We consider a class of map, recently derived in the context of cluster mutation. In this paper, we start with a brief review of the quiver context, but then move onto a discussion of a related Poisson bracket, along with the Poisson algebra of a special family of functions associated with these maps. A bi-Hamiltonian structure is derived and used to construct a sequence of Poisson-commuting functions and hence show complete integrability. Canonical coordinates are derived, with the map now being a canonical transformation with a sequence of commuting invariant functions. Compatibility of a pair of these functions gives rise to Liouville’s equation and the map plays the role of a Bäcklund transformation.

Keywords: Poisson algebra; bi-Hamiltonian; integrable map; Bäcklund transformation; Laurent property; cluster algebra

1. Introduction

Robin Bullough’s famous diagram represents a vast array of areas in mathematics and mathematical physics, together with a ‘neural network’ of connections (solid lines when established; dotted when expected) between them. This Grand Synthesis of Soliton Theory shows that some remarkable connections between seemingly disparate subjects have come about through the developments of integrable systems, which have taken place in the last 40 years or so. Of course, the diagram perpetually evolves, as dotted connections become solid and as new subject areas (with corresponding links) are added to the array. In this paper, I present some connections with subjects that did not even exist until recently!

Specifically, the present paper is concerned with integrable maps that arise in the context of cluster mutations [1]. This gives a connection to maps with the Laurent property, with the archetypical example being the Somos 4 iteration (see The On-line Encyclopedia of Integer Sequences at Sloane [2]), which arises in the context of elliptic divisibility sequences in number theory. In Fordy & Marsh [3], we considered a class of quiver that had a certain periodicity property under ‘quiver mutation’. The corresponding ‘cluster exchange relations’ then give rise to sequences with the Laurent property, which generalize many of the well-known examples.

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Dedication: this article is dedicated to the memory of Robin Bullough.

One contribution of 13 to a Theme Issue ‘Nonlinear phenomena, optical and quantum solitons’.
In this paper, I first explain some of this background, but the main emphasis will be on some associated Poisson algebras (with respect to an invariant Poisson bracket of log-canonical type). In terms of canonical variables, we obtain Hamiltonians (invariant under the action of the map) in exponential form. The compatibility of one particular pair of invariant Hamiltonians leads to Liouville’s equation for each \( q_i \), with the map now playing the role of a Bäcklund transformation, bringing us back to one of the original ideas in soliton theory.

2. The Laurent property

The Somos 4 sequence is generated by the iteration on the real line

\[
x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2, \quad \text{with} \quad x_0 = x_1 = x_2 = x_3 = 1,
\]

(2.1)

It is found that it too has the Laurent property for \( N = 4, 5, 6, 7 \), but fails at \( N = 8 \). Failure is rather simple to prove, since non-integer (rational) elements occur fairly soon in the sequence. Ad hoc proofs exist for various sequences (see [4] for a proof in the case of Somos 4), and can often be adapted to other sequences. However, a remarkable (but complicated!) proof for a very broad class of iteration was given in Fomin & Zelevinsky [5].

At about the same time, cluster algebras were introduced in Fomin & Zelevinsky [1], and it was shown that any map that arose as a cluster exchange relation necessarily had the Laurent property. Cluster algebras are an abstraction of structures that arise in the study of total positivity of matrices and in the canonical basis of a quantum group. However, for this paper, we need none of this background. Neither do we need the full definition of a cluster algebra. The most important aspect for us is the association with quivers and quiver mutation.

(a) Quiver mutation

A quiver is a directed graph, consisting of \( N \) nodes with directed edges between them. There may be several arrows between a given pair of vertices, but for cluster algebras there should be no one-cycles (an arrow that starts and ends at the same node) or two-cycles (an arrow from node \( a \) to node \( b \), followed by one
from node \(b\) to node \(a\). A quiver \(Q\), with \(N\) nodes, can be identified with the unique skew-symmetric \(N \times N\) matrix \(B_Q\) with \((B_Q)_{ij}\) given by the number of arrows from \(i\) to \(j\) minus the number of arrows from \(j\) to \(i\).

An important quiver for our discussion is the one corresponding to the Somos 4 sequence (both quiver and matrix in figure 1).

**Definition 2.1 (quiver mutation).** Given a quiver \(Q\), we can mutate at any of its nodes. The mutation of \(Q\) at node \(k\), denoted by \(\mu_k Q\), is constructed (from \(Q\)) as follows:

1. Reverse all arrows that either originate or terminate at node \(k\).
2. Suppose that there are \(p\) arrows from node \(i\) to node \(k\) and \(q\) arrows from node \(k\) to node \(j\) (in \(Q\)). Add \(pq\) arrows going from node \(i\) to node \(j\) to any arrows already there.
3. Remove (both arrows of) any two-cycles created in the previous steps.

Note that, in step 2, \(pq\) is just the number of paths of length 2 between nodes \(i\) and \(j\) that pass through node \(k\).

**Remark 2.2 (matrix mutation).** Let \(B\) and \(\tilde{B}\) be the skew-symmetric matrices corresponding to the quivers \(Q\) and \(\tilde{Q} = \mu_k Q\). Let \(b_{ij}\) and \(\tilde{b}_{ij}\) be the corresponding matrix entries. Then, quiver mutation amounts to the following formula:

\[
\tilde{b}_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k, \\
 b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + |b_{ik}|b_{kj}|) & \text{otherwise.}
\end{cases}
\]  

It is an exercise to show that, with these definitions, the Somos 4 quiver and matrix are transformed to those of figure 2, if we mutate at node 1.

(b) **Cluster exchange relations**

Given a quiver (with \(N\) nodes), we attach a variable at each node, labelled \((x_1, \ldots, x_N)\). When we mutate the quiver, we change the associated matrix...
according to formula (2.2) and, in addition, we transform the cluster variables $(x_1, \ldots, x_N) \mapsto (x_1, \ldots, \tilde{x}_k, \ldots, x_N)$, where
\[
x_k \tilde{x}_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}, \quad \tilde{x}_i = x_i \quad \text{for } i \neq k. \tag{2.3}
\]
If one of these products is empty (which occurs when all $b_{ik}$ have the same sign) then it is replaced by the number 1. This formula is called the (cluster) exchange relation. Notice that it just depends upon the $k$th column of the matrix. Since the matrix is skew-symmetric, the variable $x_k$ does not occur on the right side of equation (2.3).

After this process, we have a new quiver $\tilde{Q}$, with a new matrix $\tilde{B}$. This new quiver has cluster variables $(\tilde{x}_1, \ldots, \tilde{x}_N)$. However, since the exchange relation (2.3) acts as the identity on all except one variable, we write these new cluster variables as $(x_1, \ldots, \tilde{x}_k, \ldots, x_N)$. We can now repeat this process and mutate $\tilde{Q}$ at node $\ell$ and produce a third quiver $\tilde{\tilde{Q}}$, with cluster variables $(x_1, \ldots, \tilde{x}_k, \ldots, x_\ell, \ldots, x_N)$, with $\tilde{x}_\ell$ being given by an analogous formula (2.3).

**Remark 2.3 (involutive property of the exchange relation).** If $\ell = k$, then $\tilde{\tilde{Q}} = Q$, so we insist that $\ell \neq k$.

**Example 2.4 (the Somos 4 quiver $S_4$).** Placing $x_1, x_2, x_3, x_4$, respectively, at nodes 1–4 of quiver $S_4$ (of figure 1) gives the initial cluster. Along with the quiver mutation (leading to $\mu_1 S_4$ of figure 2), we also have the exchange relation
\[
x_1 \tilde{x}_1 = x_2 x_4 + x_3^2. \tag{2.4}
\]
This corresponds to one arrow coming into node 1 from each of nodes 2 and 4, with two arrows going out to node 3.

We can now consider mutations of quiver $\tilde{S}_4 = \mu_1 S_4$. To avoid too many ‘tildes’, let us write $\tilde{x}_1 = x_5$, so quiver $\tilde{S}_4$ has $x_5, x_2, x_3, x_4$, respectively, at nodes 1–4. Mutation at node 1 would just take us back to quiver $S_4$ (as noted in the above remark). We compare the exchange relations that we would obtain by mutating at node 2 or 3. Mutation at node 2 would lead to exchange relation
\[
x_2 \tilde{x}_2 = x_3 x_5 + x_4^2, \tag{2.5}
\]
while that at node 3 would lead to
\[ x_3 x_3 = x_2 x_2^2 + x_1^3. \] (2.6)

We see that the right-hand sides of formulae (2.4) and (2.5) are related by a shift, while formula (2.6) is entirely different. In fact, it can be seen in figures 1 and 2 that at node 2 of quiver \( \tilde{S}_4 \) is exactly the same as that at node 1 of quiver \( S_4 \), thus giving the same exchange relation. In fact, we have more. The whole quiver \( \tilde{S}_4 \) is obtained from \( S_4 \) by just rotating the arrows, while keeping the nodes fixed. It follows that mutation of quiver \( \tilde{S}_4 \) at node 2 just leads to a further rotation, with node 3 inheriting this same configuration of arrows. If at each step we relabel \( \tilde{x}_n \) as \( x_{n+4} \), the \( n \)th exchange relation can be written as
\[ x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2. \] (2.7)

This rotational property of the quiver has led to an iteration, which in this case is just Somos 4.

In Fordy & Marsh [3], we introduced and studied quivers with this type of rotational property. Consider the \( N \times N \) matrix
\[
\rho = \begin{pmatrix}
0 & \cdots & \cdots & 1 \\
1 & 0 & \cdots & \\
\vdots & \ddots & \ddots & \\
1 & 0 & \cdots & 0
\end{pmatrix}.
\]

The above rotation, which we write \( S_4 \to \tilde{S}_4 = \rho S_4 \), is achieved in the matrix formulation by
\[
\tilde{B} = \rho B \rho^{-1},
\]
with \( N = 4 \) in this case.

Consider a quiver \( Q = Q(1) \), with \( N \) nodes. We consider a sequence of mutations, starting at node 1, followed by node 2 and so on. Mutation at node 1 of a quiver \( Q(1) \) will produce a second quiver \( Q(2) \). The mutation at node 2 will therefore be of quiver \( Q(2) \), giving rise to quiver \( Q(3) \) and so on. We define a period \( m \) quiver as follows.

**Definition 2.5.** A quiver \( Q \) has period \( m \) if it satisfies \( Q(m+1) = \rho^m Q(1) \) (with \( m \) the minimum such integer). The mutation sequence is depicted by
\[
Q = Q(1) \xrightarrow{\mu_1} Q(2) \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{m-1}} Q(m) \xrightarrow{\mu_m} Q(m+1) = \rho^m Q(1),
\]
and called the periodic chain associated with \( Q \).

The corresponding matrices would then satisfy \( B(m+1) = \rho^m B(1) \rho^{-m} \).

**Remark 2.6 (the sequence of mutations).** We must perform the correct sequence of mutations. For instance, if we mutate \( \mu_3 S_4 \) at node 3, we obtain a quiver that has five arrows from node 4 to node 1, which cannot be permutation equivalent to \( Q(1) = S_4 \). As we previously saw, the corresponding exchange relation (2.6) was also different.
Remark 2.7 (periodicity and iterations). Period 1 quivers correspond to iterations on the real line. Period m quivers correspond to iterations on $\mathbb{R}^m$. The formula (2.3) consists of only two terms (additively), corresponding to ‘incoming’ and ‘outgoing’ arrows. Both the Somos 4 and Somos 5 iterations can be built in this way, but ‘not’ Somos 6 or 7, which contains three terms.

In Fordy & Marsh [3], we give a full classification of period 1 quivers, a partial classification of period 2 quivers and examples of higher period ones.

(c) Primitive quivers

In our classification of period 1 quivers, we introduced a special class of quivers, called primitives. An important feature of a primitive is that node 1 is a sink, so only step 1 of the mutation (definition 2.1) is needed. We constructed a basis of primitives for each $N$ (see figure 3 for $N = 4, 5$). The basis consists of $P_N^{(r)}$, for $1 \leq r \leq N/2$.

In our classification, the primitives are the ‘atoms’ out of which we build the general period 1 quiver for each $N$. For given $N$, we can start with an arbitrary linear combination of $P_N^{(r)}$, with integer coefficients

$$Q_N^0 = \sum_{r=1}^{[N/2]} m_r P_N^{(r)}.$$  

If all the $m_r$ have the same sign, then $Q_N^0$ already has period 1, but, otherwise, we must add ‘correction terms’, which are integer combinations of primitives with $N - 2k$ nodes ($1 \leq k \leq [N/2]$). Our classification of period 1 quivers gives the formula for these coefficients in terms of the original coefficients $m_r$. For the Somos 4 quiver, we have $m_1 = 1, m_2 = -2$, and our formula requires the addition of a further two arrows between nodes 3 and 2 (figure 4).

(d) Laurent property versus complete integrability

Each iteration we obtain through our construction is guaranteed to have the Laurent property (by the results of [1]). However, only special cases are expected to be completely integrable in any sense (see [6] for various definitions).

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Figure 4. One of $P_{4}^{(1)}$ minus two of $P_{4}^{(2)}$ plus two of $P_{2}^{(1)}$ gives $S_{4}$.

For instance, the most general 4 node, period 1 quiver corresponds to the iteration

$$x_{n}x_{n+4} = x_{n+1}^{r} + x_{n+2}^{s}.$$  \hspace{1cm} (2.9)

The iterations with $r = 1, s \in \{0, 1, 2\}$ are analysed by Hone [7], who shows that these cases are Liouville integrable (even super-integrable when $s = 0, 1$).

We first write equation (2.9) as a map on the four-dimensional space with coordinates $x_{0}, x_{1}, x_{2}, x_{3},$

$$\varphi(x_{0}, \ldots, x_{3}) = \left( x_{1}, x_{2}, x_{3}, \frac{x_{1}^{r}x_{3}^{r} + x_{2}^{s}}{x_{0}} \right).$$  \hspace{1cm} (2.10)

The log-canonical Poisson bracket $\{x_{i}, x_{j}\} = P_{ij}x_{i}x_{j}$ (see [8] for a general discussion), where

$$P = \begin{pmatrix}
0 & r & s & r(1 + s) \\
r & 0 & r & s \\
s & -r & 0 & r \\
-r(1 + s) & -s & -r & 0
\end{pmatrix}$$  \hspace{1cm} (2.11)

is invariant under the action of the map $\varphi$. This means that if $\tilde{\mathbf{x}} = \varphi(\mathbf{x})$, then $\{\tilde{x}_{i}, \tilde{x}_{j}\} = P_{ij}\tilde{x}_{i}\tilde{x}_{j}$.

**Remark 2.8.** It is an interesting fact that the matrix $P$ is (up to an overall factor) the inverse of the $B$ matrix for the corresponding quiver. The factor is the Pfaffian $(2 + s)r^{2} - s^{2}$, which actually vanishes in the Somos 4 case. Nevertheless, the matrix $P$, with $r = 1, s = 2$, is still invariant under the map.

Liouville integrability is defined in the same way as for continuous Hamiltonian systems [6]. We must first use the Casimir functions (when the Poisson matrix is degenerate) to reduce to the symplectic leaves, whose dimension is $2d$, where $d$ is the number of degrees of freedom. We then require the existence of $d$, functionally independent Hamiltonians, $h_{1}, \ldots, h_{d}$, which should be in involution (so $\{h_{i}, h_{j}\} = 0$, for all $i, j$). For the discrete case, we have the extra requirement that the functions $h_{1}, \ldots, h_{d}$ are invariants of the map. This means that the map has a system of $d$ commuting continuous symmetries (the Hamiltonian flows). In the continuous case, we can say that the Hamiltonian system is solvable, up to quadrature, but this notion is not carried to the discrete case.
In Hone [7], it is shown that, for cases \( r = 1, s = 0,1 \), there are three independent, invariant functions, out of which it is possible to construct two Poisson-commuting functions. Such systems (with additional first integrals) are known as super-integrable. In the case \( r = 1, s = 2 \) (Somos 4), the Poisson bracket is degenerate, with two Casimir functions, which are not invariant under the map. However, the action of the map on these Casimirs is by a two-dimensional integrable map (a special case of the symmetric QRT map [9]).

It is not known whether the map (2.9) is Liouville integrable for other values of \( r \) and \( s \), but some of the standard integrability tests (such as algebraic entropy [10]) indicate non-integrability.

**Remark 2.9.** Isolating and analysing the integrable cases is one of the most interesting outstanding problems.

### 3. The \( P^{(1)}_N \) iteration as a map

The following iteration corresponds to the period 1 primitive \( P^{(1)}_N \) with \( N \) nodes

\[
x_n x_{n+N} = x_{n+1} x_{n+N-1} + 1,
\]

with initial conditions \( x_i = a_i \) for \( 0 \leq i \leq N - 1 \). In Fordy & Marsh [3], it was shown that there exists a special sequence of functions

\[
J_n = \frac{x_n + x_{n+2}}{x_{n+1}}, \quad \text{satisfying} \quad J_{n+N-1} = J_n.
\]

With the given initial conditions, we have \( \{J_i = c_i : 0 \leq i \leq N - 2\} \), together with the periodicity condition, which can also be written as \( J_n = c_n \) with \( c_{n+N-1} = c_n \).

**Theorem 3.1 (linearization).** If the sequence \( \{x_n\} \) is given by the iteration (3.1), with initial conditions \( \{x_i = a_i : 0 \leq i \leq N - 1\} \), then it also satisfies

\[
x_n + x_{n+2(N-1)} = S_N x_n x_{n+N-1},
\]

where \( S_N \) is a function of \( c_0, \ldots, c_{N-2} \), which is symmetric under cyclic permutations.

Here, we restrict to the case of even \( N \). The first few \( S_N \) take the form

\[
S_2 = c_0; \quad S_4 = c_0 c_1 c_2 - c_0 - c_1 - c_2
\]

and

\[
S_6 = c_0 c_1 c_2 c_3 c_4 - c_0 c_1 c_2 - c_1 c_2 c_3 - c_2 c_3 c_4 - c_3 c_4 c_0 - c_4 c_0 c_1 + c_0 + c_1 + c_2 + c_3 + c_4.
\]

The function \( S_N \) is an invariant function of the nonlinear map (3.1), so this linearization depends upon the particular initial conditions.

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(a) The log-canonical Poisson bracket

As in the fourth-order case (2.10), write the Nth-order iteration (3.1) as a map of the space with coordinates \((x_0, \ldots, x_{N-1})\), given by

\[
\varphi(x_0, \ldots, x_{N-1}) = \left( x_1, \ldots, x_{N-1}, \frac{x_1 x_{N-1} + 1}{x_0} \right).
\] (3.4)

Again we seek an invariant Poisson bracket of log-canonical form,

\[
\{x_i, x_j\} = P_{ij} x_i x_j, \quad 0 \leq i < j \leq N - 1,
\] (3.5)

for some constants \(P_{ij}\). We seek the value of these constants for which this Poisson bracket is invariant under the action of the map \(\varphi\). Writing \(\tilde{x} = \varphi(x)\), we require

\[
\{\tilde{x}_i, \tilde{x}_j\} = P_{ij} \tilde{x}_i \tilde{x}_j.
\]

The shift structure of the map (3.4) implies a banded structure, with \(P_{i+1,j+1} = P_{ij}\), so the undetermined constants are \(P_{0j}, j = 1, \ldots, N - 1\). The precise form of \(\tilde{x}_{N-1}\) puts strong constraints on these, which can be determined up to an overall multiplicative constant.

**Lemma 3.2.** For a non-trivial Poisson bracket of the form (3.5) to be invariant under the map (3.4), we require \(N\) to be ‘even’, in which case the coefficients take the form

\[
P_{ij} = \begin{cases} 1 & \text{when } i < j \text{ and } i + j \text{ is odd,} \\ 0 & \text{when } i < j \text{ and } i + j \text{ is even.} \end{cases}
\]

This Poisson bracket is non-degenerate.

**Remark 3.3.** Again, it is an interesting fact that the matrix \(P\) is (up to an overall factor) the inverse of the \(B\) matrix for the corresponding quiver.

(b) The Poisson algebra of functions \(J_m\)

The independent functions \(J_0, \ldots, J_{N-2}\), written in terms of the coordinates \((x_0, \ldots, x_{N-1})\), are given by

\[
J_m = \frac{x_m + x_{m+2}}{x_{m+1}}, \quad m = 0, \ldots, N - 3, \quad \text{and} \quad J_{N-2} = \frac{x_0 x_{N-2} + x_1 x_{N-1} + 1}{x_0 x_{N-1}}. \] (3.6)

Under the action of the map \(\varphi\), they satisfy the cyclic conditions

\[
J_n \circ \varphi = J_{n+1}, \quad n = 0, \ldots, N - 3 \quad \text{and} \quad J_{N-2} \circ \varphi = J_0. \] (3.7)

We just need to calculate the \(N - 3\) brackets \(\{J_0, J_n\}, n = 1, \ldots, N - 3\), since all others follow through the above relations. These are easily calculated to be

\[
\begin{align*}
\{J_0, J_1\} &= 2J_0 J_1 - 2, \\
\{J_0, J_{2m-1}\} &= 2J_0 J_{2m-1}, \quad 2 \leq m \leq M - 1, \\
\text{and} \quad \{J_0, J_{2m}\} &= -2J_0 J_{2m}, \quad 1 \leq m \leq M - 2,
\end{align*}
\] (3.8)

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where \( N = 2M \). The cyclic action of \( \phi \) then implies
\[
\begin{align*}
\{ J_m, J_{m+1} \} &= 2J_m J_{m+1} - 2, \quad 1 \leq m \leq N - 3, \\
\{ J_m, J_n \} &= 2(-1)^{m+n-1} J_m J_n \quad \text{for } 1 \leq m \leq n - 2 \leq N - 4
\end{align*}
\] (3.9)
and
\[
\{ J_0, J_{N-2} \} = -2J_0 J_1 + 2,
\]
where the relation for \{ \( J_0, J_{N-2} \) \} was obtained from that of \{ \( J_{N-3}, J_{N-2} \) \} through the action of \( \phi \).

By taking cyclic sums of any function of the \( J_n \), we can build functions that are invariant under the action of \( \phi \). Since the Poisson bracket (3.5) (with \( P_{ij} \) as given in lemma 3.2) is non-degenerate (on a \( 2M \)-dimensional space), our task is to select \( M \) invariant functions that are in involution. It is, in fact, easier to work with the Poisson bracket relations (3.8) and (3.9), which define a Poisson bracket on the \((2M - 1)\)-dimensional \( J \)-space. The corresponding Poisson matrix \( P \) is the sum of two homogeneous parts: \( P = P_2 + P_0 \), each of which is itself a Poisson matrix. These therefore define a compatible pair of Poisson brackets, as follows.

**Definition 3.4 (compatible Poisson brackets).** The matrices \( P_0, P_2 \) and \( P = P_2 + P_0 \) define compatible Poisson brackets
\[
\{ f, g \}_i = \nabla f P_i \nabla g, \quad i = 0, 2, \quad \text{and} \quad \{ f, g \}_P = \nabla f (P_2 + P_0) \nabla g.
\]

We use these brackets to define a *bi-Hamiltonian ladder* [11], starting with the Casimir function of \( P_0 \) and ending with that of \( P_2 \),
\[
(P_2 + P_0)(\nabla h_M - \nabla h_{M-1} + \nabla h_{M-2} - \cdots + (-1)^{M+1} \nabla h_1) = 0, \quad (3.10)
\]
where \( h_k \) is a homogeneous polynomial of degree \( 2k - 1 \). The homogeneity property of \( P_0, P_2, h_k \) leads to equation (3.10) decoupling into a sequence of \( M + 1 \) homogeneous equations (the bi-Hamiltonian ladder),
\[
P_0 \nabla h_1 = 0, \quad P_0 \nabla h_k = P_2 \nabla h_{k-1}, \quad \text{for } 2 \leq k \leq M, \quad \text{and} \quad P_2 \nabla h_M = 0. \quad (3.11)
\]
The functions \( h_1, h_M \) are easy to find from the form of the Poisson matrices,
\[
h_1 = \sum_{k=1}^{2M-1} J_k, \quad \text{and} \quad h_M = \prod_{k=1}^{2M-1} J_k. \quad (3.12)
\]
The remaining functions, \( h_2, \ldots, h_{M-1} \), are obtained by solving the ‘central’ sequence of equations (3.11). Since \( P_0 \) has a Casimir function, we need to check that the equations are compatible, in that \( \nabla h_1 P_2 \nabla h_{k-1} = 0 \). We use the following result.

**Lemma 3.5 (bi-Hamiltonian relations).** With the Poisson brackets given by definition 3.4, the functions \( h_1, \ldots, h_M \) satisfy
\[
\{ h_i, h_j \}_0 = \{ h_i, h_{j-1} \}_2 \quad \text{and} \quad \{ h_i, h_j \}_2 = \{ h_{i+1}, h_j \}_0.
\]
Proof. The ladder relations (3.11) imply
\[
\{h_i, h_j\}_0 = \nabla h_i P_0 \nabla h_j = \nabla h_i P_2 \nabla h_{j-1} = \{h_i, h_{j-1}\}_2,
\]
and
\[
\{h_i, h_j\}_2 = \nabla h_i P_2 \nabla h_j = -\nabla h_j P_2 \nabla h_i
\]
\[
= -\nabla h_j P_0 \nabla h_{i+1} = \nabla h_{i+1} P_0 \nabla h_j = \{h_{i+1}, h_j\}_0.
\]

Lemma 3.6 (compatibility of equations (3.11)). Equations (3.11) are compatible.

Proof. The compatibility condition \(\nabla h_1 P_2 \nabla h_{k-1} = 0\) is just \(\{h_1, h_{k-1}\}_2 = 0\). The first equation is just
\[
\{h_1, h_1\}_2 = 0,
\]
which is obviously satisfied, so it is possible to solve for \(h_2\). Now suppose we have functions \(h_1, \ldots, h_{k-1}\). The compatibility condition is
\[
\{h_1, h_{k-1}\}_2 = \{h_2, h_{k-1}\}_0 = \{h_2, h_{k-2}\}_2 = \cdots = \{h_s, h_s\}_\ell = 0,
\]
for some \(s, \ell\).

To solve equations (3.11), write the equations in terms of the coordinates \((J_0, \ldots, J_{2M-3}, z_1)\) (with \(z_1 = h_1\)), after which \(P_0\) has a complete row (and column) of zeros, with the non-zero part being invertible. The above compatibility means that the final entry in the column vector \(P_2 \nabla h_{k-1}\) is zero. These calculations are straightforward and give rise to a sequence of functions of \((J_0, \ldots, J_{2M-3}, z_1)\). These are only defined up to an additive function of \(z_1\), which can be discarded. Replacing \(z_1\) by \(h_1\), we obtain the desired functions of \((J_0, \ldots, J_{2M-3}, J_{2M-2})\). We then have the following theorem of Magri [11].

Theorem 3.7 (complete integrability). The functions \(h_1, \ldots, h_M\) are in involution with respect to both of the above Poisson brackets
\[
\{h_i, h_j\}_0 = \{h_i, h_j\}_2 = 0,
\]
and hence \(\{h_i, h_j\}_P = 0\).

It then follows from Liouville’s theorem that the functions \(h_1, \ldots, h_M\) define a completely integrable Hamiltonian system.

Proof. Without loss of generality, choose \(i < j\). Then,
\[
\{h_i, h_j\}_0 = \{h_i, h_{j-1}\}_2 = \{h_{i+1}, h_{j-1}\}_0 = \cdots = \{h_k, h_k\}_\ell = 0,
\]
for some \(k, \ell\). Similarly,
\[
\{h_i, h_j\}_2 = \{h_{i+1}, h_j\}_0 = \{h_{i+1}, h_{j-1}\}_2 = \cdots = \{h_k, h_k\}_\ell = 0,
\]
for some \(k, \ell\).

(i) The Casimir function

Formula (3.10) just states that the function
\[
C = h_M - h_{M-1} + h_{M-2} - \cdots + (-1)^{M+1} h_1
\]
(3.13)
is the Casimir of the Poisson matrix \(P\), so (with respect to \(\{,\}_P\)) commutes with each \(J_i\) and hence with all functions of \(J_i\) (not just with \(h_1, \ldots, h_M\)).
Example 3.8 (the case \( N = 4 \)). Here, we have three basic functions \( J_0, J_1 \) and \( J_2 \). With \( M = 2 \), we have \( h_1 \) and \( h_2 \), given by equation (3.12).

Example 3.9 (the case \( N = 6 \)). Here, we have five basic functions \( J_0, \ldots, J_4 \). With \( M = 3 \), we have \( h_1 \) and \( h_3 \), given by equation (3.12), and

\[
h_2 = J_0 J_1 J_2 + J_1 J_2 J_3 + J_2 J_3 J_4 + J_3 J_4 J_0 + J_4 J_0 J_1.
\]

Example 3.10 (the case \( N = 8 \)). Here, we have seven basic functions \( J_0, \ldots, J_6 \). With \( M = 4 \), we have \( h_1 \) and \( h_4 \), given by equation (3.12), and

\[
h_2 = \sum_{i=0}^{6} J_i J_{i+1} (J_{i+2} + J_{i+4}) \quad \text{and} \quad h_3 = \sum_{i=0}^{6} J_i J_{i+1} J_{i+2} J_{i+3} J_{i+4},
\]

the indices here being taken modulo 6.

Remark 3.11. It can be seen from the list following theorem 3.1 that \( S_4, S_6 \) (replacing \( c_i \) by \( J_i \)) are just \( C \) of the above examples. We can use the linear difference equation (3.3) to ‘define’ \( S_N \), repeatedly using the formula

\[
x_n x_{n+N} = x_n + x_{n+1} - x_{n+M-1} + 1
\]

to rewrite this as a function of \( x_0, \ldots, x_{N-1} \).

Conjecture: the Casimir function for general \( N \) can also be written as

\[
C = \frac{x_0 + x_{2(N-1)}}{x_{N-1}} \quad \text{written in terms of} \quad x_0, \ldots, x_{N-1}.
\]

(3.14)

4. The maps in canonical coordinates

The Poisson bracket (3.5), with the \( P_{ij} \) being given by lemma 3.2, naturally separates the odd- and even-numbered variables, from which we construct, respectively, canonical variables \( p_i \) and \( q_i \) as follows:

\[
q_i = \log(x_{2(i-1)}), \quad i = 1, \ldots, M
\]

(4.1) and

\[
p_1 = \frac{1}{2} \log(x_1 x_{N-1}), \quad p_i = \frac{1}{2} \log\left(\frac{x_{2i-1}}{x_{2i-3}}\right), \quad i = 2, \ldots, M,
\]

(4.2)

where \( N = 2M \). Defining

\[
\pi_r = \sum_{i=1}^{r} p_i - \sum_{i=r+1}^{M} p_i, \quad 0 \leq r \leq M - 1, \quad \pi_M = \sum_{i=1}^{M} p_i
\]

so \( \pi_i = \log(x_{2i-1}) \) the inverse of this transformation is written

\[
x_{2r} = e^{q_{r+1}}, \quad \text{and} \quad x_{2r+1} = e^{\pi_{r+1}}, \quad 0 \leq r \leq M - 1,
\]

(4.3) and the functions \( J_k \) take the form

\[
J_{2r} = e^{-\pi_{r+1}} (e^{q_{r+1}} + e^{q_{r+2}}), \quad 0 \leq r \leq M - 2,
\]

\[
J_{2r+1} = e^{-q_{r+2}} (e^{\pi_{r+1}} + e^{\pi_{r+2}}), \quad 0 \leq r \leq M - 2
\]

(4.4)

and

\[
J_{2M-2} = e^{-q_1-\pi_M} (e^{q_1+q_M} + e^{\pi_1+\pi_M} + 1).
\]
The map \( \varphi \) equation (3.4) is canonical, now having the form

\[
\bar{q}_r = \pi_r, \quad 1 \leq r \leq M \\
\bar{p}_1 = \frac{1}{2}(q_2 - q_1 + \log(1 + e^{2p_1})), \quad \bar{p}_M = \frac{1}{2}(-q_1 - q_M + \log(1 + e^{2p_1}))
\]

and

\[
\bar{p}_r = \frac{1}{2}(q_{r+1} - q_r), \quad 2 \leq r \leq M - 1.
\]

The variables \( \pi_r \) transform as

\[
\bar{\pi}_r = q_{r+1}, \quad 1 \leq r \leq M - 1, \quad \bar{\pi}_M = -q_1 + \log(1 + e^{(\pi_1 + \pi_M)}).
\]

The functions (4.4) inherit the cyclic behaviour (3.7) under this map.

Now consider the function

\[
C = \sum_{i=1}^{M-1} e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}}) + e^{-\pi_M}(e^{q_i} + e^{-q_M}) + e^{\pi_M - q_1}
\]

\[
= \sum_{i=1}^{M-1} e^{-q_{i+1}}(e^{-\pi_i} + e^{-\pi_{i+1}}) + e^{-q_1}(e^{-\pi_1} + e^{\pi_M}) + e^{-q_1 - \pi_M}.
\]

The second line is just a re-ordering of the first, but useful.

**Lemma 4.1 (symmetry under the map (4.5)).** Under the map (4.5), the function \( C \) is invariant: \( \bar{C} = C \).

**Proof.** Using equation (4.6), it is easy to show that

\[
e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}}) \rightarrow e^{-q_{i+1}}(e^{-\pi_i} + e^{-\pi_{i+1}})
\]

and

\[
e^{\pi_M - q_i} \rightarrow e^{-q_1}(e^{-\pi_1} + e^{\pi_M}), \quad e^{-\pi_M}(e^{q_i} + e^{-q_M}) \rightarrow e^{-q_1 - \pi_M},
\]

so the first line of equation (4.7) transforms to the second, giving the result.

**Theorem 4.2 (Casimir function).** The function \( C \) is a Casimir function for the Poisson algebra of functions \( J_i \).

**Proof.** First note that

\[
\{J_0, e^{-\pi_i}(e^{-q_i} + e^{-q_{i+1}})\} = \{J_0, e^{-\pi_M}(e^{q_i} + e^{-q_M})\} = \{J_0, e^{\pi_M - q_1}\} = 0,
\]

so \( \{J_0, C\} = 0 \). Since \( C \) is an invariant function under the map (4.5), this implies that \( \{J_r, C\} = 0 \), for all \( r \), giving the result.

**Remark 4.3.** We now have three expressions for the Casimir function of the \( J \) algebra ((3.13), (3.14) and (4.7)), which coincide on all known explicit examples. However, I have no proof that these are the same.
5. The Bäcklund transformation for Liouville’s equation

Here, we consider the Hamiltonian flows generated by the Casimir $C$ and the first Hamiltonian $h_1$. Suppose these flows are, respectively, parametrized by $x$ and $t$, so we have

$$f_x = \{ f, C \}, \quad f_t = \{ f, h_1 \},$$

for any function $f(q_1, \ldots, p_M)$. Since these Hamiltonians Poisson commute, their respective flows commute, so can be considered as coordinate curves on the level surface given by $C = c_1, h_1 = c_2$.

Consider the second-order partial derivative

$$q_{ixt} = \{ \{ q_i, C \}, h_1 \} = \{ \{ q_i, h_1 \}, C \}.$$

To calculate this in general, we need the formula

$$\{ q_i, \pi_j \} = \begin{cases} 1 & \text{if } i \leq j, \\ -1 & \text{if } i \geq j + 1. \end{cases}$$

First consider $q_1$. From the definitions (4.4), we have

$$\{ q_1, J_{2r} \} = -J_{2r}, \quad \{ q_1, J_{2r+1} \} = J_{2r+1}, \quad 0 \leq r \leq M - 2 \quad (5.1)$$

and

$$\{ q_1, J_{2M-2} \} = -J_{2M-2} + 2e^{\pi_1 - q_1}.$$

Since $C$ commutes with all $J_k$,

$$\{ \{ q_1, h_1 \}, C \} = 2\{ e^{\pi_1 - q_1}, C \}.$$

We have

$$\begin{cases} \{ e^{\pi_1 - q_1}, e^{-\pi_1}(e^{-q_1} + e^{-q_2}) \} = 2e^{-2q_1}, \\ \{ e^{\pi_1 - q_1}, e^{-\pi_1}(e^{-q_i} + e^{-q_{i+1}}) \} = 0, \text{ for } i \neq 1 \quad (5.2) \end{cases}$$

and

$$\begin{cases} \{ e^{\pi_1 - q_1}, e^{-\pi_M}(e^{q_1} + e^{-q_M}) \} = 0, \quad \{ e^{\pi_1 - q_1}, e^{\pi M - q_1} \} = 0, \end{cases}$$

so

$$q_{1xt} = \{ \{ q_1, h_1 \}, C \} = 4e^{-2q_1}.$$

Now act with $\phi$ on this equation (recalling the formulae (4.5) and (4.6)) to obtain Liouville’s equation for each of $q_i, \pi_i$,

$$q_{ixt} = 4e^{-2q_i} \quad \text{and} \quad \pi_{ixt} = 4e^{-2\pi_i}, \quad i = 1, \ldots, M. \quad (5.3)$$

The Bäcklund transformation for this equation is well known [12], but here we show how to construct it from our canonical transformations (4.5) and (4.6).

Again, first consider $q_1$ and $\tilde{q}_1 = \pi_1$. Looking at the formulae (5.2), we see that $q_1 - \pi_1$ commutes with all but one term in the expression (4.7) for $C$. The remaining term gives

$$\{ q_1 - \pi_1, C \} = \{ q_1 - \pi_1, e^{-\pi_1}(e^{-q_1} + e^{-q_2}) \} = -2e^{-q_1 - \pi_1}.$$
We now use equations (5.1), together with their consequence under the map
\[
\{\pi_1, J_0\} = J_0 - 2e^{q_1 - \pi_1}, \quad \{\pi_1, J_{2r}\} = J_{2r}, \text{ for } r \neq 0,
\]
and
\[
\{\pi_1, J_{2r+1}\} = -J_{2r+1}, \quad \{\pi_1, J_{2M-2}\} = J_{2M-2},
\]
so
\[
\{q_1 + \pi_1, h_1\} = 2(e^{\pi_1 - q_1} - e^{q_1 - \pi_1}).
\]
In summary, we have shown
\[
q_{1x} - \tilde{q}_{1x} = -2e^{-q_1 - \tilde{q}_1} \quad \text{and} \quad q_{1t} + \tilde{q}_{1t} = 2(e^{q_1 - q_1} - e^{\tilde{q}_1 - q_1}), \quad (5.4)
\]
which is the Bäcklund transformation for Liouville’s equation (5.3) (for \(i = 1\)).

Again, act with \(\phi\) on these equations to obtain
\[
q_{ix} - \pi_{ix} = -2e^{-q_1 - \pi_1}, \quad q_{it} + \pi_{it} = 2(e^{\pi_1 - q_1} - e^{q_1 - \pi_1}), \quad i = 1, \ldots, M,
\]
\[
\pi_{ix} - q_{i+1x} = -2e^{-\pi_1 - q_{i+1}}, \quad i = 1, \ldots, M - 1
\]
and
\[
\pi_{it} + q_{i+1t} = 2(e^{\pi_1 - q_{i+1}} - e^{q_1 - q_{i+1}}), \quad i = 1, \ldots, M - 1.
\]

We can act again with \(\phi\), but the calculation is more complicated, since it now involves \(\tilde{\pi}_M\) (equation (4.6)). We get a relationship involving derivatives of three variables (\(\pi_M, q_1\) and \(\pi_1\)), but use equation (5.4) to eliminate derivatives of \(\pi_1 = \tilde{q}_1\) to obtain
\[
\pi_{Mx} + q_{1x} = -2(e^{q_1 - \pi_1} - e^{\pi_1 - q_1}) \quad \text{and} \quad \pi_{Mt} - q_{1t} = 2e^{-\pi_1 - q_1}.
\]
Notice that \(x, t\) seem to have reversed their roles at this step. However, the next action of \(\phi\) (again requiring more complicated manipulations) brings us full circle to the original formulae (5.4) for \(q_1, \pi_1\).

6. Conclusions

In Fordy & Marsh [3], a new class of quiver, with a certain periodicity property, was introduced and partially classified. The corresponding cluster mutation relations give rise to iterations with the Laurent property. An important open question is the classification of the subclass of such iterations that define Liouville integrable maps. The main content of this paper is the study of one particular family of such maps. The question of integrability for the general class is considered in Fordy & Hone [13].

In Fordy & Marsh [3], we noted a surprising connection between our examples of periodic quivers and those that arise in the context of quiver gauge theories [14]. Unfortunately, I had no space to describe this here, but an explanation of this is also an important open question.

This brings us back, finally, to Robin Bullough’s famous diagram. Some new boxes and connections are needed to incorporate the subject of this paper, but that is the nature of this diagram, which will grow indefinitely and become more and more complex as new discoveries are made.
Mutation-periodic quivers

References