Moment isotropy and discrete rotational symmetry of two-dimensional lattice vectors

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We present a direct proof of a theorem linking the order of moment isotropy and degree of discrete rotational symmetry for a two-dimensional set of lattice vectors. This theorem has been proved previously based on properties of sinusoidal functions. The new proof is based instead on purely linear algebraic arguments.

Keywords: isotropy; higher-order moments; rotational symmetry

1. Introduction

The lattice Boltzmann method (LBM) is one of the most active and promising fields in computational fluid dynamics (cf. [1,2]). With proper choices of a set of discrete velocity vectors, continuum hydrodynamics can be recovered at the Navier–Stokes level [3,4] and beyond (cf. [5,6]). The main theme of LBM studies is how to recover continuum fluid behaviour out of only a finite set of discrete vector values. Indeed, it is known that hydrodynamic properties can be represented by moments at various orders (cf. [7]). Deeper physics is captured if a set of lattice velocity vectors reproduce hydrodynamic moments at higher orders [8]. Hence, the key task is to ensure that the relevant moments from a set of discrete lattice velocities are identical to those from the continuum-valued microscopic velocities of real hydrodynamics. The latter by definition gives isotropic moments. Since a finite discrete vector set only admits a finite rotational symmetry, the task immediately comes down to the basic question: What degree of discrete rotational symmetry must a lattice vector set satisfy in order for a hydrodynamic moment of a given order to be isotropic? Besides direct applications in LBMs, such a question is also of interest in basic mathematics.

In this paper, we present a direct proof of a theorem linking the degree of rotational symmetry of a set of two-dimensional lattice vectors to the order of isotropy of its resulting moments. Such a theorem was proved previously based on sinusoidal functions [9]. In this paper, we provide an alternative proof that is based solely on simple linear algebraic and symmetry arguments. This alternative proof hopefully may reveal further insights into the nature of discrete systems and their macroscopic properties. We emphasize that we do not claim originality for

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identifying such a theorem, because such a concept or at least part of it is known in the community. On the other hand, we believe that the proof presented here is novel and unique, and we are not aware of any similar proofs of the theorem elsewhere.

2. Moment isotropy in two dimensions

We consider a lattice velocity set, \( C \), containing \( b \) discrete vectors in two-dimensional space. Explicitly,

\[
C = \{ c_{\alpha}; \alpha = 1, 2, \ldots, b \},
\]

and each element in the set is a constant two-dimensional vector, \( c_{\alpha} = [c_{\alpha,x}, c_{\alpha,y}] \). The vector set generates a crystallographic structure: all vectors in the set can be viewed as originating from a common point, 0. We say that \( C \) is \( B \)-fold (\( B \) being an integer) rotationally symmetric (or rotationally symmetric of degree \( B \)) if the velocity set is invariant under rotations of any integer multiple of \( 2\pi/B \). In other words, when every member vector in the set undergoes a rigid rotation of an angle multiple of \( 2\pi/B \) around 0, the rotated set of vectors yields the original set \( C \). In addition, we say that \( C \) is parity invariant if it is unchanged when every vector in the set is reversed, \( c_{\alpha} \rightarrow -c_{\alpha}, \alpha = 1, \ldots, b \). Obviously, a parity-invariant set \( C \) only admits \( B \) being even integers, while it is not parity invariant when \( B \) is odd. Note that the integer \( B \) is not equal to \( b \) in most circumstances.

An \( n \)-th-order (\( n \) being a positive integer) moment tensor constructed out of such a discrete velocity set is defined as a summation of direct products of all the vectors in \( C \):

\[
M^{(n)} \equiv \sum_{\alpha=1}^{b} c_{\alpha} c_{\alpha} \cdots c_{\alpha}. \tag{2.1}
\]

When \( n \) is odd, it is trivial to show that all odd-order moment tensors vanish for a parity-invariant lattice vector set. For even moments, say \( M^{(2n)} \), a few basic properties can be noted. First, there are \( D^n \times D^n \) number of elements (\( D = 2 \)), \( M_{i_1 i_2 \ldots i_{2n}}^{(2n)} \). Second, \( M^{(2n)} \) has a ‘super-symmetric’ form so that its elements are invariant under any permutation of its \( 2n \) sub-indices. For example, \( M_{i_1 i_2 \ldots i_{2n}}^{(2n)} = M_{i_{2n} i_{2n-1} \ldots i_1}^{(2n)} \).

The following theorem holds.

**Theorem 2.1.** For a two-dimensional parity-invariant lattice velocity set that has \( B \)-fold rotational symmetry, its \( n \)-th-order moment \( M^{(n)} \) is isotropic for \( n < B \). Also, the moment vanishes for all odd \( n \).

This theorem can be proved using sinusoidal functions (cf. [9]). Here, we give an alternative proof of this theorem using only linear algebraic and simple symmetry arguments.

The proof for vanishing odd-integer moments is trivial via applying parity arguments, so we do not need to elaborate here. For an even moment \( M^{(2n)} \), a scalar is produced by taking an inner product of it with a vector \( v \). To do this,
we define
\[ M^{(2n)} \odot \mathbf{v}^{2n} = \sum_{i_1=x}^{y} \sum_{i_2=x}^{y} \cdots \sum_{i_{2n}=x}^{y} M^{(2n)}_{i_1 i_2 \ldots i_{2n}} v_{i_1} v_{i_2} v_{i_{2n}}, \]
so that, for the specific tensor (2.1), it follows that
\[ M^{(2n)} \odot \mathbf{v}^{2n} = \sum_{a=1}^{b} (c_{a} \cdot \mathbf{v})^{2n}. \tag{2.2} \]

It is easily seen that the \( M^{(2n)} \) is isotropic if and only if, for any arbitrary two-
dimensional vector \( \mathbf{v} \), the following result is true:
\[ \sum_{a=1}^{b} (c_{a} \cdot \mathbf{v})^{2n} = A|\mathbf{v}|^{2n}, \tag{2.3} \]
where \( A \) is a real number independent of \( \mathbf{v} \), and \( |\mathbf{v}| = \sqrt{v_{x}^{2} + v_{y}^{2}} \) is the magnitude of \( \mathbf{v} \). Without proof, we mention that an \( n \)th-order isotropic tensor \( M^{(2n)} \) takes
on the following form:
\[ M^{(2n)}_{i_1 \ldots i_{2n}} \propto \Delta^{(2n)}_{i_1 \ldots i_{2n}}, \tag{2.4} \]
where \( \Delta^{(2n)}_{i_1 \ldots i_{2n}} \) is the generalized Kronecker delta function made of products of \( n \) standard Kronecker delta functions \( \delta_{ij} \) and summed over all possible distinct permutations of pairwise indices (cf. [9,10]).

Before starting on the full proof of isotropy for a \( 2n \)th-order moment tensor, we offer some insight into the basic concept. Let us first take a look at the special
case of a second-order tensor, \( M^{(2)} \), which can be expressed as a \( 2 \times 2 \) regular
matrix, \( M_{ij} \) (i, \( j = x \) or \( y \)). Obviously, by construction (see equation (2.1)), this
matrix is a real symmetric matrix, \( M_{ij} = M_{ji} \). Hence, there exists at least one
eigenvector, \( \mathbf{e} = [e_{x}, e_{y}] \), such that
\[ M^{(2)} \cdot \mathbf{e} = \lambda_{2} \mathbf{e}, \tag{2.5} \]
where \( \lambda_{2} \) is the corresponding real number eigenvalue.

If \( M^{(2)} \) is invariant under rotation by an angle \( 2\pi/B \), then another vector, \( \tilde{\mathbf{e}} \),
created by rotating \( \mathbf{e} \) by \( 2\pi/B \) is also an eigenvector having the same eigenvalue,
\( \lambda_{2} \). Apparently, as long as \( B > 2 \), \( \tilde{\mathbf{e}} \) is not collinear with \( \mathbf{e} \). Thus \( \mathbf{e} \) and \( \tilde{\mathbf{e}} \) form a
basis in two-dimensional linear space, and any arbitrary two-dimensional vector
\( \mathbf{v} \) can be constructed from a linear combination of these two vectors,
\( \mathbf{v} = a \mathbf{e} + b \tilde{\mathbf{e}}, \)
with \( a \) and \( b \) being real numbers. As a consequence, we have
\[ M^{(2)} \cdot \mathbf{v} = M^{(2)} \cdot (a \mathbf{e} + b \tilde{\mathbf{e}}) = \lambda_{2}(a \mathbf{e} + b \tilde{\mathbf{e}}) = \lambda_{2} \mathbf{v}. \tag{2.6} \]
The above result indicates that the matrix \( M_{ij} \) accepts any two-dimensional vector
as its eigenvector and all with the same eigenvalue. The only possibility is that
such a matrix is an identity matrix multiplied by a constant scalar factor \( \lambda_{2} \). That
is, \( M_{ij} \propto \delta_{ij} \). Notice, since a parity-invariant velocity set only admits rotation with
even symmetries (i.e. \( B \) is even), that the lowest rotational symmetry required
for isotropy of \( M^{(2)} \) is \( B = 4 \).
Viewing conditions (2.3) or (2.4), we have proved the following theorem.

**Theorem 2.2.** For a two-dimensional parity-invariant lattice velocity set, its second moment $M^{(2)}$ is isotropic if the set is $B$-fold rotationally invariant for $B \geq 4$.

Now we proceed to the main task of the paper, viz., to prove the general theorem for moment tensors of an arbitrary order $2n$. The framework to be used bears a strong resemblance to the regular eigenvalue problem for the special second-order moment case above. However, the key differences are that $M^{(2n)}$ does not have the usual familiar (‘second rank’) matrix form, and the $n$-fold direct product of $v$ (shown below) is not a regular vector.

The generalized vector $v^n$ is an $n$ direct product of a regular vector $v$,

$$v^n \equiv v \cdots v.$$  

(2.7)

It has $D^n$ ($D = 2$) real number of components. For example, expressing $v^2 (\equiv vv)$ in component form, it has the following four-component representation $v^2 = [v_x^2, v_xv_y, v_xy, v_y^2]$. Clearly, the middle two components have the same value. In other words, a direct product $vv$ of an arbitrary two-dimensional vector $v$ forms a single vector array of four components, but at most three of them can have different values. This observation can be generalized for $n$ direct products of a two-dimensional vector $v$: it forms a $2^n$-component single vector array, and only at most $n + 1$ number of components can be different. This means that there is a direct one-to-one mapping between $v^n$ and an $(n + 1)$-dimensional regular vector, $V^n \equiv [v^n_x, v^{n-1}_xv_y, \ldots, v_xyv^{n-1}_y, v^n_y]$. As we know from basic linear algebra, any $(n + 1)$-dimensional vector can be fully determined by $n + 1$ linearly independent vectors. The challenge is how to generate such a set of $n + 1$ linearly independent vectors from a set of $n + 1$ regular two-dimensional vectors.

**Lemma 2.3.** The $n + 1$ product vectors $V^n_\beta \equiv [v^n_{\beta,x}, v^{n-1}_{\beta,x}v_{\beta,y}, \ldots, v_{\beta,x}v^{n-1}_{\beta,y}, v^n_{\beta,y}]$ ($\beta = 1, 2, \ldots, n + 1$) form a complete basis in $(n + 1)$-dimensional space if their corresponding $n + 1$ generating regular vectors $v_\beta \equiv [v_{\beta,x}, v_{\beta,y}]$ ($\beta = 1, 2, \ldots, n + 1$) are mutually not collinear (i.e. no two vectors are parallel).

**Proof.** Assume that $v_{\beta,x} \neq 0$ for all $\beta$. Then, by a simple re-scaling we can assume that $v_{\beta,x} = 1$ for all $\beta$ and no two $v_{\beta,y}$ are equal. Then, the matrix whose rows are $V^n_\beta \equiv [1, \beta, \beta^2, \ldots, \beta^{n-1}, \beta^n]$ is a van der Monde matrix whose determinant is non-zero. Therefore, the $n + 1$ vectors $V^n_\beta$ are linearly independent and form a basis for $(n + 1)$-dimensional space.

If one of the vectors is $[0, 1]$, then the determinant is either $+/-$ the determinant of the remaining $n$-dimensional van der Monde matrix. Again, the $n + 1$ vectors are linearly independent and span the space. 

Because of the one-to-one correspondence between an $n$ product vector $v^n$ and its $(n + 1)$-dimensional vector $V^n$, $v^n$ is a vector on an $(n + 1)$-dimensional linear space, even though it has $2^n$ number of Cartesian components. Moreover, $v^n$ is completely determined in terms of $n + 1$ product vectors $v^n_\beta$ ($\beta = 1, 2, \ldots, n + 1$) that are generated from $n + 1$ non-collinear regular vectors $v_\beta$.  

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In addition to the comparison between the product vector $\mathbf{v}^n$ with a regular vector, we can also verify that $\mathbf{v}^n$ obeys the inner product rule for a vector of a single array:

$$\mathbf{v}^n \circ \mathbf{w}^n \equiv (\mathbf{v} \cdot \mathbf{w})^n = \sum_{i=1}^{2^n} v_i^n w_i^n, \quad (2.8)$$

where both $\mathbf{v}^n$ and $\mathbf{w}^n$ are $n$ product vectors, generated, respectively, out of $\mathbf{v}$ and $\mathbf{w}$, and $v_i^n$ and $w_i^n$ are the $i^{th}$ components in $\mathbf{v}^n$ and $\mathbf{w}^n$: $\mathbf{v}^n = [v_1^n, v_2^n, \ldots, v_{2^n}]$ and $\mathbf{w}^n = [w_1^n, w_2^n, \ldots, w_{2^n}].$ The last equality in equation (2.8) indicates that the inner product for the product vectors follows the standard definition if they are expressed in the forms of single arrays of $2^n$ components. Indeed, among the $2^n$ number of components, $C_m^n \equiv n!/[m!(n-m)!]$ of them have identical values that are equal to $v_x^m v_{y}^{n-m} \; (m = 0, \ldots, n).$ Therefore,

$$\sum_{i=1}^{2^n} v_i^n w_i^n = \sum_{m=0}^{n} C_m^n (v_x^m w_x^{n-m}) (w_y^m v_y^{n-m}) = \sum_{m=0}^{n} C_m^n (v_x^m w_x^m) (v_y^m w_y^{n-m})$$

$$= (v_x w_x + v_y w_y)^n. \quad (2.9)$$

Since the $2^n$-component $\mathbf{v}^n$ is not a regular vector (in $2^n$-dimensional space) but rather a vector in an $(n+1)$-dimensional space, though $\mathbf{M}^{(2^n)}$ may be interpreted as a $2^n \times 2^n$ symmetric matrix, it may not admit a conventionally defined eigenvalue problem for any product vector $\mathbf{v}^n.$ On the other hand, it is easily seen that

$$\mathbf{M}^{(2^n)} \circ \mathbf{v}^n \equiv \mathbf{M}^{(2^n)} \underbrace{\cdots \mathbf{v} \cdots \mathbf{v}}_{n \text{ times}}$$

is also a generalized vector of $2^n$ components. Without loss of generality, we can always express it as $\mathbf{M}^{(2^n)} \circ \mathbf{v}^n \equiv \lambda \mathbf{v}^n + \mathbf{h},$ where the scalar $\lambda = \lambda \mathbf{v}^n \circ \mathbf{M}^{(2^n)} \circ \mathbf{v}^n/|\mathbf{v}^n|^2,$ and $\mathbf{h} \equiv \mathbf{P} \circ \mathbf{v}^n$ is another $2^n$-component vector perpendicular to $\mathbf{v}^n,$ $\mathbf{h} \circ \mathbf{v}^n = 0.$ Here, $\mathbf{P} \equiv \mathbf{M}^{(2^n)} - \mathbf{I},$ where $\mathbf{I}$ is a $2^n \times 2^n$ unit matrix, is a ‘projection operator’ of $\mathbf{v}^n$ onto its orthogonal subspace. Notice that both $\lambda$ and $\mathbf{P}$ are in general dependent on the choice of $\mathbf{v}^n.$

**Lemma 2.4.** If the tensor $\mathbf{M}^{(2^n)}$ is invariant under rotation by an angle $\theta,$ then $\mathbf{M}^{(2^n)} \circ \mathbf{w}^n$ has the same $\lambda$ and $\mathbf{P}$ (with $\mathbf{w}^n \circ (\mathbf{P} \circ \mathbf{w}^n) = 0$) as $\mathbf{M}^{(2^n)} \circ \mathbf{v}^n$ if $\mathbf{w}^n$ is an $n$ product of $\mathbf{w} = \mathbf{R}(\theta) \cdot \mathbf{v},$ and $\mathbf{R}(\theta)$ is the rotation operator of an angle $\theta.$ In addition, if a product vector $\mathbf{u}^n$ can be constructed out of a linear combination of the $\mathbf{v}^n$ and $\mathbf{w}^n,$ then $\mathbf{M}^{(2^n)} \circ \mathbf{u}^n \equiv \lambda \mathbf{u}^n + \mathbf{P} \circ \mathbf{u}^n,$ with the same $\lambda$ and $\mathbf{u}^n \circ (\mathbf{P} \circ \mathbf{u}^n) = 0.$

**Proof.** It is straightforward to see from the definitions that $\mathbf{w}^n = (\mathbf{R}(\theta) \cdot \mathbf{v})^n \equiv \mathbf{R}^n(\theta) \circ \mathbf{v}^n.$ Let $\mathbf{M}^{(2^n)} \circ \mathbf{w}^n \equiv \lambda \mathbf{w}^n + \mathbf{P} \circ \mathbf{w}^n,$ where $\lambda = \mathbf{w}^n \circ \mathbf{M}^{(2^n)} \circ \mathbf{w}^n/|w|^2.$ The invariance of $\mathbf{M}^{(2^n)}$ shows that

$$\mathbf{w}^n \circ \mathbf{M}^{(2^n)} \circ \mathbf{w}^n = \mathbf{v}^n \circ \mathbf{M}^{(2^n)} \circ \mathbf{v}^n. \quad (2.10)$$

Also, $\mathbf{w}$ being a mere rotation of $\mathbf{v},$ it follows that $|\mathbf{w}|^2 = |\mathbf{v}|^2.$ Thus, we have $\lambda = \mathbf{P} \equiv \mathbf{M}^{(2^n)} - \mathbf{I}$ is thus equal to $\mathbf{P}.$ Since $\mathbf{w}^n \circ (\mathbf{P} \circ \mathbf{w}^n) = \mathbf{v}^n \circ (\mathbf{P} \circ \mathbf{v}^n) = 0,$ $\mathbf{P}$ is an orthogonal projection operator for both $\mathbf{v}^n$ and $\mathbf{w}^n,$ i.e. $\mathbf{P} \circ \mathbf{v}^n \perp \mathbf{v}^n$
and $P \odot w^n \perp w^n$. It also is straightforward to see via symmetry that $P \odot w^n = R^n(\theta) \odot (P \odot \nu^n)$. That is, the orthogonally projected $w^n$ is a rotation of the orthogonally projected $\nu^n$.

Next, if $u^n = a\nu^n + b w^n$, with $a$ and $b$ being real numbers, since $w = R(\theta) \cdot \nu$, it follows that $u^n = (a + b R^n(\theta)) \cdot \nu^n$. However, since $u^n$ is a product vector, it must also be expressible as an $n$ product of the regular vector $u = \rho R(\theta') \cdot \nu$, where $R(\theta') \cdot \nu$ represents a rotation by angle $\theta'$ (and $\rho$ is a constant factor $= 1$, without loss of generality). Therefore, $a + b R^n(\theta) = R^n(\theta')$ must hold. Taking these facts into account, we have

$$
M^{(2n)} \odot u^n = M^{(2n)} \odot (a\nu^n + b w^n) = a(\lambda + P \odot) \nu^n + b(\lambda + P \odot) w^n
$$

$$
= \lambda u^n + aP \odot \nu^n + bP \odot w^n
$$

$$
= \lambda u^n + (a + b R^n(\theta)) (P \odot \nu^n)
$$

$$
= \lambda u^n + R^n(\theta') \odot (P \odot \nu^n). 
$$

Hence,

$$
\lambda u^n \odot M^{(2n)} \odot u^n = \lambda|u|^2n + u^n \odot (R^n(\theta') \odot (P \odot \nu^n)).
$$

However, from $u^n = R^n(\theta') \odot \nu^n$, the second term on the right-hand side of the above becomes

$$
(R^n(\theta') \odot \nu^n) \odot (R^n(\theta') \odot (P \odot \nu^n)) = \nu^n \odot (P \odot \nu^n) = 0.
$$

Consequently, $u^n \odot M^{(2n)} \odot u^n = \lambda|u|^2n$. ■

We are now ready to prove the central result in this paper.

**Theorem 2.5.** If a parity-invariant lattice vector set $C$ is $B$-fold rotationally symmetric, then its resulting moment tensors $M^{(2n)}$ are isotropic up to order $2n \leq B - 2$.

**Proof.** Since the parity-invariant vector set has a $B$-fold rotational symmetry, moment $M^{(2n)}$ thus constructed is $B$-fold rotationally invariant. From lemma 2.4, there exist $B/2$ non-collinear vectors generated out of $2m\pi/B$ ($m = 0, 1, \ldots, B/2 - 1$) rotations from each other, so that

$$
M^{(2n)} \odot \nu^n_m \equiv \lambda\nu^n_m + P\nu^n_m, \quad m = 0, 1, \ldots, B/2 - 1,
$$

all having the same $\lambda$ and $P$. From lemma 2.3, if $B/2 \geq n + 1$, then we can select $n + 1$ out of these $B/2$ product vectors to form a complete basis, so that any arbitrary product vector $\nu^n$ can be expressed as a linear combination of them, $\nu^n = \sum_{a=1}^{n+1} a\nu^n_a$, where $a\nu^n_a$ are some real numbers. Hence, for an arbitrary product vector $\nu^n$, we have

$$
\nu^n \odot M^{(2n)} \odot \nu^n = \lambda|\nu|^{2n},
$$

(2.12)

according to lemma 2.4. Notice again, in order to achieve the above conclusion, that we must have $B/2 \geq n + 1$, so that $M^{(2n)}$ is isotropic for $2n \leq B - 2$. ■

Explicitly, we see that the fourfold rotationally symmetric square lattice gives moment isotropy up to second order, the sixfold rotationally symmetric hexagonal lattice gives moment isotropy up to fourth order, and the eightfold rotationally symmetric octagonal lattice up to sixth order, etc.
3. Discussion

In this paper, we have presented a new proof of the theorem that relates the degree of rotational symmetry of a two-dimensional lattice vector set and the order of isotropy of its moment tensors. The theorem has been proved previously using sinusoidal functions (cf. [9]). On the other hand, the new proof presented here is based only on simple linear algebraic and symmetry arguments.

Notice that the theorem proved in detail here is only for situations that satisfy parity invariance. On the other hand, this theorem is easily generalized to include non-parity-invariant lattice sets, e.g. when the degree of rotational symmetry $B$ is odd. All the steps in the above proof are carried out without modifications, except that the final condition, $B/2 \geq n + 1$, needs to be replaced by $B \geq n + 1$, since all $B$ vectors are not collinear rather than just half of them. Hence, we have a more general theorem.

**Theorem 3.1.** If a lattice vector set $C$ is $B$-fold rotationally symmetric, then its resulting moment tensors $M^{(2n)}$ are isotropic up to order $2n \leq B - 2$ when $B$ is even, and isotropic up to order $2n \leq 2B - 2$ when $B$ is odd.

The above property presents a rather interesting paradox: higher symmetry by parity produces lesser moment isotropy. For a specific example, a threefold symmetric triangular lattice admits fourth-order moment isotropy, while a fourfold square lattice only admits second-order isotropy; a fivefold pentagonal lattice admits eighth-order isotropy, while the sixfold hexagonal lattice only admits fourth-order isotropy. On the other hand, non-parity-invariant lattices only give isotropic (i.e. vanishing) odd-order moments up to $n \leq B - 2$ (cf. [9]).

Another question pertaining to the theorem is whether or how such a property in two dimensions may be extended to higher dimensions. There are apparent difficulties in making this extension. For instance, there are only a finite number of regular polytopes for dimensions greater than two [11]. Hence, there does not seem to exist a similar relationship that links the rotational symmetry to tensor isotropy at arbitrary orders. On the other hand, it is still of interest to show why some specific polytopes obey tensor isotropy to certain known orders. For example, through direct verification [8], we know that both icosahedron and dodecahedron symmetries admit moment tensor isotropy up to fourth order in three-dimensional space, while cube and octahedron symmetries admit isotropy only up to second order. Directly observing geometric structures from the viewpoint of rotational symmetry, we find that each of the former contains 12 rotationally symmetric axes of degree 5, while the highest degree of rotational symmetry along any axis in the latter two is 4. According to the theorem above, we recognize that tensor moments projected onto two-dimensional planes perpendicular to these axes have isotropy up to 4 (or higher) or 2, respectively. The question then is why icosahedron and dodecahedron structures admit fourth-order moment isotropy in the full three-dimensional space? Or, more generally, why and under what conditions does a polytope admit a given order of moment isotropy in its full dimensional space, if such an order of isotropy is satisfied for finite rotational axes?

It is even less straightforward to conceptualize rotational symmetry for polytopes in dimensions higher than three. Yet, by direct examination, we know that both the so-called 4DFCHC polytope [12] and its reciprocal [9] admit
moment isotropy up to fourth order, while an equally weighted combination of the two increases the order of moment isotropy to sixth. Henceforth, it is of particular interest to understand the underlying reasons from a generalized rotational symmetry point of view.

The authors wish to dedicate this work to the memory of our dear friend and collaborator Isaac Goldhirsch.

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