Effects of imperfections on localized bulging in inflated membrane tubes

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The problem of localized bulging in inflated membrane tubes shares the same features with a variety of other localization problems such as formation of kink bands in fibre-reinforced composites and layered structures. This type of localization is known to be very sensitive to imperfections, but the precise nature of such sensitivity has not so far been quantified. In this paper, we study effects of localized wall thinning/thickening on the onset of localized bulging in inflated membrane tubes as a prototypical example. It is shown that localized wall thinning may reduce the critical pressure or circumferential stretch by an amount of the order of the square root of maximum wall thickness reduction. As a typical example, a 10 per cent maximum wall thinning may reduce the critical circumferential stretch by 19 per cent. This square root law complements the well-known Koiter’s two-thirds power law for subcritical periodic bifurcations. The relevance of our results to mathematical modelling of aneurysm formation in human arteries is also discussed.

Keywords: imperfection sensitivity; membrane tubes; localization; bifurcation; aneurysm

1. Introduction

Most studies on elastic buckling have focused on buckling into sinusoidal patterns. Buckling of an Euler strut and a cylindrical thin shell typifies two possible scenarios in this case: one is supercritical and the other subcritical. Subcritical bifurcations are known to be sensitive to imperfections. For elastic structures, Koiter [1] (see also Hutchinson & Koiter [2]) demonstrated that geometrical or material imperfections in the form of a buckling mode may reduce the critical load by an amount of the order of the two-thirds power of the imperfection amplitude. This is now known as the two-thirds power law for imperfection-sensitive structures and it provided an explanation on why the critical loads recorded in actual experiments on compressed cylindrical shells were much lower than the theoretical values predicted by the linear buckling analysis.

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Elastic localization

Sinusoidal buckling patterns may localize, either owing to the existence of a small or large parameter in the linearized problem or owing to nonlinear modulation. For instance, when a thick-walled cylindrical shell is everted, a linear buckling analysis would predict sinusoidal buckling (wrinkling) localized near the inner surface because of the largeness of the critical mode number in the circumferential direction [3] (also A. Juel 2009, personal communication, with photos showing that localization may also take place along the circumferential direction due to infinitely many nearby modes interacting resonantly). The other localization mechanism, localization owing to nonlinear modulation, has been systematically studied in recent decades by Hunt and co-workers (see [4,5] and references therein).

The present paper is concerned with another type of localization that does not involve any sinusoidal oscillations at all. Examples of this type include necking of a polymeric strip under stretching [6,7], Lüders band formation in steel strips [8,9], localized bulging in inflated membrane tubes [10–12], kink band formation in fibre-reinforced composites [13–19] and layered structures [20,21], and stress-induced phase transformations [22–24]. For each of these problems, the actual load deformation diagram typically contains four distinctive sections: a section of uniform deformation that is terminated when the load reaches a maximum, a snap-back section, growth of the localized deformation as the load decreases, and propagation of the localized deformation at a constant value of the load. The nature of the latter constant load value is now well understood: it can be determined by the Maxwell equal area rule. It is also known that the stage corresponding to the snap-back section is usually abrupt and that the load maximum is very sensitive to geometrical and material imperfections. However, much less is known about the nature of the load maximum. For instance, it is not yet completely understood how the load maximum can be determined when there are no imperfections and how the imperfection sensitivity can be quantified when imperfections are present. For the necking problem, it was shown by Mielke [25] that the load maximum corresponds to a bifurcation at zero mode number (see also Fu [26] and Dai et al. [27]). For the tube inflation problem, recent studies by Fu et al. [28] and Pearce & Fu [29] show that the load maximum also corresponds to a bifurcation at zero mode number, but it may or may not equal the load maximum in uniform inflation depending on how the tube is loaded axially. The occurrence of the latter load maximum is usually referred to as limit-point instability [30–32]. For kink band formation in fibre-reinforced composites, the load maximum is believed to correspond to a loss of ellipticity [33–35].

In this paper, we aim to quantify imperfection sensitivity for the class of problems listed above. This will be achieved through an analysis of the model problem of inflation of a membrane tube. This is a simple enough problem for which almost the entire inflation process can be described analytically, and there also exists an extensive literature on its numerical and experimental aspects (see Kyriakides & Chang [12], Pamplona et al. [36] and Shi & Moita [37] and references therein).

This paper can be viewed as the fourth of a series of studies devoted to an improved understanding of the tube inflation process. In the first of this series [28], the initial bulging/necking was recognized as a bifurcation problem and the corresponding bifurcation condition was derived using two different methods. The two subsequent papers—Pearce & Fu [29] and Fu & Xie [38]—then addressed...
the question of characterization of the fully nonlinear bulging solutions and their stability properties under pressure or volume control. The present paper is a first study on the effects of material and geometrical imperfections. We shall focus on the case when the membrane wall suffers from a localized thinning or thickening. This is particularly pertinent to the continuum-mechanical modelling of aneurysm formation in arterial walls, where it is believed that localized wall thinning owing to pathological changes is a precursor of aneurysm formation, and aneurysm repairs invariably result in localized material or geometrical imperfections.

The rest of this paper is organized as follows. In the following section, we write down the governing equations and summarize previously known results for perfect membrane tubes. In §3, we carry out a weakly nonlinear analysis for the case when the membrane tube suffers localized wall thinning or thickening. We derive the amplitude equation for the radius variation along the axis and use its solutions to quantify the effects of imperfections. The paper is concluded with some additional remarks and a discussion on the relevance of our results to the mathematical modelling of aneurysm formation in arteries.

2. Governing equations and known results for a perfect membrane tube

We consider the problem of inflation of an infinite cylindrical membrane tube that is incompressible, isotropic and hyperelastic. The tube is assumed to have variable thickness $H$ and uniform mid-plane radius $R$ before inflation. To simplify analysis, we assume that the remote axial stretch is maintained at unity all the time, but our results can easily be modified for other cases such as that of closed ends or when the remote axial stretch is maintained at a non-unit constant value (the latter case would pertain to human arteries). We also assume that when the tube is inflated by an internal pressure, the inflated configuration maintains axial symmetry. Thus, in general, the axisymmetric deformed configuration may be described by

$$r = r(Z) \quad \text{and} \quad z = z(Z),$$  \hspace{1cm} (2.1)

where $Z$ and $z$ are the axial coordinates of a representative material particle before and after inflation, respectively, and $r$ is the mid-plane radius after inflation (figure 1).

Since the deformation is axially symmetric, the principal directions of stretch coincide with the lines of latitude, the meridian and the normal to the deformed surface. Thus, the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = \sqrt{r'^2 + z'^2} \quad \text{and} \quad \lambda_3 = \frac{\text{deformed wall thickness}}{H},$$  \hspace{1cm} (2.2)

where the indices $(1,2,3)$ are used for the latitudinal, meridional and normal directions, respectively, and the primes indicate differentiation with respect to $Z$. In the following analysis, we use $R$ as the unit of length, which is equivalent to setting $R = 1$.

The principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$ in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i \hat{W}_i - p, \quad i = 1,2,3 \ (\text{no summation}),$$  \hspace{1cm} (2.3)

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where $\hat{W} = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ is the strain energy function, $\hat{W}_i = \partial \hat{W} / \partial \lambda_i$, and $p$ is the pressure associated with the constraint of incompressibility [39]. Using the incompressibility constraint $\lambda_1 \lambda_2 \lambda_3 = 1$ and the membrane assumption $\sigma_3 = 0$, we find

$$\sigma_i = \lambda_i W_i, \quad i = 1, 2 \text{ (no summation)},$$

(2.4)

where $W(\lambda_1, \lambda_2) = \hat{W}(\lambda_1, \lambda_2, \lambda_1^{-1} \lambda_2^{-1})$ and $W_1 = \partial W / \partial \lambda_1$, etc. In our numerical illustrations, we shall assume that the membrane material is described by the Ogden strain energy function

$$W = \sum_{r=1}^{3} \frac{\mu_r}{\alpha_r} (\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3),$$

(2.5)

where $\alpha_1 = 1.3, \alpha_2 = 5.0, \alpha_3 = -2.0, \mu_1 = 1.491, \mu_2 = 0.003$ and $\mu_3 = -0.023$ are material constants given by Ogden [40], and the $\mu$'s have been scaled by the ground state shear modulus.

By considering equilibrium of an infinitesimal volume element in the $r$- and $z$-directions, respectively, we obtain

$$\left( \frac{HW_2 r'}{\lambda_2} \right)' + Pr' - HW_1 = 0 \quad (2.6)$$

and

$$\left( \frac{HW_2 z'}{\lambda_2} \right)' - Prr' = 0. \quad (2.7)$$

These two equations are equivalent to

$$(HW_2)' - \frac{H}{\lambda_2} r' W_1 = 0 \quad (2.8)$$

and

$$Hz' W_1 + \frac{HW_2}{\lambda_2} (r' z'' - z' r'') - Pr\lambda_2^2 = 0, \quad (2.9)$$

which represent equilibrium of an infinitesimal volume element in the meridional and normal directions, respectively. With the help of these two sets of equations
and the additional relations $r' = \lambda_2 \sin \phi, z' = \lambda_2 \cos \phi$ (figure 1), we may obtain

$$
\begin{align*}
\lambda_1' &= \lambda_2 \sin \phi, \\
\lambda_2' &= \frac{W_1 - \lambda_2 W_{12}}{W_{22}} \sin \phi - \frac{H' W_2}{H W_{22}} \\
\phi' &= \frac{W_1}{W_2} \cos \phi - \frac{P \lambda_1 \lambda_2}{H W_2},
\end{align*}
$$

(2.10)

and

$$
W_{12} = \frac{v_2}{v_1 v_2},
$$

where $W_{12} = \frac{v_2}{v_1 v_2},$ etc. This system of equations will be integrated numerically to obtain fully nonlinear solutions.

The equilibrium equation (2.7) can be integrated straight away to yield

$$
\frac{H W_2 z'}{\lambda_2} - \frac{1}{2} P r'^2 = \text{constant} \equiv C_2,
$$

(2.11)

which reflects the fact that the resultant in the $Z$-direction at any cross section must be a constant. Mathematically, this corresponds to invariance of the governing equations with respect to translations in $z.$ When $H$ is a constant, the equations are also invariant with respect to translations in $Z,$ leading to another integral

$$
W - \lambda_2 W_2 = \text{constant} \equiv C_1.
$$

(2.12)

This integral was first obtained by Pipkin [41] by integrating (2.8) directly.

We now summarize what is known for a perfect membrane tube, that is, when $H$ is a constant. On taking the limit $Z \to \infty$ in (2.6), (2.11) and (2.12), we obtain

$$
P = \frac{W_1^{(\infty)}}{r_\infty}, \quad C_2 = W_2^{(\infty)} - \frac{1}{2} P r_\infty^2 \quad \text{and} \quad C_1 = W^{(\infty)} - W_2^{(\infty)},
$$

(2.13)

where the superscript $(\infty)$ signifies evaluation at

$$
\lambda_1 = r_\infty \quad \text{and} \quad \lambda_2 = 1,
$$

(2.14)

$r_\infty$ being the radius at infinity.

Equations (2.11) and (2.12) with (2.13) always admit the trivial solution (2.14) as a solution, but as $r_\infty$ is increased from unity (with the corresponding pressure calculated according to (2.13)1), other non-trivial solutions may bifurcate from this trivial solution. For a bifurcated solution symmetric about $Z = 0,$ we have $r'(0) = 0$ so that

$$
\lambda_1(0) = r_0 \quad \text{and} \quad \lambda_2(0) = z_0',
$$

(2.15)

where $r_0 = r(0)$ and $z_0' = z'(0).$ On evaluating (2.11) and (2.12) at $Z = 0,$ we obtain two algebraic equations that can be solved numerically to find $r_0$ and $z_0'$ in terms of the single control parameter $r_\infty.$ Figure 2 shows the dependence of $r_0$ on $r_\infty$ for the Ogden material model.

Once $r_0$ and $z_0'$ are known for each $r_\infty,$ we may integrate the first-order system of differential equations (2.10) subjected to the initial conditions (2.15) and $\phi(0) = 0.$ It is found that only the solid segment in figure 2 corresponds to localized solutions. Figure 2 can be interpreted as follows. Uniform inflation would start from the origin, where $\lambda_1 = \lambda_2 = 1,$ and then follow the horizontal axis where $r_0 \equiv r_\infty.$ As inflation reaches the bifurcation point B, where $r_\infty = r_{cr},$ a bulge will initiate at $Z = 0$ and its subsequent growth follows the curve BT. Growth
in the radius $r_0$ stops at the turning point $T$, where $dr_0/dr_\infty \to \infty$. It can be shown [29] that the latter condition implies that the right-hand side (r.h.s.) of $(2.10)_3$ vanishes and thus the corresponding point $(\lambda_1, \lambda_2, \phi) = (r_0, z'_0, 0)$ becomes another fixed point of the dynamical system $(2.10)$ (in addition to the fixed point $(r_\infty, 1, 0)$). In dynamical system theory terms, each localized solution before point $T$ is reached corresponds to a homoclinic orbit, whereas at $T$, the localized solution has flattened out at $Z = 0$ and represents two heteroclinic solutions (kink solutions) stitched together at $Z = 0$. If the tube is inflated further, the localization will propagate in both directions, while the radius at $Z = 0$ remains unchanged.

It was shown by Fu et al. [28] that if $r = r_\infty + y(Z)$ and $|y| \ll 1$, then $y$ must satisfy the differential equation

$$(y')^2 = \omega(r_\infty)y^2 + \gamma(r_\infty)y^3 + O(y^4), \quad (2.16)$$

where explicit expressions for $\omega(r_\infty)$ and $\gamma(r_\infty)$ in terms of the strain energy function can be found in the studies of Fu et al. [28] and Fu & Il’ichev [42], respectively. It can be seen from (2.16) that there is a bifurcation when $\omega(r_\infty) = 0$, the first root of which then defines the $r_{ct}$ appearing in the previous paragraph. Writing $r_\infty = r_{ct} + \epsilon r_1$, $y = \epsilon y_1 + O(\epsilon^2)$, where $\epsilon$ is a small positive constant and $r_1$ is an $O(1)$ constant, then to leading order (2.16) gives

$$\frac{d^2y_1}{d\xi^2} = \omega'_{ct} r_1 y_1 + \frac{3}{2} \gamma_{ct} y_1^2, \quad \xi = \sqrt{\epsilon} Z, \quad (2.17)$$

which has a localized solution given by

$$y_1 = -\frac{\omega'_{ct} r_1}{\gamma_{ct}} \text{sech}^2 \left( \frac{1}{2} \sqrt{\omega'_{ct} r_1 \xi} \right), \quad (2.18)$$

where $\omega'_{ct} = d\omega(r_{ct})/dr_{ct}$, $\gamma_{ct} = \gamma(r_{ct})$. This solution is valid provided $\omega'_{ct} (r_\infty - r_{ct}) > 0$. For all materials that we have considered so far, $\omega'_{ct}$ corresponding to the first bifurcation point is always negative and so bifurcation into the above
localized solution must necessarily be subcritical. For the Ogden material, we have \( \omega'_{cr} = -3.2329, \gamma_{cr} = -1.3369 \) and so the localized solution (2.18) represents a bulge (since \( y_1 > 0 \)).

We observe that for subcritical bifurcations into sinusoidal patterns, a weakly nonlinear near-critical analysis that takes into account amplitude modulation would lead to an equation similar to (2.17), but with the quadratic term replaced by a cubic term (see [43] or [44]). As a result, the \( \text{sech}^2 \) on the r.h.s. of (2.18) would be replaced by the less-localized \( \text{sech} \).

Since any localized bulging/necking solutions must tend to the uniform state \( r = r_\infty \) in the limit \( Z \to \pm \infty \), for a tube with variable thickness \( H(Z) \) whose variation is localized (2.16) is valid sufficiently away from the site of localized wall thinning/thickening. Thus, we have \( r' \sim -\sqrt{\omega(r_\infty)(r - r_\infty)} \) as \( Z \to \pm \infty \). The (symmetric) localized bulging/necking solutions can then be determined by integrating the system (2.10) from \( Z = 0 \) towards \( \infty \) subject to the initial conditions

\[
\lambda_1(0) = r_0, \quad \lambda_2(0) = z'_0 \quad \text{and} \quad \phi(0) = 0,
\]

where \( r_0 \) is to be guessed in our shooting procedure and the constant \( z'_0 \) is related to \( r_0 \) by the integral (2.11). Thus, for each specified \( r_\infty \) and a guess for \( r_0 \), we solve (2.11) to find the corresponding \( z'_0 \). We iterate on \( r_0 \) so that the decay condition

\[
r'(L) + \sqrt{\omega(r_\infty)(r(L) - r_\infty)} = 0 \tag{2.19}
\]

is satisfied for a sufficiently big positive number \( L \). As localized solutions correspond to homoclinic orbits in the phase plane that usually separate unbounded solutions from periodic solutions, a reasonable starting guess for \( r_0 \) is required to ensure the success of the above shooting method. We start from a case for which the imperfection is of small amplitude and \( r_\infty \) is close to the bifurcation.
Figure 4. Results for the case $h_0 = 1, \epsilon = 0.2$. (a) Comparison of amplitude diagrams for a perfect membrane tube (dashed line) and for an imperfect membrane tube (solid lines, weakly nonlinear result given by (3.6); solid dots, fully nonlinear result obtained by integrating (2.10)). (b) Three solutions of $r(Z)$ for an imperfect membrane tube when $r_\infty = 1.55$ (solid lines, fully nonlinear result; dashed lines, weakly nonlinear result).

value. The weakly nonlinear result to be presented in the following section then provides a very good initial guess for $r_0$. Once a solution is found, we increase the amplitude of imperfection and decrease $r_\infty$ in small steps, using the solution from the previous step as the initial guess for the current step. A selection of such fully nonlinear numerical solutions are presented as dots in figures 3 and 4, and will be discussed together with weakly nonlinear solutions in §3.

3. Effects of localized wall thinning

The weakly nonlinear solution (2.18) indicates that when $r_\infty - r_{cr}$ is of order $\epsilon$, the localized bulging solution has an $O(\epsilon)$ amplitude and varies over an $O(1/\sqrt{\epsilon})$ length scale. We now assume that the variable thickness $H$ takes the form

$$H = H_0(1 + \epsilon^2 h(\xi)), \quad h(\pm \infty) \rightarrow 0,$$

where $H_0$ is the constant wall thickness at infinity and the function $h(\xi)$ is to be prescribed. The order of deviation of the wall thickness from the constant value $H_0$ is dictated by the requirement that $h(\xi)$ must appear in our final amplitude equation.

When $H$ is not constant, the integral (2.12) no longer holds and so the simple procedure of Fu et al. [28] for deriving (2.16) cannot be applied. Instead, we shall use the equilibrium equation (2.6) and integral (2.11) to derive the amplitude equation.

We thus look for an asymptotic solution of the form

$$r = r_{cr} + \epsilon r_1 + \epsilon^2 y_1(\xi) + \epsilon^2 y_2(\xi) + \cdots \quad \text{and} \quad z' = 1 + \epsilon v_1(\xi) + \epsilon^2 v_2(\xi) + \cdots,$$

where $y_1(\xi), y_2(\xi), v_1(\xi), v_2(\xi), \ldots$ are to be determined at successive orders of approximations. As we are interested only in localized solutions, the asymptotic expansion for the pressure is simply the Taylor series of (2.13)$_1$. 

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On substituting (3.2) into (2.6) and (2.11), and equating the coefficients of $\varepsilon$, we find a relation between $y_1(\xi)$ and $v_1(\xi)$ and reproduce the bifurcation condition $\omega(r_{cr}) = 0$, whereas on equating the coefficients of $\varepsilon^2$, we find a relation between $y_2(\xi)$ and $v_2(\xi)$ and the amplitude equation

$$\frac{d^2 y_1}{d\xi^2} = \omega'(r_{cr}) r_1 y_1 + \frac{3}{2} \gamma_{cr} y_1'' + \zeta h(\xi),$$

(3.3)

where the only new coefficient $\zeta$ is given by

$$\zeta = W_{1,2}^{-1} W_{2,2}^{-1} (W_1 W_{22} + W_1 W_2 - W_2 W_{12})|_{\lambda_1 = r_{cr}, \lambda_2 = 1},$$

(3.4)

and takes the value 2.0328 for the Ogden material. The amplitude equation (2.17) for a perfect membrane tube is recovered by setting $h(\xi) = 0$. For each specified $h(\xi)$, the amplitude equation (3.3) is to be integrated subject to the decay condition $y_1(\pm \infty) \rightarrow 0$.

There are obviously three special cases, corresponding to $h(\xi)$ being a constant multiple of $y_1^2$, $y_1$, $y_1''$, respectively, for which closed-form localized solutions can be obtained as (3.3) can then be reduced to the form (2.17) by substitution of coefficients or rescaling of $\xi$. We first assume that

$$h(\xi) = \frac{3}{2} d_1 y_1^2,$$

where $d_1$ is a constant. By making the substitution $\gamma_{cr} \rightarrow \gamma_{cr} + \xi d_1$ in (2.18), we obtain the solution

$$r - r_\infty = \epsilon y_1 = - \frac{\omega'_{cr} \epsilon r_1}{\gamma_{cr} + \xi d_1} \mathrm{sech}^2 \left( \frac{1}{2} \sqrt{\omega'_{cr} r_1 \xi} \right).$$

(3.5)

Denoting $h(0)$ by $h_0$, we have

$$h_0 = \frac{3}{2} d_1 y_1^2(0) = \frac{3 d_1 (\omega'_{cr} r_1)^2}{2 (\gamma_{cr} + \xi d_1)^2},$$

which can be solved to express $d_1$, and hence $h(\xi)$ and $y_1(\xi)$, in terms of $h_0$. We then obtain

$$r_0 - r_\infty = \epsilon y_1(0) = - \frac{\omega'_{cr} (r_\infty - r_\epsilon)}{2 \gamma_{cr}} \left[ 1 \pm \sqrt{1 - \frac{8 \gamma_{cr} h_0 \zeta \epsilon^2}{3 \omega'_{cr}^2 (r_\infty - r_\epsilon)^2}} \right].$$

(3.6)

When $h_0 < 0$, which corresponds to localized wall thinning, the graph of (3.6) in the $(r_\infty, r_0 - r_\infty)$ plane is a parabola opening to the left (figure 3a), similar to the classical amplitude diagrams for imperfection-sensitive structures. As we move away from the bifurcation value $r_{cr}$, we have $(r_{cr} - r_\infty)/\epsilon \rightarrow \infty$. The upper and lower branches then have the asymptotic behaviour

$$r_0 - r_\infty \sim - \frac{\omega'_{cr} (r_\infty - r_{cr})}{\gamma_{cr}},$$

(3.7)

respectively. We note that the upper asymptotic limit is the amplitude of the localized solution for a perfect membrane tube (see equation (2.18)), whereas the lower asymptotic limit is of the same order as the original wall thinning. In figure 3a, we have shown the weakly nonlinear solution (3.6) together with a
portion of figure 2 near the bifurcation point. On the same figure, we have also used dots to indicate a set of fully nonlinear solutions obtained by integrating the system (2.10); these fully nonlinear solutions will be discussed in detail later.

We see that localized wall thinning has the effect of reducing the bifurcation value of $r_\infty$, and hence the bifurcation pressure. The reduced bifurcation value of $r_\infty$ corresponds to the turning point of the parabola in figure 3a, and is given by

$$r_\infty = r_{cr} + 2 \sqrt{\frac{2}{3}} \frac{\epsilon \gamma_{cr} \zeta h_0}{\omega'_{cr}}.$$  (3.8)

Since $\gamma_{cr} < 0$ and $\zeta > 0$ for the Ogden material, the above expression is real only if $h_0 < 0$, that is, only if the wall suffers localized thinning. In this case, the r.h.s. reduces to $r_{cr}(1 - 0.4935 \sqrt{-\epsilon h_0})$ so that the bifurcation value of $r_\infty$ is reduced by an amount proportional to the square root of the amplitude of localized thinning. As an illustrative example, when $\epsilon^2 h_0 = -0.1$, which corresponds to a 10 per cent maximum wall thinning, the reduction would be 16 per cent, which would bring the critical stretch from 1.6873 down to 1.4240.

When $h_0 > 0$, which corresponds to localized wall thickening, the two branches of the weakly nonlinear solution (3.6) correspond to the two solid lines in figure 4a where again the dashed line signifies the result for the perfect tube and the solid dots correspond to fully nonlinear solutions obtained by integrating (2.10). We note that the solid lines beyond the critical value $r_\infty = r_{cr}$ are meaningless as the corresponding weakly nonlinear solution (3.5) is no longer real. We had originally expected that in a small neighbourhood of $r_\infty = r_{cr}$, the fully nonlinear solution would be well approximated by the weakly nonlinear solution (3.6). But figure 4a shows that the upper branch of the fully nonlinear solution comes down and then turns left, instead of proceeding along the weakly nonlinear solution further. As a result, for values of $r_\infty$ less than 1.567 approximately, the fully nonlinear solution has three branches. The top and bottom branches correspond to localized bulging and necking, respectively, and are well approximated by the weakly nonlinear theory, whereas the middle solution has no counterpart in the weakly nonlinear theory. Further examination of the profiles corresponding to the middle branch reveals that each of them has a double hump structure, with the two humps symmetrically located on the two sides of $Z = 0$. The valley at $Z = 0$ has zero depth at the turning point (where the top and the middle branches join) and it gets deeper and deeper as we move left on the middle branch (figure 5). As $r_\infty$ decreases, we are confident that the top branch tends to that for the perfect tube and the bottom branch tends to the trivial state, but so far we have not yet been able to determine the fate of the middle branch. The reason is that the solution becomes increasingly sensitive to the guess of $r(0)$ and, as a result, it gets more and more difficult to decide whether the solution for $r(Z)$ is a true localized solution, a periodic solution or an unbounded solution. For instance, for the last point on the middle branch, a variation in $r(0)$ of the order of $10^{-7}$ can change the solution from one type to another.

The second special case that we consider corresponds to $h(\xi) = d_2 y_1(\xi)$, where $d_2$ is a constant. In this case, the effect of imperfection can be scaled out by the simple substitution

$$\omega'_{cr} r_1 \rightarrow \omega'_{cr} r_1 + d_2 \xi.$$
Figure 5. Profiles of $r(Z)$ corresponding to the middle branch for $r_{\infty} = 1.56, 1.45, 1.3$.

Thus, the localized solution is given by

$$y_1 = -\frac{\omega'_c r_1 + d_2 \zeta}{\gamma_c} \operatorname{sech}^2 \left( \frac{1}{2} \sqrt{\omega'_c r_1 + d_2 \zeta} \right).$$

Denoting $h(0)$ again by $h_0$, we have

$$h_0 = d_2 y_1(0) = -\frac{d_2(\omega'_c r_1 + d_2 \zeta)}{\gamma_c},$$

which can be solved to yield

$$\omega'_c r_1 + d_2 \zeta = \frac{\omega'_c r_1}{2} \left( 1 \pm \sqrt{1 - \frac{4 \gamma_c h_0 \zeta}{\omega'_c r_1^2}} \right).$$

It then follows that

$$r_0 - r_{\infty} = \epsilon y_1(0) = -\frac{\omega'_c (r_{\infty} - r_c)}{2 \gamma_c} \left( 1 \pm \sqrt{1 - \frac{4 \gamma_c h_0 \zeta^2}{\omega'_c (r_{\infty} - r_c)^2}} \right).$$

This expression has the same structure as (3.6). The two solutions are real only if the expression under the radical is non-negative, that is

$$r_{\infty} \leq r_c + \frac{2 \epsilon \sqrt{\gamma_c h_0 \zeta}}{\omega'_c},$$

where the r.h.s. defines the new critical principal stretch which may be compared with the r.h.s. of (3.8). For the Ogden material, the new critical stretch takes the form

$$r_c (1 - 0.604421 \sqrt{-\epsilon^2 h_0}).$$
where $-\epsilon^2 h_0$ represents maximum wall thinning. This type of wall thinning is slightly more effective in reducing the critical principal stretch. For instance, a 10 per cent maximum wall thinning would now reduce the critical stretch by 19 per cent.

Finally, we consider the case when $h(\xi)$ is proportional to $y''_1$ and write $h(\xi) = d_3 y''_1$, where $d_3$ is a constant. The last term in (3.3) can be eliminated by rescaling $\xi$ and we have

$$y_1 = -\frac{\omega'_r r_1}{\gamma_{cr}} \text{sech}^2 \left( \frac{1}{2} \sqrt{\frac{\omega'_r r_1}{1 - d_3 \xi}} \right).$$

(3.14)

From $h_0 = d_3 y''_1(0)$ we obtain

$$d_3 = \frac{2 \gamma_{cr} h_0}{2 \xi \gamma_{cr} h_0 + \omega'_r r_1^2}, \quad \omega'_r r_1 = \frac{\omega'_r r_1^2 + 2 \xi \gamma_{cr} h_0}{\omega'_r r_1}.$$ (3.15)

When $h_0 < 0$, the above solution is valid for all $r_1 < 0$ and so this kind of localized thinning has no effect on the onset of localized bulging. When $h_0 > 0$, the expression under the radical in (3.14) is positive only if (assuming $r_1 < 0$)

$$r_1 < -\frac{\sqrt{2\xi(-\gamma_{cr}) h_0}}{|\omega'_r|},$$

that is,

$$r_\infty \leq r_{cr} + \frac{\epsilon \sqrt{2(-\gamma_{cr}) h_0 \xi}}{\omega'_r}.$$ (3.16)

Since this is also the condition under which the expression for the imperfection is real, this case is of no practical interest.

We note that the wall thickness variation function $h(\xi)$ associated with the exact solutions (3.6) and (3.11) is dependent on $r_\infty$ through the appearance of $r_1$ in the argument of the sech function. Thus, for instance, the weakly and fully nonlinear results in figure 3a correspond to $h(\xi) = -\text{sech}^2(0.8990 \sqrt{r_{cr} - r_\infty} Z)$, which is continually changing as $r_\infty$ is varied. To show that this weak dependence on $r_\infty$ does not qualitatively change the main characteristics of the amplitude diagrams, we now consider the case when $h(\xi)$ is given by $h(\xi) = -e^{-\xi^2}$, which does not vary with respect to $r_\infty$. In this case the amplitude equation (3.3) can be solved numerically subject to the initial conditions $y_1(0) = k$, $y'_1(0) = 0$, where $k$ is iterated so that the solution satisfies the decay condition

$$y'_1(\xi) + \sqrt{\omega'(r_{cr})} r_1 y_1(\xi) \to 0 \text{ as } \xi \to \infty.$$  

We vary $r_1$, find the associated $y_1(0)$ and then compute $r_\infty$ and $r_0 - r_\infty$. With $h(0) = -1$, the small parameter $\epsilon$ can be taken as the square root of maximum relative wall thinning (see (3.1)). It is found that the amplitude diagram is similar
to the solid parabolic curve in figure 3a, but with the turning point occurring at \( r_1 = -0.8683 \). The corresponding reduced critical stretch is then given by

\[
r_{cr} - 0.8683\epsilon = r_{cr}(1 - 0.5146\epsilon),
\]

which may be compared with (3.13).

4. Conclusion

In this paper, we have used the simple problem of tube inflation to quantify the effect of material and geometrical imperfections on the critical load for the onset of localization. For imperfections in the form of localized wall thinning, it is found that they can reduce the critical circumferential stretch by an amount that is of the order of the square root of the imperfection amplitude. The proportionality coefficient is dependent on the form of the imperfection, but this square root rule is universal. This result complements the well-known Koiter’s two-thirds power rule for subcritical bifurcations into periodic patterns and demonstrates that bifurcation into localized patterns is much more susceptible to imperfections than bifurcation into periodic patterns.

We note that effects of imperfections have also previously been studied numerically by Kyriakides & Chang [12] on a tube whose radial movement is restricted at the two ends, and they came to the conclusion that imperfections have little effect on the initiation pressure. This result does not contradict our conclusion as restricted ends can be viewed as an imperfection (in the sense that bulging occurs as soon as the tube is inflated) and so essentially Kyriakides & Chang [12] were comparing one imperfection against another.

Our study is also motivated by possible application in the mathematical modelling of aneurysm formation in human arteries. All existing studies on aneurysm modelling have focused on the evolution of aneurysms after they have already formed, with a view to guide the clinician as to whether the risk of bursting outweighs the risk of repairs (see Watton et al. [45], Watton & Hill [46] and references therein). We believe that our present study might be relevant to understanding the process leading to the initial formation of aneurysms. One scenario is that pathological changes in arteries change the constitutive behaviour and cause localized thinning, which in turn makes localized bulging possible. Some preliminary studies based on simple models for arterial walls have shown that the onset pressure for localized bulging is unrealistically high in the absence of imperfections. If localized wall thinning can reduce the critical circumferential stretch significantly, it can then bring the onset pressure to more realistic values.

Another aspect of our results that is relevant to aneurysm modelling is that localized wall thickening is not necessarily beneficial. On the one hand, it can equally reduce the onset pressure, and on the other hand, a twin-hump aneurysm can be excited at much lower pressure values; see the turning point and the middle branch in figure 4a, respectively. Of course, the actual arterial wall constitution is much more complicated than the single-walled isotropic model used in the present study [47], but our present result is at least indicative of what might be expected when more elaborate models are used.

Since this paper was submitted, it has subsequently been demonstrated by Fu et al. [48], using a realistic constitutive model, that the initial onset of aortic...
abdominal aneurysms can indeed be modelled as a bifurcation phenomenon, and that in the bifurcation interpretation of aneurysm formation, the imperfection sensitivity determined in this paper plays a pivotal role.

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References


