I summarize various tests of general relativity on astrophysical scales, based on the large-scale structure of the universe but also on other systems, in particular the constants of physics. I emphasize the importance of hypotheses on the geometric structures of our universe while performing such tests and discuss their complementarity as well as their possible extensions.

Keywords: general relativity; equivalence principle; constants

1. Introduction: gravity and the cosmological model

(a) Dark energy models

During the past decade, the need to test general relativity on astrophysical scales has been boosted by the quest to understand the physical origin of the late time acceleration of the cosmic expansion, as now established by various cosmological probes. This is, however, only the continuation of what has been performed in the Solar System for about a century in order to extend, as much as possible, the domain of validity of this theory of gravitation (see Uzan [1], and Einsenstaedt [2] for a historical perspective).

The main difficulty in extending the tests of general relativity performed in the Solar System to astrophysical and cosmological scales is that our knowledge of the universe (such as the determination of distances, times, distribution of matter, etc.) depends strongly on the construction of a cosmological model [3,4]. Such a construction relies on four main hypotheses:

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(H1) a theory of gravity,
(H2) a description of the matter contained in the universe and the non-
gravitational interactions,
(H3) a symmetry hypothesis, and
(H4) a hypothesis on the global structure, i.e. the topology, of the universe.

These hypotheses are indeed not on the same footing. H1 and H2 refer to the
physical theories and are not sufficient as their equations cannot be solved
without making simplifying assumptions such as the symmetries (H3) of the
solutions describing our universe on large scales. H4 is then an assumption
on some global properties of these cosmological solutions, with the same
local geometry.

Our reference cosmological model, the $Λ$CDM ($Λ +$ cold dark matter) model,
assumes that gravity is described by general relativity (H1) and that the universe
contains the fields of the standard model of particle physics plus some dark matter
and a cosmological constant, the latter two having no physical explanation at the
moment. Note that in the cosmological context this involves an extra assumption
since what will be required by the Einstein equations is the effective stress–energy
tensor of matter averaged on large scales. It thus implicitly refers to an (not
defined) averaging procedure [5]. It deeply involves the Copernican principle (H3),
without which the Einstein equations cannot be solved, and also assumes that
the spatial sections are simply connected (H4). H2 and H3 also imply that the
description of standard matter reduces to a mixture of pressureless and radiation
perfect fluids.

(b) Checking the hypotheses

The number of models to account for dark energy and dark matter has been
flourishing with the main problem that most of them contain free functions that
can be tuned in such a way that their predictions are infinitesimally closed to
those of a $Λ$CDM model. It is thus, in general, impossible to rule them out, but
just to exhibit their high level of fine-tuning and their need for extra structure
for little gain in explanation. Accepting one of these models would require much
more than a better fit to data.

This has motivated the developments of tests of the four cosmological
hypotheses instead of working out the cosmological signatures of many models.
These tests are null tests in the sense that any significant violation would
indicate the need to extend at least one of these hypotheses. They would, in
particular, indicate whether the new structures to consider are physical degrees of
freedom or geometrical degrees of freedom (e.g. going beyond a pure homogeneous
space–time).

Among these tests, it has been shown that general relativity can be tested
by using observations of the large-scale structure of the universe [1,6], as
first proposed by Uzan & Bernardeau [7] ten years ago or by studying the
fundamental constants [8–11]. We shall discuss them below. A test of Maxwell
electromagnetism through the distance duality relation has also been constructed
by Uzan et al. [12]. Several tests of the Copernican principle have been recently
proposed [13–16] and the topology of the universe can be tested using the cosmic
microwave background [17–20].
These tests have been related to the typical signatures of universality classes of dark energy models based on the type of new degrees of freedom and their couplings (figure 1a). It is important to perform all these tests together since most (if actually not all in their present form) tests of general relativity implicitly assume the validity of the Copernican principle (see Uzan [4]), probably their most severe limitation.

2. Testing and modifying general relativity

(a) General relativity and its Solar System tests

Einstein’s general relativity is based on two independent hypotheses. First, it rests on the Einstein equivalence principle, which includes the universality of free fall, local position and local Lorentz invariances. We refer to Will [22] for a detailed description of these principles. In its weak form, the equivalence principle can be mathematically implemented by assuming that all matter fields, including gauge bosons, are minimally coupled to a single metric tensor $g_{\mu \nu}$. This metric defines
the lengths and times measured by laboratory rods and clocks so that it is often called the physical metric. This implies that the action for any matter field, $\psi$, say, is of the form

$$S_{\text{matter}}[\psi, g_{\mu\nu}].$$

This so-called metric coupling ensures, in particular, the validity of the universality of free fall. The second building block of general relativity is related to the dynamics of the gravitational sector, assumed to be dictated by the Einstein–Hilbert action

$$S_{\text{gravity}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R,$$

where $g_{\mu\nu}^*$ is a massless spin-2 field called the Einstein metric. General relativity assumes that both metrics coincide, $g_{\mu\nu} = g_{\mu\nu}^*$ (which implements the equivalence principle in its strong form). It is indeed possible to design theories in which this is not the case (e.g. scalar–tensor theories), so that general relativity is one out of a large family of metric theories. The variation of the total action with respect to the metric yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $T_{\mu\nu} \equiv (2/\sqrt{-g}) \delta S_{\text{matter}}/\delta g_{\mu\nu}$ is the matter stress–energy tensor. The coefficient $8\pi G/c^4$ is determined by the weak-field limit of the theory that should reproduce the Newtonian predictions.

The assumption of a metric coupling (2.1) is actually well tested in the Solar System. First, it implies that all non-gravitational constants are space–time independent, which has been tested to a very high accuracy in many physical systems (see below). Second, isotropy has been tested from the constraint on the possible quadrupolar shift of nuclear energy levels [23] proving that different matter fields couple to a unique metric tensor at the $10^{-27}$ level. Third, the universality of free fall can be tested by comparing the acceleration of two test bodies in an external gravitational field. The parameter $\eta_{12} \equiv 2|a_1 - a_2|/|a_1 + a_2|$ can be constrained experimentally, e.g. by comparison of Earth-core-like and Moon-mantle-like bodies gave [24] $\eta_{\text{Earth, Moon}} = (0.1 \pm 2.7 \pm 1.7) \times 10^{-13}$. The lunar laser ranging (LLR) experiment [25], which compares the relative acceleration of the Earth and the Moon in the gravitational field of the Sun, also set the constraints $\eta_{\text{Earth, Moon}} = (-1.0 \pm 1.4) \times 10^{-13}$. Since the core represents only 1/3 of the mass of the Earth, and since the Earth mantle has the same composition as the Moon, one loses a factor 3 so that this constraint is actually similar to the one obtained in the laboratory. Further constraints are summarized in §2.2.1 of Uzan [11]. The latter bound also contains a contribution from the gravitational binding energy, so that it constrains the strong equivalence principle. When the laboratory results are combined with the LLR, one gets a constraint on the strong equivalence principle parameter $\eta_{\text{SEP}} = (3 \pm 6) \times 10^{-13}$. Fourth, the Einstein effect (or gravitational redshift) has been measured at the $2 \times 10^{-4}$ level [26].

The dynamics of general relativity can be tested in the Solar System by using the parametrized post-Newtonian (PPN) formalism [22]. The formalism assumes that gravity is described by a metric and that it does not...
involve any characteristic scale. In its simplest form, it reduces to the two Eddington parameters entering the metric of the Schwarzschild metric in isotropic coordinates

\[ g_{00} = -1 + \frac{2Gm}{rc^2} + 2\beta_{\text{PPN}} \left( \frac{2Gm}{rc^2} \right)^2 \quad \text{and} \quad g_{ij} = \left( 1 + 2\gamma_{\text{PPN}} \frac{2Gm}{rc^2} \right) \delta_{ij}. \]

Indeed, general relativity predicts \( \beta_{\text{PPN}} = \gamma_{\text{PPN}} = 1 \). These two phenomenological parameters are constrained (i) by the shift of the Mercury perihelion [27] which implies that \( |2\gamma_{\text{PPN}} - \beta_{\text{PPN}} - 1| < 3 \times 10^{-3} \), (ii) by the LLR experiment [25] which implies that \( |4\beta_{\text{PPN}} - \gamma_{\text{PPN}} - 3| = (4.4 \pm 4.5) \times 10^{-4} \), and (iii) by the deflection of electromagnetic signals which are all controlled by \( \gamma_{\text{PPN}} \). For instance, the very long baseline interferometry [28] implies that \( |\gamma_{\text{PPN}} - 1| = 4 \times 10^{-4} \) while the measurement of the time-delay variation to the Cassini spacecraft [29] sets \( \gamma_{\text{PPN}} - 1 = (2.1 \pm 2.3) \times 10^{-5} \). The PPN formalism does not allow testing finite range effects that could be caused, for example, by a massive degree of freedom. In that case, one expects a Yukawa-type deviation from the Newton potential, of amplitude \( \alpha \) on a characteristic scale \( \lambda \). The constraints on \((\lambda, \alpha)\) are summarized in Hoyle et al. [30] which typically shows that \( \alpha < 10^{-2} \) on scales ranging from a millimetre to Solar System size.

(b) In which regime and how can we modify general relativity?

(i) Beyond the Solar System

Before investigating gravity beyond general relativity, let us try to sketch the regimes in which these modifications may (or shall) appear. We can distinguish the following regimes.

— Weak–strong field regimes can be characterized by the amplitude of the gravitational potential. For a spherical static space–time, \( \Phi = GM/rc^2 \). It is of the order of \( \Phi_\odot \sim 2 \times 10^{-6} \) at the surface of the Sun and equal to 1/2 for a black hole.

— Small–large distances. Such modifications can be induced by a massive degree of freedom that will induce a Yukawa-like coupling. While constrained on the size of the Solar System, there is no constraint on scales larger that \( 10 \text{h}^{-1} \text{Mpc} \).

— Low–high acceleration regimes are of importance to discuss galaxy rotation curves and (galactic) dark matter, as suggested by the modified Newtonian dynamics (MOND) phenomenology [31] mostly because the Tully–Fischer law tells that dark matter cannot be explained by a modification of general relativity at a fixed distance.

— Low–high curvature regimes distinguish possible extensions of the Einstein–Hilbert action. For instance, a quadratic term of the form \( \alpha R^2 \) becomes significant compared with \( R \) when \( GM/r^3c^2 \gg \alpha^{-1} \) even if \( \Phi \) remains small. In the Solar System, \( R_\odot \sim 4 \times 10^{-28} \text{cm}^{-2} \).

In cosmology, there are various possible regimes in which one may modify general relativity. The dark matter problem can be accounted
for by a modification of Newton gravity below the typical acceleration
\(a_0 \sim 10^{-8} \text{ cm s}^{-2}\) \([31]\). This corresponds to
\[
\Phi R < a_0^2 \sim 3 \times 10^{-31} R_\odot.
\] (2.4)

The curvature of the homogeneous universe is, using the Friedmann equation,
\[
R_{\text{FL}}(z) = 3H_0^2[\Omega_{\text{mat}0}(1 + z)^3 + 4\Omega_{\Lambda0}],
\] (2.5)
from which we deduce that \(R_{\text{FL}} \sim 10^{-5} R_\odot\) at the time of nucleosynthesis, \(R_{\text{FL}} \sim 10^{-20} R_\odot\) at the time of decoupling and \(R_{\text{FL}} \sim 10^{-28} R_\odot\) at \(z = 1\). The curvature scale associated with the cosmological constant being \(R_\Lambda = (1/6)A\), the cosmological constant problem corresponds to a low curvature regime,
\[
R < R_\Lambda \sim 1.2 \times 10^{-30} R_\odot.
\] (2.6)

The fact that the limits (2.4) and (2.6) intersect illustrates the coincidence problem, that is \(a_0 \sim cH_0\) and \(\Omega_{\text{mat}0} \sim \Omega_{\Lambda0}\). Note that both arise on curvature scales much smaller than those probed in the Solar System (figure 1b).

For cosmological perturbations, the gravitational potential at the time of decoupling \((z \sim 10^3)\) is of the order of \(\Phi \sim 10^{-5}\). During the matter era, the Poisson equation imposes that \(\Phi\) remains almost constant so that we never expect a potential larger than \(\Phi \sim 10^{-5}\) on cosmological scales. We are thus always in a weak field regime. The characteristic distance scale is fixed by the Hubble radius \(c/H_0\). The curvature perturbation associated with the large-scale structures is, in the linear theory, of the order of \(\delta R = (6/a^2)\Delta \Phi \sim 3H_0^2 \Omega_{\text{mat}0}(1 + z)^3 \delta_{\text{mat}}(z)\).

Since at redshift zero, \(\langle \delta_{\text{mat}}^2 \rangle = \sigma_s \sim 1\) in a ball of radius of 8 Mpc, we conclude that \(\langle \delta R^2 \rangle^{1/2} \sim 3H_0^2 \Omega_{\text{mat}0} \sigma_s\), while \(R_{\text{FL}} = 3H_0^2 \Omega_{\text{mat}0}\) if \(\Lambda = 0\). This means that the curvature perturbation becomes of the order of the background curvature at a redshift \(z \sim 0\), even if we are still in the weak field limit. This implies that the effect of large-scale structures on the background dynamics may be non-negligible. This effect has been argued to possibly be at the origin of the acceleration of the universe \([5]\), but no convincing formalism to describe this backreaction has been constructed yet. Note that in this picture, the onset of the acceleration phase is determined by the amplitude of the initial power spectrum.

In conclusion, to address the dark energy or dark matter problem by a modification of general relativity, we are interested in modifications on large scales (typically Hubble scales), low acceleration (below \(a_0\)) or small curvature (typically \(R_\Lambda\)), as summarized in figure 1b.

(ii) Theoretical constructions

In modifying general relativity, we shall demand the following of the new theory. (i) It does not contain ghosts, i.e. degrees of freedom with negative kinetic energy. The problem with such a ghost is that the theory would be unstable. In particular, the vacuum can decay to an arbitrary amount of positive energy (standard) gravitons whose energy would be balanced by negative energy ghosts. (ii) It has a Hamiltonian bounded from below. Otherwise, the theory would be unstable, even if one cannot explicitly identify a ghost degree of freedom. (iii) The new degrees of freedom are not tachyon, i.e. do not have a negative mass. (iv) It is compatible with local tests of deviation from general relativity, in particular in the Solar System.
Then, we can either modify the Einstein–Hilbert action while leaving the coupling of all matter fields to the metric unchanged, or modify the coupling(s) in the matter action. The possibilities are numerous [6,22,32] and we cannot give an extensive review of the models here. We shall thus mention the different possibilities.

— Modifying the Einstein–Hilbert action. The simplest example of higher order gravity models based on a quadratic action is

$$S_{\text{gravity}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} [R + \alpha C_{\mu\nu\rho\sigma}^2 + \beta R^2 + \gamma GB], \quad (2.7)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $GB \equiv R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$ is the Gauss–Bonnet term. $\alpha$, $\beta$ and $\gamma$ are three constants with dimension of an inverse mass square. GB does not contribute to the local field equations of motion. The action (2.7) gives a renormalizable theory of quantum gravity at all orders provided $\alpha$ and $\beta$ are non-vanishing; however, such theories contain ghosts since the graviton propagator $1/(p^2 + \alpha p^4)$ can be decomposed as $1/(p^2 + \alpha p^4) = 1/p^2 - 1/(p^2 + (1/\alpha))$. The first term is nothing but the standard propagator of the usual massless graviton. The second term correspond to an extra-massive degree of freedom with mass $\alpha^{-1}$ and its negative sign indicates that it carries negative energy: it is a ghost. Moreover, if $\alpha$ is negative, this ghost is also a tachyon! The only viable such modification arises from $\beta R^2$, which introduces a massive spin-0 degree of freedom. This conclusion can be generalized to theories involving an arbitrary function of the metric invariants, $f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma})$. They are generically not stable theories with the exception of $f(R)$ theories.

— Modifying the matter action. Many other possibilities, known as bi-metric theories of gravity, arise if one assumes that $g_{\mu\nu} \neq g_{\mu\nu}^*$. Instead one can postulate that the physical metric is a combination of various fields, e.g. $g_{\mu\nu} = g_{\mu\nu}^*[\phi, A_\mu, B_{\mu\nu}, \ldots] = A^2(\phi)[g_{\mu\nu}^* + \alpha_1 A_\mu A_\nu + \alpha_2 g_{\mu\nu}^* g^{\alpha\beta} A_\alpha A_\beta + \ldots]$. As long as these new fields enter quadratically, their field equation is $\Box A = AT$ (where $T$ is the matter source and $\Box$ is the d’Alembertian operator), so that matter cannot generate them if their background value vanishes. On the other hand, if their background value does not vanish then these fields define a preferred frame and Lorentz invariance is violated. Such modifications have, however, drawn some attention, in particular in the construction of field theories reproducing the MOND phenomenology [31]. Their mathematical consistency and stability were investigated in depth in the excellent analysis of Esposito-Farèse & Bruneton [32] who argue that no present theory passes all available experimental constraints while being stable and admitting a well-posed Cauchy problem. They also notice that while couplings of the form $g_{\mu\nu}[g_{\mu\nu}^*, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \ldots]$ seem to lead to well-defined theories in vacuum (in particular when linearizing around a Minkowsky background), they are unstable inside matter. The case in which only a scalar partner, $g_{\mu\nu} = A^2(\phi)g_{\mu\nu}^*$, is introduced leads to consistent field theories and is the safest way to modify the matter coupling.
Higher dimensional theories of gravity, including string theory, predict non-metric coupling as those discussed in §1. Many scalar fields, known as moduli, appear in the dimensional reduction to four dimensions. As a simple example, let us consider a five-dimensional space–time and assume that gravity is described by the Einstein–Hilbert action

\[ S = \frac{1}{12\pi^2 G_5} \int \tilde{R} \sqrt{|\tilde{g}|} \, d^5 x, \]  

(2.8)

where we denote by a bar quantities in five dimensions to distinguish them from the analogous quantities with no bar in four dimensions. Decomposing the metric into a symmetric tensor part, \( g_{\mu\nu} \), with 10 independent components, a vector part, \( A_\alpha \), with four components and finally a scalar field, \( \phi \), to complete the counting of the number of degrees of freedom \((15 = 10 + 4 + 1)\) and compactifying on a circle, the action (2.8) reduces to the four-dimensional action

\[ S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \phi \left( R - \frac{\phi^2}{4M^2} F_{\alpha\beta} F^{\alpha\beta} \right), \]  

(2.9)

where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) and with \( G = 3\pi \tilde{G}_5 / 4V_5 \). The scalar field couples explicitly to the kinetic term of the vector field. This coupling cannot be eliminated by a redefinition of the metric and induces a variation of the fine-structure constant as well as a violation of the universality of free fall [8,11]. Such dependencies of the masses and couplings are generic for higher dimensional theories and in particular string theory. While these tree-level predictions of string theory are in contradiction with experimental constraints, many mechanisms can reconcile it with experiment. In particular, quantum loop corrections may modify the coupling in such a way that it has a minimum [33]. The scalar field can thus be attracted towards this minimum during the cosmological evolution, so that the theory is attracted towards general relativity. Another possibility is to invoke an environmental dependence, as can be implemented by the chameleon mechanism [34].

3. Equivalence principle

The first aspect of general relativity that can be tested on astrophysical scales is the equivalence principle. The connection lies in the fact that the mass of any composite body, starting for example from nuclei, includes the mass of the elementary particles that constitute it (this means that it will depend on the Yukawa couplings and on the Higgs sector parameters), but also a contribution, \( E_{\text{binding}} / c^2 \), arising from the binding energies of the different interactions (i.e. strong, weak and electromagnetic) including gravitational for massive bodies. Thus, the mass of any body is a complicated function of all the constants, \( m[\alpha_i] \).

The action for a point particle is given by

\[ S_{\text{matter}} = - \int m_A[\alpha_j] c \sqrt{-g_{\mu\nu}(x)} v^\mu v^\nu \, dt, \]  

(3.1)
where $\alpha_j$ is a list of constants and $A$ in $m_A$ recalls that the dependency in these constants is \textit{a priori} different for bodies of different chemical composition. Variation of this action gives the equation of motion 

$$u^\nu \nabla_\nu u^\mu = - \left( \sum_i \frac{\partial \ln m_A}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial x^\beta} \right) (g^\beta \mu + u^\beta u^\mu). \quad (3.2)$$

A test body will thus not enjoy a geodesic motion. In the Newtonian limit $g_{00} = -1 + 2\Phi_N/c^2$, and at first order in $v/c$, the equation of motion of a test particle reduces to

$$a = g_N + \delta a_A \quad \text{and} \quad \delta a_A = -c^2 \sum_i f_{A,i} (\nabla \alpha_i + \dot{\alpha}_i \frac{v_A}{c^2}), \quad (3.3)$$

where we have introduced the sensitivity of the mass $A$ with respect to the variation of the constant $\alpha_i$ by $f_{A,i} = \frac{\partial \ln m_A}{\partial \alpha_i}$. This shows that if the constants depend on time then there must exist an anomalous acceleration that will depend on the chemical composition of the body $A$. The anomalous acceleration is generated by the change in the binding energies, in the Yukawa couplings and in the Higgs sector parameters, so that the $\alpha_i$-dependencies are \textit{a priori} composition-dependent. Hence, any variation of the fundamental constants will entail a violation of the universality of free fall: the total mass of the body being space-dependent, an anomalous force appears if energy is to be conserved.

Various systems have been used to set constraints on the time variation of the fundamental constants, and in particular on the fine-structure constant (figure 2a). All the constraints and relations with the modifications of general relativity are extensively discussed in Uzan [8,11]. An indication that the numerical value of any constant has drifted during the cosmological evolution would be a sign in favour of models of the classes C and D.

4. Large-scale structure of the universe

Two roads can be followed to constrain the deviations from general relativity. Either one defines a class of gravity models that contain general relativity in some limit and then confronts it to cosmological data to determine how close to general relativity, in this particular space of theories, the theory of gravity should sit. We first discuss the example of scalar–tensor theories. Or, one tries to quantify the allowed deviations from the reference model while being as much as can be model-independent. The strategy is then to exhibit consistency relations between different observables, which must hold in our $\Lambda$CDM reference model. This was first proposed in the particular case of the Poisson equation in Uzan & Bernardeau [7].

(a) Model-based constraints: example of scalar–tensor theories

The case in which only a scalar partner to the graviton is introduced leads to consistent field theories and is the safest way to modify the matter coupling. Gravity is then mediated not only by a massless spin-2 graviton, but also by
a spin-0 scalar field that couples universally to matter fields (this ensures the universality of free fall). In the Einstein frame, the action takes the form

$$S = \frac{1}{16\pi G_\ast} \int d^4x \sqrt{-g_\ast} \left[ R_\ast - 2g_\ast^{\mu\nu} \partial_\mu \varphi_\ast \partial_\nu \varphi_\ast - 4V(\varphi_\ast) \right] + S_\text{matter}[A^2(\varphi_\ast) g_\mu^\nu; \psi],$$

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where \( A(\varphi_s) \) is a coupling function. The kinetic terms have been diagonalized, so that the spin-2 and spin-0 degrees of freedom of the theory are perturbations of \( g_{\mu\nu}^* \) and \( \varphi_s \), respectively.

This action defines an effective gravitational constant \( G_{\text{eff}} = G_* A^2 \). This constant does not correspond to the gravitational constant effectively measured in a Cavendish experiment that is given by

\[
G_{\text{cav}} = G_* A_0^2 (1 + \alpha_0^2), \tag{4.1}
\]

where the first term \( G_* A_0^2 \) corresponds to the exchange of a graviton, while the second term \( G_* A_0^2 \alpha_0^2 (\alpha(\varphi_s) \equiv \frac{d\ln A}{d\varphi_s} \) characterizing the coupling of the scalar field to matter) is related to the long-range scalar force.

In such a case, one can improve tests of the deviations from general relativity by combining the local constraints with cosmological constraints arising from big bang nucleosynthesis [35,36], cosmic microwave background [37], weak lensing [38] and equation of state obtained from supernovae [39,40]. This allows with to combine the properties of the homogeneous solution (attraction mechanism), large-scale structure both in the linear and nonlinear regime.

\[ \text{(b) Model-independent test of general relativity} \]

(i) Original idea

The first idea to test the predictions of general relativity using the large-scale structure of the universe, and in particular, galaxy catalogue and the then very recent weak-lensing observations, was proposed in 2001 in Uzan & Bernardeau [7] by Francis Bernardeau and myself.

Extracting constraints on deviations from general relativity is difficult mostly because large-scale structures entangle the properties of matter and gravity. On sub-Hubble scales however, one can construct tests reproducing those in the Solar System.

This relies on the very simple idea that if gravity is well described by general relativity then, on sub-Hubble scales, focusing only on scalar perturbations, the space–time metric can be written as

\[
ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Psi) a^2(t) \gamma_{ij} dx^i dx^j, \tag{4.2}
\]

where \( \Phi \) and \( \Psi \) are the two gravitational potentials. The perturbation equations then reduce to the conservation of the matter stress–energy tensor (continuity and Euler equations)

\[
\dot{\delta}_m = -\frac{\theta_{\text{mat}}}{a} \quad \text{and} \quad \dot{\theta}_{\text{mat}} + H \theta_{\text{mat}} = -\frac{1}{a} \Delta \Phi. \tag{4.3}
\]

The Einstein equations reduce to the Poisson equation

\[
\Delta \Psi = 4\pi G \rho_{\text{mat}} a^2 \delta_{\text{mat}} \tag{4.4}
\]

and

\[
\Phi - \Psi = 0, \tag{4.5}
\]

which arises from the fact that the matter anisotropic stress is negligible.

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This shows that the two gravitational potentials have to coincide, which is related to the fact that $\gamma_{\text{PPN}}=0$ in general relativity, and their spectrum has to be proportional to the matter power spectrum, as first explained in Uzan & Bernardeau [7]. If the Poisson equation holds, then the power spectra of the gravitational potential $P_\Phi(k)$ and of the matter distribution $P_m(k)$ must be related by

$$k^4 P_\Phi(k, a) = \frac{9}{4} \Omega_{\text{mat}}(H^2_0 a^{-2} P_{\text{mat}}(k, a),$$

(4.6)

whatever the cosmological scenario. This means that the scale dependence of the two spectra are related in a very specific way. In particular, if the Poisson equation is modified, the change of shape of the matter power spectrum is model-dependent, but the fact that the two spectra differ is a model-independent conclusion. In particular, such a relation can be tested by comparing weak lensing data to galaxy surveys, if the scale dependence of the bias is mild, as expected from numerical simulation, since $C_{\text{kk}}$ and $P_{\text{gg}}$ give access to $P_\Phi$ and $P_{\text{mat}}$. Note also that it has a trivial generalization: if the fields are all proportional to the same stochastic variable (which is the case for adiabatic initial conditions), then $P_\Phi P_{\text{mat}} = \sqrt{P_{\Phi} P_{\text{mat}}}$, which again can be tested using galaxy–galaxy lensing.

Let us recall that the deviation from the standard behaviour of the matter power spectrum is model-dependent (it depends, in particular, on the cosmological parameters), but that the discrepancy between the matter and gravitational potential Laplacian power spectra is a direct signature of a modification of general relativity. The main limitation is the little control we have on the clustering of dark matter and on its bias.

(ii) **Towards a post-$\Lambda$CDM parametrization**

In order to go further than this original idea, one needs to construct the most general extension of this set of equations, still assuming that we are dealing with a metric theory of gravity. This idea to construct such a post-$\Lambda$CDM parametrization on sub-Hubble scales was first proposed in Uzan [6], following the analysis of the particular case of scalar–tensor theory by Schimd *et al.* [38] (where the form of the functions appearing below was explicitly given).

As long as we stick to the linear regime, these extensions can be implemented by modifying the previous equations as

$$\dot{\theta}_{\text{mat}} = -\frac{\theta_{\text{mat}}}{a} \quad \text{and} \quad \dot{\theta}_{\text{mat}} + H\theta_{\text{mat}} = -\frac{1}{a} \Delta \Phi + S_{\text{de}}.$$  

(4.7)

The term $S_{\text{de}}(k, a)$ encodes the new long-range force between the new degree of freedom and standard matter (and dark matter!). Then, we need to write down the Einstein equations. First, we can generalize the Poisson equation, written in Fourier space, as

$$-k^2 \Psi = 4\pi G F(k, H) \rho_{\text{mat}} a^2 \delta_{\text{mat}} + \Delta_{\text{de}}.$$  

(4.8)

The first term $F(k, H)$ accounts for a scale dependence of the gravitational interaction, while $\Delta_{\Phi}$ accounts for a possible clustering of the new degree of freedom, and in particular of dark energy if there is no modification of general relativity (this shows at this stage that the Poisson equation can be modified without modification of general relativity if dark energy can cluster; also care
needs to be taken in the case of massive neutrinos that can enter on the r.h.s. of this equation). Then, there is the possibility to have an effective anisotropic stress so that
\[ \Delta(\Phi - \Psi) = \Pi_{\text{de}}. \]  
(4.9)
It follows that the deviation from general relativity is encoded in the four functions \((S_{\text{de}}, F, \Delta_{\text{de}} \text{ and } \Pi_{\text{de}})\) which, in the case of the \(\Lambda\)CDM model, reduces to \((0, 1, 0, 0)\) and in the case of dark energy to \((0, 1, \Delta_{\text{de}}, \Pi_{\text{de}})\), even though in most cases \(\Delta_{\text{de}}\) and \(\Pi_{\text{de}}\) are negligible. Their expression for quintessence, scalar–tensor and Dvali–Gabadadze–Porrati models can be found in Uzan [6]. For the same reason that, in the case of a general relativity modification, at the background level \(P_{\text{de}}\) and \(\rho_{\text{de}}\) can depend on \(H\), the quantities \(S_{\text{de}}, \Delta_{\text{de}}\) and \(\Pi_{\text{de}}\) can depend on \(\Phi\) and \(\Psi\), while \(F\) is a function of the background quantities only.
Several similar approaches have since been designed in the literature to reproduce this formalism (changing its notations) and I refer to the recent review [1] (and in particular its §4.2.2) for their relation to my original formulation. In particular, most of these works propose interesting parametrizations of the unknown function introduced here in order to constrain them with cosmological data. Indeed, such a phenomenological description cannot be complete as we have no equation to describe the evolution of the new degrees of freedom. At the background level, this gap is often filled by assuming a parametrization of the dark energy equation of state. The difficulty is to introduce parametrizations of the perturbation equations, which are consistent with the one of the background dynamics since both should arise from the same theoretical modification of general relativity. This aspect has been overlooked so that most formalisms span a space of parameters much larger than the one corresponding to well-defined theories.

The original idea of Uzan & Bernardeau [7] was also extended to multiple cosmological probes such as the use the galaxy–velocity correlation and galaxy–galaxy lensing, which give access to \(\langle \delta g \delta_{\text{mat}} \rangle \propto bf(\delta_{\text{mat}}^2)\) and \(\langle \delta g \kappa \rangle \propto b\langle \delta_{\text{mat}} \Delta(\Phi + \Psi) \rangle\) so that the ratio of these two quantities is expected to be independent of the bias, at least in the regime of linear biasing (see §4.5.2 of Uzan [1]). All these works are thus starting from the constitutive relations that exist in \(\Lambda\)CDM, and thus assuming general relativity valid, to construct from large-scale structure survey some tests that will indicate the violation of one of these relations. They are now in a stage that they form a convincing set of tools to test the deviations from general relativity by means of the large-scale structure.

5. Conclusions

The past decade has witnessed tremendous progress in the possibility of using cosmological data to test deviations from general relativity, in particular using the large-scale structure of the universe.
It is important to be aware that the large-scale structure observations (and the power of weak lensing in this discussion) are not the ultimate Grail mostly because the interpretation of the cosmological data depends on the whole cosmological model. For instance, a large-scale violation of the Copernican principle may modify the previous constructions. It was, for instance, shown in Dunsby et al. [14] that the velocity–density relation can be drastically modified when homogeneity...
does not hold, even if all other observations coincide with those of $\Lambda$CDM. This urges towards the construction of a test of general relativity that is independent of the Copernican principle, an issue that remains unexplored. Note, however, that the test of the equivalence principle based on the fundamental constants does not depend on this hypothesis.

One also has to disentangle the effect of the initial conditions and the effect of the nonlinear dynamics. The recent developments of second-order perturbation theory and the possibility to compute accurately the deviation from Gaussianity in the cosmic microwave background [41] or in weak lensing [42] may, however, offer a new route to test general relativity.

The combination of cosmological data with local constraints allows one to significantly increase the bounds, but it is a priori difficult to guess which system is the most constraining. For instance, in the case of scalar–tensor theories, big-bang nucleosynthesis can set sharper constraints than local tests. For the time variation of fundamental constants, atomic clocks and violation of the universality of free fall are, at the moment, the most sensitive probes, but quasar absorption should soon become competitive. Note also that there exist classes of models enjoying a violation of general relativity that cannot be detected locally, but only with cosmological observations [43].

While the use of large-scale structure has attracted huge attention and gathered a community interested in testing general relativity (and in the broad sense theoretical physics) on astrophysical scales, we should not restrict ourselves to this probe, and widen our tools, in particular, to free ourselves from the actual Copernican view of the universe. We have alluded to such possibilities (constant/duality distance/nucleosynthesis). Other interesting windows include multi-messenger astronomy, and in particular the propagation of gravity waves that can become a powerful tool to test bimetric theories or stellar dynamics [44].

While primarily motivated by dark energy, these researches are unavoidable in learning more about the domain of validity of general relativity, which we know is not the final theory of gravity.

References


