \( N_I \) is small, equations (4.3) and (5.4) reduce to equations (1.3) and (1.4), which yields the Kolmogorov decay law, \( u^2 \sim t^{-10/7} \). When \( N_I \) is large, on the other hand, inertia is unimportant and diffusive Alfvén waves increase \( \ell_// \) but leave \( \ell_\perp \) unchanged on times of order \( \tau \). The high-\( N_I \) case is, therefore, governed by
\[
\frac{du^2}{dt} = -\beta \left( \frac{\ell_\perp}{\ell_//} \right)^2 \frac{u^2}{\tau},
\]

plus the constraints imposed by equation (4.3) and \( \ell_\perp = \text{const} \). This yields the well-known expressions
\[
u^2 = u_0^2 \left[ 1 + \frac{2\beta t}{\tau} \right]^{-1/2} \quad \text{and} \quad \ell_// = \ell_0 \left[ 1 + \frac{2\beta t}{\tau} \right]^{1/2}.
\]

For intermediate \( N_I \), however, equations (4.3) and (5.4) between them contain three unknowns, \( u^2, \ell_\perp \), and \( \ell_// \), and so the problem is under specified. To close the system Davidson [3] proposed
\[
\frac{d}{dt} \left( \frac{\ell_//}{\ell_\perp} \right) = \frac{2\beta}{\tau}.
\]

This is exact for \( N_I \rightarrow 0 \) and \( N_I \rightarrow \infty \), and may be thought of as an interpolation formula for intermediate \( N_I \). Finally, integrating equations (5.4) and (5.7), subject to the constraint of equation (4.3), and assuming isotropic initial conditions,
yields the decay laws
\[
\frac{u^2}{u_0^2} = \hat{t}^{-1/2} \left[ 1 + \left( \frac{7\alpha}{15\beta} \right) (\hat{t}^{3/4} - 1) N_0^{-1} \right]^{-10/7},
\]
\[
\frac{\ell_\perp}{\ell_0} = \left[ 1 + \left( \frac{7\alpha}{15\beta} \right) (\hat{t}^{3/4} - 1) N_0^{-1} \right]^{2/7},
\]
and
\[
\frac{\ell_/}{\ell_0} = \hat{t}^{1/2} \left[ 1 + \left( \frac{7\alpha}{15\beta} \right) (\hat{t}^{3/4} - 1) N_0^{-1} \right]^{2/7},
\]
where \(N_0\) is the initial value of \(N_I\), and \(\hat{t} = 1 + 2\beta(t/\tau)\). These expressions reduce to Kolmogorov’s decay law for small \(N_0\), and to equation (5.6) at large \(N_0\).

We now turn to the case of high-\(R_m\) turbulence evolving in the presence of a uniform, imposed field. Here, the energy associated with the induced field \(b\) cannot be neglected, and so a term \(b^2/\rho\mu\) must be added to the left of equation (5.1). However, Alfven waves travelling along the mean field tend to promote an equipartition of energy between \(u^2\) and \(b^2/\rho\mu\). Moreover, the dissipation of energy in such turbulence is observed to be of the order of \(u^3_\perp/\ell_\perp\), as in hydrodynamic turbulence [16]. Thus, our energy equation takes the familiar form
\[
\frac{d u_\perp^2}{dt} = -\alpha \frac{u_\perp^3}{\ell_\perp}, \quad \alpha \sim 1,
\]
where \(u_\perp^2 = \frac{1}{2}u^2_\perp\). At high \(R_m\) the ratio of the integral length scales, \(\ell_//\ell_\perp\), is set by the so-called ‘critical-balance’ condition, which requires \(\ell_//\ell_\perp \sim V_A/\ell_\perp\), where \(V_A\) is the Alfven wave speed associated with the mean field, \(V_A = B_0/\sqrt{\rho\mu}\). (See [16] for a discussion of critical balance.) We can now integrate equation (5.9), subject to the constraint of equation (4.3), and this yields
\[
\frac{u_\perp^2}{u_0^2} = \left[ 1 + \frac{3\alpha u_0 t}{5 \ell_0} \right]^{-5/3},
\]
and
\[
\frac{\ell_\perp}{\ell_0} = \left[ 1 + \frac{3\alpha u_0 t}{5 \ell_0} \right]^{1/6} \quad \text{and} \quad \frac{\ell_/}{\ell_0} \sim \frac{V_A}{u_0} \left[ 1 + \frac{3\alpha u_0 t}{5 \ell_0} \right],
\]
where \(u_0\) and \(\ell_0\) are the initial values of \(u_\perp\) and \(\ell_\perp\). Once again, our angular momentum constraint has provided simple decay laws.

6. The decay of rapidly rotating turbulence

Let us now consider freely decaying turbulence subject to a background rotation \(\Omega\). It is observed that rotation tends to suppress the rate of decay of energy and we wish to quantify this. The relative strength of the inertial and Coriolis forces is given by the Rossby number \(Ro = u/\Omega_\ell\). In laboratory experiments of such turbulence, the initial Rossby number is usually chosen to be large. This ensures...
that no inertial waves are generated by the grid used to create the turbulence. However, as the turbulence decays $Ro$ falls, and once the Rossby number reaches $Ro \sim 1$, the turbulence starts to generate inertial waves. These propagate along the rotation axis at the group velocity $c_g \sim \Omega \ell$, causing the large eddies to elongate in the direction of $\Omega$. Thus $\ell_{//}$ grows as $\ell_{//} = \ell_0 (1 + \kappa t)$, where $\kappa \sim 1$ and $\ell_0$ is the value of $\ell_{//}$ at the instant when the inertial waves first appear [17]. This predicted linear growth in $\ell_{//}$ finds support in the experiments of both Jacquine et al. [18] and Staplehurst et al. [19]. It follows from constraint (4.3) that, in freely decaying, rapidly rotating turbulence, 

$$u_\perp^4 \ell_{//}^4 = u_\perp^4 \ell_0^4 (1 + \kappa \Omega t)^{-1},$$  

(6.1)

where $u_0 = u_\perp(t = 0)$, and $t = 0$ corresponds to the time at which columnar eddies first appear and $\ell_{//}$ starts to grow linearly in time. Note that $u_0$ and $\ell_0$ are constrained to satisfy $u_0/\ell_0 \sim 1$, as $Ro \sim 1$ at $t = 0$. Let us now suppose that the rate of dissipation of energy, $du_\perp^2/dt$, is uniquely determined by the integral scales, $u_\perp$ and $\ell_{//}$, and by $\Omega$: 

$$\frac{du_\perp^2}{dt} = F(u_\perp, \ell_{//}, \Omega).$$  

(6.2)

Then dimensional analysis demands 

$$\frac{du_\perp^2}{dt} = -G \left( \frac{u_\perp}{\Omega \ell_{//}} \right) \frac{u_\perp^3}{\ell_{//}},$$  

(6.3)

where $G$ is some unknown function. Since experiments show that rotation suppresses dissipation, $G$ must be an increasing function of $u_\perp/\Omega \ell$, and the simplest option for $G$ which is consistent with the experimental evidence is $G(\chi) \sim \chi$ for $\chi \leq 1$. This is tantamount to requiring $du_\perp^2/dt \sim \Omega^{-1}$, and indeed Squires et al. [20] proposed $du_\perp^2/dt \sim \Omega^{-1}$ in a numerical study of rotating turbulence. Adopting this simple functional form for $G$, equation (6.3) becomes 

$$\frac{du_\perp^2}{dt} = -\alpha \frac{u_\perp^4}{\Omega \ell_{//}^3}, \quad R_0 \leq 1,$$  

(6.4)

for some dimensionless constant $\alpha$ of order unity. Integrating equation (6.4) subject to the constraint of equation (6.1) yields, for large $u_0 t/\ell_0$,

$$u_\perp^2 \sim \frac{\Omega^2 \ell_0^2}{1 + \kappa \Omega t}.$$  

(6.5)

This is interesting because the decay data of Staplehurst et al. [19], which have been replotted in figure 1, seem to follow a power law reasonably close to $u_\perp^2 \sim (\Omega t)^{-1}$. Estimate (6.5) is also consistent with the measurements of Jacquin et al. [18], which show $u^2 \sim t^{-n}$, with $n \approx 1.4$ for $Q = 0$, and $n \approx 0.81 \to 1.08$ for $Ro \sim 1$. Morize & Moisy [21], on the other hand, found a more rapid decay of energy. However, their experiment was of limited height, with columnar eddies spanning the domain. In such a situation, Ekman layers form, which augment the dissipation of energy. So it would seem that equation (6.5) is consistent with the limited experimental data currently available. Curiously, equation (6.5) is quite different to the prediction of Squires et al. [20] who, based on numerical
experiments, and on the assumption that $I$ is conserved, proposed $u_{\perp}^2 \sim t^{5/7}$, $\ell_{//} \sim t^{6/5}$ and $\ell_{\perp} \sim t^{1/5}$. However, their analysis is not self-consistent, since their proposed scalings demand $I \sim t^{9/7}$, which is far from constant.

7. The decay of stratified turbulence

Finally, we consider turbulence evolving in the presence of a uniform background stratification. Let $N$ represent the Brunt–Väisälä frequency, where $N^2$ is proportional to the imposed density gradient, and let $Fr = u_{\perp}/N\ell_{\perp}$ be the corresponding Froude number. For low $Fr$ the vertical integral scale in stratified turbulence, $\ell_{//}$, is limited by the constraint

$$\frac{u_{\perp}}{N\ell_{//}} = C \sim 1,$$

which comes from the need for inertia to balance the buoyancy force. Moreover, the horizontal kinetic energy is observed to decay as

$$\frac{du_{\perp}^2}{dt} = -\alpha \frac{u_{\perp}^3}{\ell_{\perp}}, \quad \alpha \sim 1.$$

(Here, $\alpha$ and $C$ are constants of order unity.) Expressions (7.1) and (7.2) are proposed in, for example, Brethouwer et al. [22]. Integrating expression (7.2),
subject to the constraints imposed by constraint (4.3) and expression (7.1), yields

$$\frac{u_\perp^2}{u_\perp^2_0} = \left[ 1 + \frac{7\alpha u_0 t}{8} \right]^{-8/7}$$

(7.3)

and

$$\ell_\perp = \left[ 1 + \frac{7\alpha u_0 t}{8} \right]^{3/7} \quad \text{and} \quad \ell_\parallel = \frac{1}{C N\ell_0} \left[ 1 + \frac{7\alpha u_0 t}{8} \right]^{-4/7}.$$  

(7.4)

Yet again, our angular momentum constraint (4.3) has determined the rate of energy decay. However, unlike MHD and rotating turbulence, where our predicted decay laws have been tested against experimental data or numerical simulations, predictions (7.3) and (7.4) have yet to be put to the test, though they are not incompatible with the numerical simulations of Staquet & Godeferd [23], who estimate $u_\perp^2 \sim t^{-n}$, $n \approx 1.0$.

8. A comparison of the different systems

There are a number of interesting similarities between stratified and high-$R_m$ MHD turbulence. In both cases they satisfy

$$u_\perp^2 \ell_\perp^4 \ell_\parallel = \text{const.},$$  

(8.1)

$$\frac{du_\perp^2}{dt} = -\frac{u_\perp^3}{\ell_\perp}, \quad \alpha \sim 1$$  

(8.2)

and

$$\frac{u_\perp}{f\ell_{\text{min}}} \sim 1,$$  

(8.3)

where $\ell_{\text{min}}$ is the smaller of $\ell_\parallel$ and $\ell_\perp$ and $f$ is the characteristic frequency of wave propagation, $f = N$ or else $f = V_A/\ell_\parallel$. The similarities become less obvious as we move to rotating or low-$R_m$ MHD turbulence. In rotating turbulence, for example, $R_0 = u_\perp/2\ell_{\text{min}}$ starts out of order unity, which is reminiscent of equation (8.3), but then $R_0$ falls as the energy decays. Moreover, the energy decay law does not take the form of equation (8.2), but rather is equation (6.4). The differences are even stronger in low-$R_m$ MHD turbulence, where there is no analogue of equation (8.3).

These differences probably reflect the fundamentally different ways in which these systems develop anisotropy in the large-scale eddies. In stratified turbulence, it is generally believed that the characteristic pancake-like eddies form through a nonlinear mechanism, driven by $u \cdot \nabla u$. In rotating turbulence, on the other hand, the large-scale columnar eddies seem to form by quasi-linear inertial wave propagation [17,19], in which energy is preferentially pumped along the rotation axis by inviscid inertial waves. Finally, in low-$R_m$ MHD turbulence structure formation is intrinsically a dissipative process, in which certain wavevectors are preferentially dissipated by the magnetic field [3]. Since different physical processes drive the anisotropy in these systems, we should not be surprised that their behaviour is different.

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9. A comparison with Saffman turbulence

All of the discussion so far has focused on \( E(k \to 0) \sim k^4 \) turbulence. The case of Saffman turbulence, where \( E(k \to 0) \sim k^2 \), has been considered in Davidson [11] and we close by comparing the two classes of turbulence. In Saffman turbulence we must replace equation (1.2) in isotropic turbulence by

\[
L = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle \, d\mathbf{r} = \text{const.}, \quad (9.1)
\]

and equation (4.3) by

\[
u_\perp^2 \ell_\perp^2 \ell_// = \text{const.} \quad (9.2)
\]

in rotating, stratified and MHD turbulence. The other details remain unchanged. It turns out that this makes no difference in the case of rotating turbulence, but it does change the decay laws for stratified and low-\( R_m \) MHD turbulence. For example, in stratified turbulence the energy decay exponent in equation (7.3) changes from \( 8/7 \) to \( 4/5 \),

\[
\frac{u_\perp^2}{u_0^2} = \left[ 1 + \frac{5\alpha u_0 t}{4 \ell_0} \right]^{-4/5}, \quad (9.3)
\]

and it is intriguing that the numerical experiments of Staquet & Godeferd [23] lie between the two. In low-\( R_m \) MHD turbulence the energy decay given by equation (5.8) is replaced by

\[
\frac{u_\perp^2}{u_0^2} = \hat{t}^{-1/2} \left[ 1 + \left( \frac{5\alpha}{9\beta} \right) (\hat{t}^{3/4} - 1) N_0^{-1} \right]^{-6/5}, \quad (9.4)
\]

a prediction which has yet to be put to the test.

10. Conclusions

We have made explicit predictions for the rate of decay of energy in MHD, rotating and stratified turbulence. These predictions are based on the assumption that the long-range interactions induced by the Biot–Savart law may be neglected, which seems reasonable in the light of the numerical data for isotropic turbulence. The predictions for low-\( R_m \) MHD turbulence have been verified by DNS, and those for rotating turbulence are consistent with the limited experimental data currently available. It would be interesting, therefore, to produce data against which the proposed decay laws for stratified turbulence could be tested.

The underlying reason for the weakness of the long-range interactions in these homogeneous systems remains an important, open question.

References


