We compute trajectories of fluid particles in a water wave that propagates with a constant shape at a constant speed. The Stokes drift, which asserts that fluid particles are pushed forward by a wave, is proved using a new method. Numerical examples with various gravity and surface tension coefficients are presented.

Keywords: partial differential equations; nonlinear analysis; water wave

1. Introduction

We consider two-dimensional progressive water waves that propagate with a constant shape at a constant speed. Fluid motion is assumed to be irrotational. We are interested in the study of trajectories of fluid particles in the stationary coordinate system. Trajectories in a coordinate system attached to the wave are easily computed by drawing contours (level curves) of the stream function. Examples are abundant [1]. On the other hand, we need to take some care to compute trajectories in the stationary coordinate system. Longuett-Higgins [2] gives some examples for gravity waves, but we do not know examples of capillary–gravity waves. The first objective of this paper is to give some new examples. We draw trajectories of gravity, capillary–gravity and pure capillary waves.

It is well known (see Lamb [3] or Milne-Thomson [4]) that fluid particles in linearized water waves of small amplitude move on a circle or an ellipse, depending on whether the depth of water is infinite or finite, respectively. Therefore, the fluid particle does not move on average, whereas the wave itself propagates with a non-zero speed. This is, however, a proposition that is valid only approximately. In fact, Stokes [5] discovered that a particle trajectory is not closed.

Recently, Constantin [6] mathematically proved that none of the trajectories in the gravity waves are closed. It was also proved that trajectories in some linearized water waves are not closed either. See Constantin and co-workers [7,8] for the case of finite and infinite depth, respectively. The second objective of this paper is to prove non-closedness of trajectories by a method different from that of Constantin and co-workers [6,7]. Our proof may be of some use because of its simplicity.

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One contribution of 13 to a Theme Issue ‘Nonlinear water waves’.
This paper is organized as follows. In §2, we give an account of the background of the present problem. Following this, we prepare some notations and a mathematical framework in §3. In §4, we study Crapper’s pure capillary waves. Gravity and capillary–gravity waves are computed in §5. Constantin’s theorem is proved in §6. Finally, concluding remarks are presented in §7.

2. Background

For historical material on water waves, we recommend Craik [9]. Here, we summarize what is necessary for this paper.

Fluid particles in linearized gravity waves of small amplitude move on a circle or an ellipse. This was first recognized by Green [10, p. 280]. Stokes [5] then considered a higher order approximation. In the second-order approximation, he found a term that is proportional to the time variable, hence the particle trajectory does not close, but the fluid particle moves, on average, in the same direction as the wave. This phenomenon is now called the Stokes drift. If the reader looks at the figures in Van Dyke [11, p. 110] carefully, then he/she will find some trajectories which are not closed.

Longuett-Higgins [2] considered gravity waves of any amplitude and computed approximately the trajectories of particles. The paper also presents some laboratory experiments, which seem to agree with his theory.

From the viewpoint of modern mathematicians, the argument of Stokes is not a rigorous proof of non-closedness of particle trajectories. It is Constantin [6] who first presented a rigorous proof of the Stokes drift. His result has many variants [7,8,12–14], which we will comment on in the following sections of this paper.

Recently, the paper of Chen et al. [15] came to our attention. The authors of this paper computed approximate solutions of fifth order and compared them with their laboratory experiments. The agreement seems to be very good, see also Umeyama [16].

These studies are concerned with irrotational waves. Particles in rotational waves behave quite differently. In fact, all the trajectories in Gerstner’s wave (discovered by Gerstner [17], and later independently by Rankine [18]) are circles, no matter how large the wave amplitude may be, see earlier studies [3,4,19,20].

3. Preliminaries

In the following text, we restrict ourselves to particles in irrotational waves. In the moving frame, where the wave profile looks stationary, we take \((x, y)\) as the coordinates. We set \(z = x + iy\) and identify the complex plane with the \((x, y)\)-plane. In the stationary coordinate system, we take \((X, Y)\). They are related by \(X = x + \gamma t, Y = y\), where \(\gamma\) is the propagating speed of the wave profile. As we do not lose generality by normalizing \(\gamma = -1\) and setting the wavelength as \(2\pi\), we do so henceforth. Accordingly, we have \(X + t + iY = x + iy\). Hereafter, we use the notation of Okamoto & Shōji [1]. The reason that we use \(\gamma = -1\) rather than \(\gamma = 1\) is that we required many formulae in Okamoto & Shōji [1], where \(\gamma = -1\) was employed, and we prefer compatibility with Okamoto & Shōji [1] to that of earlier studies [6–8,12,13,15,19,21,22]. These earlier studies employed \(\gamma = 1\).
The fluid is assumed to be incompressible and inviscid. The motion of fluid is assumed to be irrotational. Throughout this paper, except in §6b, we consider only progressive waves on water of infinite depth. Using the complex potential $f = U + iV$, where $U$ is the velocity potential and $V$ is the stream function, we define $\zeta$ as

$$\zeta = \exp(-if).$$  \hspace{1cm} (3.1)

Our normalization implies that $U$ and $V$ are so normalized that $|U| \leq \pi, -\infty < V \leq 0$. Accordingly, $\zeta$ runs in and on the unit disc of the complex plane, namely, $|\zeta| \leq 1$. We define $\zeta = x + iy$, which is conformally in one-to-one correspondence with $\zeta$ and $f$. For later use, we define $\rho$ and $\sigma$ by $\zeta = \rho e^{i\sigma}$. We now set

$$i \log \frac{df}{dz} = \omega = \theta + i\tau,$$  \hspace{1cm} (3.2)

which is regarded as an analytic function of $\zeta$. Here, $\omega$ is defined by the equality on the left-hand side. The equality on the right-hand side implies just that the real and imaginary parts of $\omega$ are denoted by $\theta$ and $\tau$, respectively. $\omega$ satisfies that $\omega(0) = 0$, which simply implies that the velocity tends to unity as $\zeta \to 0$, i.e. as $y \to -\infty$.

It is shown by Okamoto & Shōji [1] that two-dimensional progressive waves are characterized by the following Levi-Civita equation:

$$e^{2\tau} \frac{d\tau}{d\sigma} - pe^{-\tau} \sin \theta + q \frac{d}{d\sigma} \left( e^{\tau} \frac{d\theta}{d\sigma} \right) = 0 \quad (0 \leq \sigma \leq 2\pi).$$  \hspace{1cm} (3.3)

Note that $\theta$ and $\tau$ in (3.2) are regarded as a function of $\zeta$ defined in the unit disc of the complex plane. $\theta$ and $\tau$ in (3.3) denote $\theta(1, \sigma)$ and $\tau(1, \sigma)$, respectively. As an analytic function is uniquely determined by its boundary values, $\theta(\rho, \sigma)$ and $\tau(\rho, \sigma)$ are uniquely determined once $\theta(1, \sigma)$ and $\tau(1, \sigma)$ are given. As $\tau$ is an imaginary part of the analytic function $\theta + i\tau$, $\tau$ can be written as the Hilbert transform of $\theta$. Accordingly, we have

$$\tau(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(s) \cot \frac{\sigma - s}{2} \, ds,$$

where $\tau(\sigma) = \tau(1, \sigma)$ and $\theta(\sigma) = \theta(1, \sigma)$. $p$ and $q$ in (3.3) are given by

$$p = \frac{gL}{2\pi \gamma^2} \quad \text{and} \quad q = \frac{2\pi T}{mg \gamma^2 L}.$$

Here, $L$ is the wavelength, $\gamma$ is the propagation speed, $g$ is the acceleration due to gravity, $T$ is the surface tension and $m$ is the density of the fluid. As we have normalized with $\gamma = -1$ and $L = 2\pi$, $p$ is simply the acceleration due to gravity.

The issue of existence of the solutions was discussed in earlier studies [1,23].

4. Crapper’s waves

Crapper [24] discovered that the pure capillary wave is an exact solution when gravity is neglected ($p = 0$) and the surface tension is the only force acting on the...
fluid surface. It is given by
\[ \frac{df}{dz} = \left( \frac{1 + A \zeta}{1 - A \zeta} \right)^2 \quad (\zeta \in \mathbb{C}, |\zeta| \leq 1), \quad (4.1) \]
where \(-1 < A < 1\) is a real parameter that is related to \(q\) in the following way:
\[ q = \frac{1 + A^2}{1 - A^2}, \]
see Okamoto & Shôji [1]. As (3.1) gives us
\[ \frac{d\zeta}{df} = -i \zeta, \quad (4.2) \]
(4.1) implies that
\[ \frac{dz}{d\zeta} = i \left( \frac{1 - A \zeta}{1 + A \zeta} \right)^2 = i \left( \frac{1 - 4A}{(1 + A \zeta)^2} \right). \quad (4.3) \]
Integrating this, we obtain
\[ z = i \left( \log \zeta + \frac{4}{1 + A \zeta} - 4 \right). \quad (4.4) \]
Here, the integral constant is chosen so that the free surface of the trivial solution (the one with \(A = 0\)) becomes the line segment \(y = 0, -\pi \leq x < \pi\). The free surface in the moving frame is obtained if we set \(\zeta = e^{i\sigma}\) in (4.4). If \(0 < \rho < 1\) is fixed and \(\sigma\) runs in \(0 \leq \sigma < 2\pi\), then a particle trajectory below the free surface in the moving frame is obtained. We draw in figure 1 some wave profiles \((\rho = |\zeta| = 1)\) and particle trajectories in the water \((0 < \rho < 1)\). If \(|A| > 0.45467 \cdots\), the wave profile has self-intersection points, and it is an unphysical solution. We must therefore consider those solutions only with \(|A| < 0.45467 \cdots\), see Okamoto & Shôji [1].

Once the trajectory \(z = z(t)\) in the moving frame is given, we can obtain the corresponding \(\zeta = \zeta(t)\) by applying the inverse mapping of (4.4). But the concrete expression of the inverse mapping does not seem to be available, and we proceed as follows. Using (4.3), we have
\[ \frac{dz}{dt} = i \left( \frac{1 - A \zeta}{1 + A \zeta} \right)^2 \frac{d\zeta}{dt}. \quad (4.5) \]
On the other hand, we can obtain the complex velocity using (4.1). The velocity in complex number notation is
\[ \left( \frac{1 + A \zeta}{1 - A \zeta} \right)^2, \]
where the bar denotes the complex conjugate. This must be equal to \(dz/dt\). Hence,
\[ \left( \frac{1 + A \zeta}{1 - A \zeta} \right)^2 = \frac{i(1 - A \zeta)^2}{\zeta(1 + A \zeta)^2} \frac{d\zeta}{dt}. \]
Figure 1. Crapper’s waves in the moving coordinates: (a) $A = 0.1$, (b) $A = 0.25$, (c) $A = 0.44$ and (d) $A = 0.47$. $\rho = 0.1, 0.2, 0.4, 0.6, 0.8, 1$. The broken line shows the level $y = 0$. The arrows indicate the directions of the particles. They move from left to right.

We therefore obtain

$$\frac{d\zeta}{dt} = \frac{\zeta|1 + A\zeta|^4}{i|1 - A\zeta|^4}. \quad (4.6)$$

If we represent $\zeta(t)$ in polar coordinates as $\zeta(t) = \rho(t)e^{i\sigma(t)}$, then (4.6) becomes

$$\frac{d\sigma}{dt} = -\left(\frac{1 + 2A\rho \cos \sigma + A^2\rho^2}{1 - 2A\rho \cos \sigma + A^2\rho^2}\right)^2 \quad \text{and} \quad \frac{d\rho}{dt} = 0. \quad (4.7)$$

We can solve this as

$$-t = \frac{-4b}{1 - b^2} \frac{\sin \sigma}{1 + b \cos \sigma} + \frac{8b^2}{(1 - b^2)^{3/2}} \tan^{-1}\left(\frac{1 - b}{1 + b} \tan\left(\frac{\sigma}{2}\right)\right) + \sigma \quad (4.8)$$

where $b$ is defined by

$$b = \frac{2A\rho}{1 + A^2\rho^2}.$$
(As $\rho$ is independent of $t$ and is a constant for a given particle, we regard it as a parameter.) We have chosen the integral constant so that $t=0$ corresponds to $\sigma = 0$. Or, instead, we may choose $\sigma' = \pi - \sigma$ to have

\[
\begin{align*}
t &= \frac{4b}{1-b^2} \sin \sigma' + \frac{8b^2}{(1-b^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1+b}{1-b}} \tan\left(\frac{\sigma'}{2}\right)\right) + \sigma' \\
(0 < \sigma' < 2\pi).
\end{align*}
\]

Here, the integral constant is adjusted so that $t=0$ at $\sigma'=0$. When we derive (4.9), we should note an identity $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$. It is easily seen that we can choose a branch of $\tan^{-1}$, so that the right-hand side of (4.9) becomes smooth at $\sigma' = \pi$.

We are now ready to draw trajectories in the stationary frame. Using (4.4) and $X + iY = x - t + iy$, we have

\[
\begin{align*}
X &= \frac{8b^2}{(1-b^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1+b}{1-b}} \tan\left(\frac{\sigma}{2}\right)\right) - \frac{1+b^2}{1-b^2} \frac{2\sin \sigma}{1+b^2} + b \cos \sigma', \\
Y &= \frac{2\sqrt{1-b^2}}{1+b \cos \sigma} - 2 + \log \rho
\end{align*}
\]

or

\[
\begin{align*}
X &= -\frac{8b^2}{(1-b^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1+b}{1-b}} \tan\left(\frac{\sigma'}{2}\right)\right) - \frac{1+b^2}{1-b^2} \frac{2\sin \sigma'}{1-b^2} - b \cos \sigma', \\
Y &= \frac{2\sqrt{1-b^2}}{1+b \cos \sigma} - 2 + \log \rho
\end{align*}
\]

either will do, but we mostly use (4.12) and (4.11) in what follows.

We now compute some trajectories. We suppose that one of the crests of the wave starts from the $y$-axis at $t=0$. We then consider the motion of several particles that lie on the $y$-axis at $t=0$. We compute trajectories for $0 \leq \sigma' \leq 2\pi$. This implies that we draw trajectories until the particle comes to the same height as the initial position. Figure 2 shows the trajectories in the case of $A = 0.44$. We see that the particle on the free surface takes a long journey. Figure 3 shows some trajectories when $A = 0.25$ and $A = 0.1$. No trajectory in these figures is closed, and the particles drift leftwards. As the wave is periodic, the trajectories actually look like figure 4.

When $A = 0.44$, we plot a trajectory in figure 5 that lies somewhat deep, $\rho = 0.02$. It is not closed, but is very much like a circular curve. Note that decreasing $\rho$ implies increasing the depth and reducing the nonlinear effect. Therefore, figure 5 complies with linear theory.

We now define the drift distance as

\[
D(\rho) = X(\sigma' = 2\pi) - X(\sigma' = 0).
\]

This is the distance between two consecutive crests in the stationary frame. As is seen from (4.12), it can be expressed as

\[
D(\rho) = -\frac{8\pi b^2}{(1-b^2)^{3/2}} = -32\pi A^2 \rho^2 \frac{1 + A^2 \rho^2}{(1 - A^2 \rho^2)^{3/2}}.
\]
Figure 2. Particle trajectories in the stationary coordinates in $0 \leq \sigma' \leq 2\pi$. $A = 0.44$. $\rho = 0.1, 0.2, 0.4, 0.6, 0.8, 1$. Particles move from the right to the left.

Figure 3. The same as in figure 2, but (a) $A = 0.25$ and (b) $A = 0.1$.

Figure 4. Particle trajectories, $A = 0.44, \rho = 0.4, 0.2$.

Note that $Y(\sigma' = 0) = Y(\sigma' = 2\pi)$, hence the trajectory is closed if and only if $D(\rho) = 0$. Actually, $D(\rho) < 0$ for all $\rho \in (0, 1]$, which implies that the fluid particle drifts towards the left—the same direction as the wave. A parameter defined by

$$\bar{y} = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(\sigma) \, d\sigma = \log \rho$$

may be more intuitive than $\rho$, and let us employ this as the parameter representing the depth. By this, we can regard $D = D(\bar{y})$ as being defined in $\bar{y} \in (-\infty, 0]$. Its graph is drawn in figure 6. As $D(\bar{y})$ decreases exponentially as $\bar{y} \to -\infty$, the drift is very small in deep water.

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We next verify that the trajectories are almost circular in deep water. We expand (4.12) and (4.11) in small $b$ to have

$$
\begin{align*}
X &= -2b \sin \sigma' - 4b^2 \sigma' - b^2 \sin 2\sigma' + O(b^3) \\
Y &= 2b \cos \sigma' + 2b^2 \cos^2 \sigma' + \log \rho - b^2 + O(b^3)
\end{align*}
$$

If we take only $O(b)$ terms, we have a circle. The term $-4b^2 \sigma'$ is responsible for the Stokes drift.
Trajectories of fluid particles

Figure 7. Particle trajectories in $0 \leq t \leq 2\pi$. $A = 0.25, \rho = 1, 0.8, 0.6, 0.4, 0.2, 0.1$ and $(X|_{t=2\pi}, Y|_{t=2\pi})$ with various $\rho$ are plotted together. (a) The $y$-axis is mapped, at $t = 2\pi$, onto the grey line. (b) $(X, Y)$ with $t = 2\pi k/10 \ (k = 1, 2, \ldots, 10)$ with $0.05 \leq \rho \leq 1$.

Figure 8. (a,b) The same as in figure 7, but $A = 0.44$.

So far we have considered trajectories from one crest to the next crest. But the time that elapsed during this motion is different from trajectory to trajectory. For instance, the particle on the free surface in figure 2 travels a long distance and takes a lot of time. The travel time is considerably larger than $2\pi$. On the other hand, the particle lying in deep water comes to the next crest in a time that is close to $2\pi$. Therefore, it would be interesting to consider trajectories for a fixed time interval, say, $0 \leq t \leq 2\pi$. Figure 7a shows parts of trajectories for $0 \leq t \leq 2\pi$. Figure 7b shows positions of particles at $t = 2\pi k/10 \ (k = 0, 1, \ldots, 10)$, which started the $y$-axis at $t = 0$. Figure 8 shows the same data for a different $A$.

5. Gravity and capillary–gravity waves

We now study trajectories in the case of general $(p, q)$. We consider only symmetric waves. Accordingly, $\theta$ is assumed to be odd in $\sigma \in (-\pi, \pi)$. We start
with (3.3). By the Fourier spectral method, we obtain its approximate solution,

$$\theta(1, \sigma) = \sum_{n=1}^{N} a_n \sin n\sigma. \quad (5.1)$$

For some solutions, $N = 128$ is enough, but some gravity waves requires a large $N$ such as $N = 1024$ [1]. With the coefficients $a_n$ in hand, we compute

$$\omega(\rho, \sigma) = \theta(\rho, \sigma) + i\tau(\rho, \sigma) = \sum_{n=1}^{N} a_n \rho^n \sin n\sigma - i \sum_{n=1}^{N} a_n \rho^n \cos n\sigma$$

$$= \sum_{n=1}^{N} (-i) a_n (\rho e^{i\sigma})^n = -i \sum_{n=1}^{N} a_n \zeta^n$$

and

$$\frac{df}{dz} = \exp(-i\omega) = \exp \left( - \sum_{n=1}^{N} a_n \zeta^n \right).$$

As in the case of Crapper’s wave, we have

$$\frac{d\zeta}{dz} = -i\zeta \exp \left( - \sum_{n=1}^{N} a_n \zeta^n \right) \quad \text{or} \quad \frac{dz}{d\zeta} = \frac{i}{\zeta} \exp \left( \sum_{n=1}^{N} a_n \zeta^n \right). \quad (5.2)$$

The velocity is represented as

$$\exp \left( - \sum_{n=1}^{N} a_n \zeta^n \right) = \frac{dz}{dt} = \frac{i}{\zeta} \exp \left( \sum_{n=1}^{N} a_n \zeta^n \right) \frac{d\zeta}{dt}.$$ 

Consequently,

$$\frac{d\zeta}{dt} = -i\zeta \exp \left( - \sum_{n=1}^{N} a_n (\bar{\zeta}^n + \zeta^n) \right) = -i\zeta \exp \left( -2 \sum_{n=1}^{N} a_n \rho^n \cos n\sigma \right)$$

$$= -i\zeta \exp \left( -i(\omega - \bar{\omega}) \right).$$

This gives us

$$\frac{d\rho}{dt} \equiv 0 \quad \text{and} \quad \frac{d\sigma}{dt} = -\exp \left( -\sum_{n=1}^{N} 2a_n \rho^n \cos n\sigma \right).$$

Therefore, $\rho$ is constant along an individual trajectory of a fluid particle. Furthermore,

$$-t = \int_{\sigma}^{\sigma_0} \exp \left( \sum_{n=1}^{N} 2a_n \rho^n \cos ns \right) \, ds \quad (-\pi \leq \sigma \leq \pi). \quad (5.3)$$
Instead of $\sigma$, we may use $\sigma' = \pi - \sigma$ and (5.2) to obtain
\[
\frac{d}{d\sigma'}(x + iy) = \exp \left( \sum_{n=1}^{N} a_n(-\rho)^n \cos n\sigma' - i \sum_{n=1}^{N} a_n(-\rho)^n \sin n\sigma' \right)
\]
\[= \exp(i\omega(\rho, \pi - \sigma')) \tag{5.4}\]
and
\[
\frac{dt}{d\sigma'} = \exp \left( 2 \sum_{n=1}^{N} a_n(-\rho)^n \cos n\sigma' \right) = \exp (i\omega - i\overline{\omega}). \tag{5.5}\]

If the amplitude of the wave is not very large, we may integrate (5.5) by a simple quadrature rule. Once we have $x, y, t$ as a function of $\sigma'$, then we can draw
\[X + iY = -t + x + iy. \]

For a fixed $\rho$, we compute $Y(\rho, 0)$ by
\[Y(\rho, 0) = -\int_{\rho}^{1} \frac{1}{s} \exp \left( \sum_{n=1}^{N} a_n(-s)^n \right) \, ds. \]

With these formulae we can easily compute trajectories. Figures 9–13 are examples. Longuett-Higgins [2] computes some gravity waves, but it seems to us that trajectories of capillary–gravity waves have not been computed before.

Figure 9 shows trajectories of fluid particles when $(p, q) = (0.842, 0)$. Numerical data in (5.1) are borrowed from Okamoto & Shōji [1]. Trajectories of five particles $\rho = 0.2 \times k (k = 1, 2, \ldots, 5)$ are plotted. Those on the left-hand side are trajectories in stationary coordinates. They start from the $y$-axis and move leftwards. Those on the right-hand side (thinner ones) are trajectories in moving coordinates. They start from the $y$-axis and move rightwards until they come to $x = 2\pi$. Figure 10 is a blow-up of the trajectory of $\rho = 0.2$.

In the same fashion, we draw figures 11–13, where solutions with different $(p, q)$ are chosen, but we take the same $\rho$, i.e. $\rho = 0.2, \ldots, 1$. 

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Figure 10. A particle trajectory deep in a gravity wave, \((p, q) = (0.842, 0)\), \(\rho = 0.2\).

Figure 11. Particle trajectories on and in a capillary–gravity wave: (a) \((p, q) = (0.73, 0.28)\) and (b) the trajectory of \(\rho = 0.6\).

Figure 12. Particle trajectories on and in a capillary–gravity wave, \((p, q) = (0.79, 0.5)\).
6. Trajectory is not closed

This section is devoted to a mathematical proof of Constantin’s theorem that a particle trajectory is not closed.

(a) The case of infinite depth

Theorem 6.1. Suppose that the depth is infinite. Then, for any $p$ and $q$, we have $X|\sigma'=2\pi < X|\sigma'=0$. Also, $X|\sigma'=0 - X|\sigma'=2\pi$ is monotonically increasing in $\rho$.

Proof. What we have to prove is

$$X|\sigma=\pi - X|\sigma=-\pi > 0.$$

Note first that (5.4) implies that

$$-x|_{\sigma=-\pi} + x|_{\sigma=\pi} = \int_{-\pi}^{\pi} \text{Re} \left[ \exp(i \omega(\rho, \sigma)) \right] d\sigma,$$

and (5.3) implies that

$$-t|_{\sigma=\pi} + t|_{\sigma=-\pi} = \int_{-\pi}^{\pi} |\exp(i \omega(\rho, \sigma))|^2 d\sigma.$$

As $X = x - t$, our goal is to prove

$$\int_{-\pi}^{\pi} \text{Re} \left[ \exp(i \omega(\rho, \sigma)) \right] d\sigma < \int_{-\pi}^{\pi} |\exp(i \omega(\rho, \sigma))|^2 d\sigma. \quad (6.1)$$

The left-hand side is computed as follows:

$$\int_{-\pi}^{\pi} \text{Re} \left[ \exp(i \omega(\rho, \sigma)) \right] d\sigma = \text{Re} \left[ \int_{-\pi}^{\pi} \exp(i \omega(\rho, \sigma)) d\sigma \right]$$

$$= \text{Re} \left[ \int_{|\xi|=\rho} \exp(i \omega(\xi)) \frac{d\xi}{i \xi} \right] = 2\pi.$$

The right-hand equality follows from Cauchy’s integral formula and $\omega(0) = 0$. (Note that $\omega(0) = 0$ is a consequence of the fact that $df/dz \to 1$ as $y \to -\infty$. See (3.2).)
Next, we expand the analytic function \( \exp(i\omega(\rho, \sigma)) \) into the Taylor expansion,
\[
\exp(i\omega(\rho, \sigma)) = 1 + a_1 \zeta + a_2 \zeta^2 + \cdots.
\]
Then,
\[
\int_{-\pi}^{\pi} |\exp(i\omega(\rho, \sigma))|^2 \, d\sigma = 2\pi(1 + |a_1|^2 \rho^2 + |a_2|^2 \rho^4 + \cdots).
\]
Accordingly, unless \( a_1 = a_2 = \cdots = 0 \), the inequality (6.1) holds true. Monotonicity in \( \rho \) is obvious.

**Corollary 6.2.** The absolute value of the drift distance is positive, and is the largest on the free surface (\( \rho = 1 \)).

(b) The case of finite depth

If we wish to prove that \( X|_{\sigma=\pi} - X|_{\sigma=-\pi} > 0 \), in the case of finite depth, we need some modification on the proof above. Let us recall some formulae in Okamoto & Shōji [1]. If the depth is specified, then we are given a parameter \( \eta \in (0, 1) \) and a function \( \omega = \theta + i\tau \) that is defined in the annulus \( \eta \leq |\zeta| \leq 1 \) and is analytic in \( \eta < |\zeta| < 1 \). They satisfy the equation (3.3) on the outer boundary \( \rho = 1 \), and \( \theta \) vanishes on the inner boundary \( \rho = \eta \). Note that we do not lose generality [1] if we assume that
\[
\int_{-\pi}^{\pi} \tau(\rho, \sigma) \, d\sigma = 0.
\]
With these formulae, we have to prove (6.1), which we can write as
\[
\int_{-\pi}^{\pi} e^{-\tau} \cos \theta \, d\sigma < \int_{-\pi}^{\pi} e^{-2\tau} \, d\sigma. \tag{6.2}
\]

**Theorem 6.3.** Unless \( \theta \equiv 0 \), (6.2) holds true.

**Proof.** To prove (6.2), we first note that
\[
\int_{-\pi}^{\pi} e^{-\tau} \cos \theta \, d\sigma < \int_{-\pi}^{\pi} e^{-\tau} \, d\sigma
\]
unless \( \theta \equiv 0 \). Second, the Hölder inequality gives us
\[
\left( \int_{-\pi}^{\pi} e^{-\tau} \, d\sigma \right)^2 \leq 2\pi \int_{-\pi}^{\pi} e^{-2\tau} \, d\sigma.
\]
We finally use Jensen’s inequality with a convex function \( x \mapsto e^x \) to obtain
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\tau} \, d\sigma \geq \exp \left( -\frac{1}{2\pi} \int_{-\pi}^{\pi} \tau \, d\sigma \right) = 1.
\]
Combining these inequalities, we can conclude that (6.2) is valid unless \( \theta \equiv 0 \).
from ours. Ours is also valid in the presence of surface tension. It seems to us that the proof in earlier studies [6,14] requires an assumption that the wave profile has one and only one relative maximum. We do not need to assume this. As numerical examples of gravity waves that possess more than one relative maxima in its profile are known [1], our proof may be worth being recorded. (The capillary–gravity waves in §5 are also such examples.) A theorem in the case of solitary waves was considered by Constantin & Escher [12].

Water waves with underlying uniform current are considered in Constantin & Strauss [13]. We can also consider them with underlying shear flow, see Okamoto & Shōji [1, ch. 7]. These constitute further problems, but we leave them to the reader.

7. Conclusion

None of the particle trajectories in the stationary coordinates are closed, although they are very close to a closed circular curve if the fluid particle lies deeply. Fluid particles are pushed out by the wave in the same direction as the wave. The drift distance is the largest on the water surface and decreases exponentially and monotonically in depth.

As Gerstner’s wave has been a unique example of explicitly written trajectories, our explicit formula for particle trajectory of Crapper’s waves may be worthy of notice. This analysis can be viewed as one of the particle dynamics considered in many situations in earlier studies [25,26].

We are naturally led to a question: if we consider a rotational wave, what is a necessary and sufficient condition for a vorticity distribution to produce closed trajectories only? Our method is restricted to irrotational waves, and does not seem to work in the case of rotational waves. For rotational waves, see Okamoto & Shōji [1, ch. 7] and other recent papers [19,20,22,27–29]. Waves on a sloping beach [21], too, may well give us an interesting problem. We would like to pose this problem to the reader.

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