Modulated oscillations in many dimensions

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Modulated oscillations are described via their time-varying amplitude and frequency. For multivariate signals, there is structure in the signal beyond this local amplitude and frequency defined for each signal component, in turn describing the commonality of the components. The multivariate structure encodes how the common oscillation is present in each component signal. This structure will also be evolving. I review the special case of the representation of both bivariate and trivariate oscillations. Additionally, existing results on the general multivariate oscillation are covered. I discuss the difference between a model of a multivariate oscillation compared with other common signal models of phenomena observed in several channels, and how their properties are different. I show how for the multivariate signal the global dimensionality of the signal is built up from local one-dimensional contributions, and introduce the purely unidirectional signal, to quantify how any given signal is different from the closest such signal. I illustrate the properties of the derived representation of the multivariate signal with synthetic examples, and discuss the representation of data from observations in physical oceanography.

1. Introduction

Oscillations are a fundamental type of signal, arising in many fields of observations, such as systems biology, econometrics, acoustics and physics [1–4]. They are a natural form of variation because they describe fluctuations around a mean value and so can stably persist over time, without meandering very far away from previously visited values. The basic description of an oscillation involves its amplitude and the period of repetition, and in addition to serving as a representation of a signal, the two quantities can also be used as a basis for parameter estimation and classification of the signal.
To encompass more varied signals arising in nature than simple constant-frequency and amplitude oscillations, it becomes necessary to permit the variation in the signal to increase. I plot an oceanographic example of a pair of float time series in figure 1 showing an initial stage where the two components are oscillatory before $t = 252$ and later map out meandering motion. To get a better idea of their oscillatory structure, I also show their Fourier transform in figure 2, as well as the Fourier transform of the signals before $t = 252$. We see the global presence of many frequencies, but a considerable prevalence of frequencies near $\omega \approx 1.5$ before $t = 252$. 

**Figure 1.** A bivariate oscillation observed in an oceanographic signal: (a) the northward velocity and (b) the eastward velocity. A solid vertical line has been added to show $t = 252$ in both plots.

**Figure 2.** A bivariate oscillation observed in an oceanographic signal (Fourier transform): (a) the northward velocity and (b) the eastward velocity. Solid lines, full record; dotted lines, initial record.
Two features are apparent from these plots: firstly, the overall change in amplitude of the oscillations, and secondly, a drift in the instantaneous period of the oscillations. It is my wish to describe this pair of signals as a single phenomenon. To be able to do so, two things need to be understood: how to combine multiple channels containing oscillations into one object to model, and how to describe the evolution of this object as one rather than as two separate phenomena. Reverting back to the figure, for the bivariate signal this corresponds to how the relative amplitudes evolve, and how the joint phase characteristics of the two signals change together over time as well as how to parametrize them as simply and intuitively as possible.

The main tool for describing time-varying oscillations is as a modulated oscillation [5,6]. This corresponds to modelling a signal with evolving amplitude and frequency over time, just like the two signals in figure 1 appear to. In multiple dimensions, we need to understand coherent oscillations as viewed across several channels, and the observation of such oscillations can be found in practical applications, for example, as drifter and float time series in oceanography [7], as trivariate records of acceleration in seismology [8], as neuroscience measurements, such as electroencephalography, as magnetoencephalography [9], as blood flow [10], as climate signals [11] and as the generic application of acoustic transducers [12].

To understand multivariate oscillations, I start by reviewing the concept of a modulated oscillation in one dimension [5], as well as the recently introduced concept of a modulated oscillation in two dimensions [7,13] and three dimensions [14] jointly with the more general basics of the n-dimensional oscillation [15]. When observing more than one signal, our understanding of the properties and relationships between the multiple signals needs to be modelled, as well as any joint time variation. In this instance, ideas from multivariate statistics (see e.g. [16] or [17]) meet time series and signal processing, and the modulated oscillation [18, ch. 1].

My idea of a multivariate modulated oscillation is the combined model for the time evolution of a single oscillatory signal, joined with evolving relationships between signals. As a simple example of the additional structure in a multivariate signal, I show again data from an oceanographic float in figure 3a, discussed as two separate signals earlier in this section. As can be seen from the figure, the combination of the north and east velocities are mapping out an ellipse in space, and as time moves forward the ellipse has decreasing radius, as is also obvious from the figure of superpositions of trajectories in figure 3b. The properties of the phenomenon we are observing, the float being trapped in a vortex, are described both by the geometry of the signal (such as the geometry of the circled ellipse describing the vortex that is circled by the measurement device), and by the frequency of the circling. These properties can be parametrized in terms of a bivariate oscillation [13], and the parameters of the ellipse are natural summaries of the observed signal [7].

Unlike for independent replicates of vector-valued random variables, the time-domain structure of multiple time series corresponding to a multivariate oscillation is ‘connected’ (smooth) in time, and this structure must be modelled, quantified and understood. Because we are dealing with oscillations, it is natural to tackle this problem in the Fourier domain. An approach would be first to Fourier transform all the observed signals in time and then implement dimensionality reduction over the component series in frequency [19, ch. 9]. This has the double benefit, for a stationary set of series, e.g. those time series whose statistical properties are invariant to the exact time of observation, of creating independent realizations at each frequency, and converting phase and time shifts into multiplicative operations in frequency, thus simplifying further analysis. The joint time-evolving structure of the signals treated here is more complicated, as their frequency content is evolving, their joint evolution must be understood, and the valid set of behaviours that lead to an oscillation must be determined.

We need to decide how to quantify or parametrize the time evolution of the multivariate oscillation to understand what forms of time variation are stable, and what choices of the parameters lead to non-oscillatory phenomena so that they may be excluded (cf. [20]). We can contrast this with one-dimensional signals where the stability is easier to quantify. For a single oscillation, the amplitude and frequency modulation describe the basic time frequency
Figure 3. An example of a simple bivariate oscillation in float data from oceanography. (a) The trajectories in space–time, with the z-axis as time; (b) the Lissajous plot of the same trajectory. The time axis has been normalized for display convenience.

characteristics of the signal, as the amplitude determines whether a signal component can be considered ‘present’ at a given time, while the instantaneous frequency governs what frequencies are present locally in time. In addition to these two quantities, other quantities derived from them describe the evolution of the local frequency and the amplitude, such as the chirp rate and the instantaneous bandwidth. The instantaneous bandwidth quantifies the degree of amplitude modulation, which can be used, to the first order, to quantify if it is suitable to describe a single signal as a modulated oscillation.

Building on, and reviewing, the previous work [13,15], I introduce new time-domain summaries of the multivariate structure of the oscillations. The model of a multivariate oscillation implies that we can always write the signal locally as one dimensional. I describe how to approximate the signal by a simple unidirectional approximation globally, and how to quantify the signal’s departure from the globally unidirectional. I calculate how the global dimensionality of the multivariate signal is built up over time by unidirectional contributions. This forms a nice contrast to how signals are built up in dimensionality in an unobserved components model, where the multivariate structure is time invariant but where the temporal variation of the common structure is not constrained at all, unlike the case of an oscillation that is locally smooth.

As a final note, this paper is about describing oscillations observed across many channels observed over time—it is not about isolating or extracting these components. I shall assume that this pre-processing step has already been carried out and the signal components estimated separately, using, say, discrete wavelets [21,22], continuous wavelets [15,23] or the empirical mode decomposition [24,25], and instead discuss the representation of the object in terms of its time-varying characteristics once estimated, rather than its intrinsic extraction. To post-process the output of such methods or any of the other forms of signal analysis (see also [26]), the model of the multivariate oscillation needs to be more clearly formulated and understood.
2. Modulated oscillations

(a) Amplitude-modulated, frequency-modulated signals and the analytic signal

Before I can discuss modulated oscillations in several dimensions, let me start by describing amplitude modulation (AM) and frequency modulation (FM) [5,6,27,28] in a univariate signal. Such a signal can be written as

\[ x(t) = a(t) \cos(\phi(t)), \]

where \( a(t) \) is the amplitude modulation and \( \phi(t) \) is the time-varying phase function, a result of frequency modulation. Clearly, there are no unique \( a(t) \) and \( \phi(t) \) only that satisfy equation (2.1), and we need to understand why one assignment should be preferred. By calculating the analytic signal \( x_+(t) \) from \( x(t) \) [18,29], the so-called canonical amplitude \( a(t) \) and phase \( \phi(t) \) from \( x(t) \) can be defined by

\[ x_+(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega = a(t) e^{i\phi(t)} = x(t) + iy(t). \]

It follows that unless we are given \( a(t) \) and \( \phi(t) \) explicitly and if we are only presented with \( x(t) \), there is no unambiguous method of retrieving the two functions \( a(t) \) and \( \phi(t) \). In this paper, I choose to use the canonical amplitude and phase [27,29]. Alternatives in using the analytic signal include the Teager–Kaiser method [30], Mandelstam’s method [31], Shekel’s method [32], (semi-)parametric models of amplitude and phase [33] or probabilistic alternatives [34]. Any of these methods yields a second signal, \( y(t) \), to complement \( x(t) \) into a complex-valued function \( x_+(t) = x(t) + iy(t) \), or equivalently give an amplitude \( a(t) = \sqrt{x^2(t) + y^2(t)} \) and a phase \( \phi(t) = \tan^{-1}(y/x) \), such that \( x_+(t) = a(t)e^{i\phi(t)} \). Three important rates of change describing the oscillation \( x_+(t) \) are given by

\[ \omega(t) \equiv \phi'(t), \quad \nu(t) \equiv \frac{\phi'(t)}{a(t)} \quad \text{and} \quad \xi(t) \equiv \frac{\phi''(t)}{a(t)} + i\omega'(t), \]

which are the instantaneous frequency \( \omega(t) \) [35], the instantaneous bandwidth \( \nu(t) \) [18,36,37] and what I refer to as the instantaneous curvature \( \xi(t) \) [20], respectively. These paramaters, \( a(t), \phi(t), \omega(t), \) etc., are all characterizations of local (time) structure of \( x(t) \). These parameters would be used to describe the modulated oscillation: \( a(t) \) signifying the local amplitude or energy of the signal, \( \omega(t) \) the local frequency content, \( \nu(t) \) determining the local stability of the signal. A signal that ‘looks like’ an oscillation would be expected to have \( |\nu(t)| \ll |\omega(t)| \) as well as \( |\xi(t)| \ll |\omega(t)|^2 \), as otherwise the signal is varying too rapidly to resemble an oscillation [20].

Normally, one wishes to recover \( a(t), \phi(t) \) and thus \( \omega(t) \) from an observed \( x(t) \) to describe the time-frequency properties of \( x(t) \) with these parameters. One may think of \( a(t) \), the magnitude, as determining the presence of the oscillation at time point \( t \). If \( a(t) > h \), some predefined value related to the noise in the system, then one may consider the oscillation of frequency \( \omega(t) \) as ‘present’ at time point \( t \) (see also [21]). As an example of these concepts, I show an oscillation with time-varying frequency \( \omega(t) \) in figure 4. If we wish to represent its presence in time frequency, because the signal is an oscillation, it is nominally concentrated to its instantaneous frequency. The oscillation here has an instantaneous frequency that meanders across time, as seen from figure 4a. If we look at the oscillation in time (figure 4c) or in frequency (figure 4b), the description is less precise, and becomes ‘smeared out’. In many practical problems, the instantaneous frequency path itself characterizes the type of signal we are investigating [38], and can be used for classification, especially when combined with the time-varying amplitude estimate. In slightly more general terms, we visualize the description of the signal as a time-varying oscillation as the location of a particle orbiting a circle \((x(t),y(t))\) with radius \(a(t)\) and orbital frequency \(\omega(t)\).
To foreshadow later, more complicated, elliptical structure, I define the geometry in phase space to be the curve

$$G_x = \left\{ (x, y) : x = \frac{x(t)}{a(t)}, \ y = \frac{y(t)}{a(t)}, \ \forall t \right\}. \quad (2.4)$$

Here $G_x$ maps out the ‘shape’ of the oscillations taken in aggregate. As an example of this, I refer to figure 5. Here we see the univariate modulated oscillation in frequency (figure 5a), where the spread of the signal in frequency is due to the time variation of the amplitude and instantaneous frequency. I also display the oscillation in time $x(t)$ as a solid line, the extra component $y(t)$ as a dotted line, and the envelope of the complex signal as a thick solid line (figure 5b). The geometry of the univariate signal is just a circle, displayed in figure 5c, and I show the moments describing the time evolution of the signal in figure 5d. As the instantaneous frequency dominates the other time-varying descriptors, the description of the signal as a time-varying oscillation is appropriate.

Finally, some warning should be issued as to when we can usefully describe a zero-mean signal as an oscillation with a time-varying amplitude and frequency. Equation (2.1) is hiding many issues, despite its superficial simplicity, and putting in an arbitrary $a(t)$ and $\phi(t)$ in the equation means that the same functions that defined the oscillation will not be recovered from equation (2.2). For this reason, if these parameters are too time variable, the description as an oscillation is not appropriate. The rationale for this is that the $\omega(t)$ defined in equation (2.2) from

\[\text{Figure 4. A univariate modulated oscillation. This diagram shows (a) the simplistic understanding of the oscillation in time-frequency, concentrated to a single time-varying frequency curve, and its magnitude in (b) frequency and (c) time. The instantaneous frequency describes the local time-frequency characteristics of the signal. (Online version in colour.)}\]
Figure 5. A univariate modulated oscillation. (a) The Fourier transform of $x(t)$ for a finite sample length signal. (b) The modulated oscillation $x(t)$ and its conjugate signal $y(t)$ defined from the analytic signal, as well as the envelope and its negative defined from the magnitude of the complex signal. (c) The shape mapped out by $G_x$ defined from the analytic signal, or the circular ‘geometry’ of the univariate signal. (d) The instantaneous frequency (solid), bandwidth (dotted) and chirping (dot-dashed) of the signal.

The analytic signal is strongly linked to the mean frequency of $x(t)$, $\tilde{\omega}_x$, and in fact Cohen [18] shows that

$$\tilde{\omega}_x = \frac{\int_0^\infty \omega |X(\omega)|^2 \, d\omega}{\int_0^\infty |X(\omega')|^2 \, d\omega'} = \frac{\int_{-\infty}^{\infty} a^2(t) \omega(t) \, dt}{\int_{-\infty}^{\infty} a^2(t') \, dt'},$$

so that the mean frequency of the signal is averaging to the weighted average of the frequency defined from equation (2.2). This means in general that the derivative of the phase as specified in the model in equation (2.1) does not satisfy a relationship of the type in equation (2.5). Beware of the fact that here the same notation (which is an abuse of notation) is used for the amplitude and phase defined from the analytic signal, and the true functions, specified in equation (2.1).

(b) Elliptically modulated signals

To model bivariate signals such as freely drifting instruments, Lilly & Gascard [7] as well as Lilly & Olhede [13] defined the elliptically modulated signal. These authors were treating the analysis of two oscillations, oscillating with the same local period as a single object, circling an ellipse rather than a circle over time. An elliptically modulated signal maps out a ‘frozen’ ellipse in time via the equation (see [13, eqn (18)]):

$$z(t) = x_1(t) + i x_2(t) = e^{i \theta(t)} [a(t) \cos \phi(t) + i b(t) \sin \phi(t)].$$
Here \( \theta(t) \), \( a(t) \) and \( b(t) \) specify the ‘frozen’ ellipse parameters, and every \( 2\pi/\omega_\phi(t) \) with \( \omega_\phi(t) = (d/dt)\phi(t) \) the ellipse is traversed by the signal. This ‘frozen’ ellipse can be compared with the ‘frozen’ circle with radius \( a(t) \) that is mapped out by the analytic signal calculated from a single AM/FM signal. An ellipse has more structure than a circle, with a major axis length \( |a(t)| \), a minor axis length \( |b(t)| \) and the angle of incidence between the \( x \)-axis and the major axis, given by \( \theta(t) \).

In parallel to the geometry defined for the univariate signal, I define

\[
G_x = \left\{ (x,y) : x = \frac{x(t)}{\|x_+(t)\|}, \ y = \frac{y(t)}{\|x_+(t)\|}, \ \forall t \right\},
\]

where

\[
\|x_+(t)\|^2 = x_1^2(t) + x_2^2(t).
\]

An important characteristic of \( G_x \) is the signed linearity \([7,39]\),

\[
\lambda(t) = r_z a^2(t) - b^2(t)
\]

where \( r_z \) takes the value +1 or -1 depending on the sign of \( b(t) \). The signed linearity contains information that with \( \|x_+(t)\|^2 \) and \( \theta(t) \) can be used to describe the geometry at any point rather than using \( a(t) \) and \( b(t) \). I retain \( \phi(t) \) to describe the oscillatory structure of \( x_1(t) \) and \( x_2(t) \). If \( \lambda(t) = 0 \) (\( |b(t)| = |a(t)| \)) then the ellipse is a circle, while if \( |\lambda(t)| = 1 \) (\( |b(t)| = 0 \)) then the motion is purely linear as described by Lilly & Olhede [13]. One may define the joint instantaneous frequency of the entire bivariate signal as defined at any time point by

\[
\omega_\phi(t) = \omega_\phi(t) + r_z \sqrt{1 - \lambda^2(t)} \omega_\phi(t)
\]

\[
\omega_\phi(t) = \frac{\text{Im}[x_+^1(t)(d/dt)x_+(t)]}{\|x_+(t)\|^2},
\]

where \( \omega_\phi(t) \) is known as the orbital frequency and \( \omega_\phi(t) = (d/dt)\theta(t) \) is the precession rate. This yields one instantaneous frequency per signal, rather than producing two frequencies describing the signals separately. Similarly to the one-dimensional case, one can define an instantaneous bandwidth square as

\[
\omega_\phi^2(t) = \left| \frac{d \ln \kappa(t)}{dt} \right|^2 + \frac{1}{1 - \lambda^2(t)} \left| \frac{d \lambda(t)}{dt} \right|^2 + \lambda^2(t) \omega_\phi^2(t)
\]

\[
\omega_\phi^2(t) = \omega_\phi^2(t) + \omega_\phi^2(t) + \omega_\phi^2(t),
\]

where \( \omega_\phi^2(t) \) is the squared amplitude bandwidth, \( \omega_\phi^2(t) \) is the deformation bandwidth and \( \omega_\phi^2(t) \) is the precession bandwidth. The total bandwidth \( \omega_\phi(t) \) describes the spread in frequency not due to the time-varying joint frequency.

I plot a bivariate signal in figure 6. We see in figure 6a that there is support in both negative and positive frequencies; that the real and imaginary parts of the signal have different magnitudes (cf. figure 6b); that a precessing ellipse is mapped out (cf. figure 6c); and where figure 6d summarizes the evolving local description and the variation in the signal. The dominant contribution to the variation in the signal is the instantaneous frequency displayed in figure 6d as a solid line, where the instantaneous bandwidth is displayed as a dotted line. In addition, the three contributions to the instantaneous bandwidth, the precession rate, the amplitude rate of change and the deformation of the ellipse, are included as dashed lines. Because the amplitude rate of change is dominating the change of the ellipse except near \( t = 0 \), the instantaneous bandwidth and amplitude change lines are overlapping and look like a dash-dotted line. The ellipse is not deforming, and so the deformation bandwidth of the ellipse is everywhere zero. The non-zero line just above this is the precession bandwidth, and as can be seen from figure 6c, the ellipse is continuously precessing in time.
Figure 6. A bivariate modulated oscillation. (a) The Fourier transform for a finite sample length signal. (b) The modulated oscillation $x(t)$ and the signal $y(t)$, as well as the amplitude $\kappa(t) = \|x_+ (t)\|/\sqrt{2}$ and its negative $-\kappa(t)$. (c) The evolving normalized ellipses mapped out by the analytic signal, precessing in time, corresponding to the points in $G_x$. (d) The instantaneous frequency (solid), the instantaneous bandwidth (dotted), stemming from its three components (dashed) and the chirp-rate (dash-dotted).

(c) Multivariate oscillations

A common model in multivariate time series is to assume that a small set of signals are common to, or found in, all observations [2,40,41]. This is an unobserved components model corresponding to

$$x(t) = \sum_{j=1}^{J} u_j s_j(t).$$

(2.12)

This equation is stating that the multivariate signal $x(t)$ is a linear combination of $J$ signals $\{s_j(t)\}_{j=1}^{J}$ where each signal $x_n(t)$ is given by assigning weight $u_{jn}$ to the unobserved component $s_j(t)$. Equation (2.12) alone describes none of the dynamic properties of the signals $s_j(t)$, and does not describe any dynamic evolution between the signals, as $\{u_{jk}\}$ are all constant in time. For simplicity, $s_j(t)$ and $s_k(t)$ are assumed orthogonal when $j \neq k$. To underscore the dynamics of the signal, moving the signal forward in time by $\tau$ points would yield

$$x(t + \tau) = \sum_{j=1}^{J} u_j b_j(t, \tau) s_j(t), \quad \tau \in [0, T],$$

(2.13)

with $b_j(t, \tau) = s_j(t + \tau)/s_j(t)$ if $s_j(t) \neq 0$ for all $j$. For any multivariate signal with arbitrary unobserved components $\{s_j(t)\}$, there is no particular reason for the $b_j(t, \tau)$ to take any given (simple) smooth form because $s_j(t)$ is not assumed to satisfy any particular temporal form. The unobserved components model of equation (2.12) demands strict constancy across the weights,
but no special model for \( s_j(t) \) is posited across time. In contrast, the models I intend to use for the multivariate \( x(t) \) are intended for oscillatory signals, constrained to follow the same local oscillation, which puts significant structural constraints on \( s_j(t) \) and thus \( b_j(t, \tau) \).

I need to define the multivariate analytic signal \( x_+(t) \) from the vector-valued \( x(t) \) to describe the signal simply and understand its frequency content. The signal \( x(t) \) has dimension \( N \) where its analytic extension has a Fourier representation of

\[
x_+(t) = \frac{1}{2\pi} \int_0^{\infty} 2X(\omega)e^{i\omega t} \, d\omega,
\]

where \( X(\omega) \) is the Fourier transform of \( x(t) \). We see that \( x_+(t) \) is built up from the differential contributions at each frequency of a positively rotating phasor \( e^{i\omega t} \) with a vector-valued weight \( 2X(\omega) \).

I shall assume only one common component present in the oscillations and write the analytic signal of \( x(t) \) as

\[
x_+(t) = \|x_+(t)\| e^{i\phi(t)} u_1(t),
\]

where \( \|x_+(t)\| \in \mathbb{R}^+, \phi(t) \in \mathbb{R} \) and \( u_1(t) \in \mathbb{C}^N \), with \( u^H(t)u(t) = 1 \) for any given \( t \in \mathbb{R}^+ \). Comparing equation (2.12) with (2.15), we see that if we take \( s_1(t) = \|x_+(t)\| e^{i\phi(t)} \) the equations now have a very similar form, even if the latter description (equation (2.15)) initially seems to have a smaller dimensionality, as it only involves a single complex \( N \)-vector \( u_1(t) \). Of course if \( u_1(t) \) changes in time (as indicated by its argument) then many directions are mapped out over time, and the signal lives in a subspace of larger dimensionality. Thus, neither of the two models (equation (2.12) or (2.15)) subsumes the other, if observed in continuous time. Another possible model that I could have used is

\[
x_+(t) = \sum_{j=1}^J x_j e^{i\omega_j(t)+i\phi_j} u_j.
\]

This model limits the components present in the channels of \( x_+(t) \) to a number of time-invariant frequencies \( \omega_j \) whose weighting in each channel is also constant in time. The advantage of the model is that it is naturally analysed in the Fourier domain, and that no time tracking of frequencies needs to be carried out, as each \( e^{i\omega_j(t)+i\phi_j} \) has a fixed frequency content limited to \( \omega_j \). The actual signals that follow this model are fewer, and the lack of ability to track time channels will limit the practical utility of the model.

I wish to model the signal elements of \( x_+(t) \) as oscillations evolving together, and define the \( n \)th component amplitude and phase by using a subscript of \( n \) component by component analogously to (2.2). The joint oscillating structure, viz. their joint instantaneous frequency, is then [13, eqn 53]

\[
\omega_X(t) = \frac{\sum_{n=1}^N a_n^2 \omega_n(t)}{\sum_{n=1}^N a_n^2(t)} = \sqrt{\frac{X_+^H(t)X_+(t)}{\|x_+(t)\|^2}}.
\]

We must model the full joint variability of the signals across time. Lilly & Olhede [15, eqn 7] proposed a model of

\[
x_+(t+\tau) = e^{i\omega_X(t)\tau} \left[ x_+(t) + \tau \bar{x}_1(t) + \frac{1}{2} \tau^2 \bar{x}_2(t) + \varepsilon_3(t, \tau) \right], \quad \tau \in [0, T],
\]

where the intrinsic deviation vectors \( \{\bar{x}_p(t)\} \), and the error term \( \varepsilon_3(t, \tau) \), are assumed small for small to moderate \( \tau \), and determined in [15]. Only if \( \|x_+(t+\tau) - \varepsilon(t)\| \) is small can we reasonably think of \( x(t) \) as a multivariate oscillation slaved to the same instantaneous frequency and this implies that we can map \( \tau \) steps into the future mainly by locally oscillating using \( e^{i\omega_X(t)\tau} \) with small correction terms. I note that equations (2.12) and (2.15) are instantaneous specifications of multivariate structure, while equations (2.13) and (2.18) are evolving that structure as we observe the signal across time. If the individual \( s_j(t) \) are unstructured, such an evolution will not be simple, but for oscillations we can write down equation (2.18), and quantify a joint evolution
of the geometry of the multiple signals in $x_+(t)$. The deviation vectors take a simple form; for example, $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are given by

$$\tilde{x}_1(t) = x_+(t) + i\omega(t)x_+(t)$$

and

$$\tilde{x}_2(t) = x_+(t) + i\omega(t)x_+(t) - \omega_2^2(t)x_+(t),$$

and yield the multivariate analogues of the univariate instantaneous bandwidth $\nu_n(t)$ and curvature $\xi_n(t)$ via their normalized magnitudes

$$\nu_x(t) \equiv \frac{\|\tilde{x}_1(t)\|}{\|x_+(t)\|} \quad \text{and} \quad \xi_x(t) \equiv \frac{\|\tilde{x}_2(t)\|}{\|x_+(t)\|}.$$  

Compare these expressions with equations (2.3) that give the equivalent description for a univariate signal. The normalized magnitudes quantify, respectively, the first- and second-order deviations, aggregated over all signal components, of the local behaviour of the signal $x_+(t)$ from that expected for pure (not time-varying) oscillation. The quantities $\omega_x(t)$, $\nu_x(t)$ and $\xi_x(t)$ are referred to as the instantaneous moments and are in direct analogy with equations (2.3).

Let us illustrate these concepts with a trivariate example (figures 7 and 8). We have two components, one local to $t = 1000$, and the other local to $t > 2000$. The real part of the signal is plotted as solid, while its Hilbert transform is plotted as a dotted line. The signals also have different frequency content, as can be observed from their zero-crossing structure, as seen in figure 7. I show their trajectory in three-space on figure 8. Clearly, these signals are mapping out a subset of the full space of $\mathbb{R}^3$ over time, and I wish to describe their path as mapped over time. The commonality of the signals is described via their joint instantaneous frequency $\omega_x(t)$ and via the instantaneous bandwidth $\nu_x(t)$, as well as their instantaneous curvature $\xi_x(t)$. However, to describe their multivariate structure more precisely over time, we need some more careful understanding of the higher-dimensional structure present in the signal and its evolution in time.

3. Multivariate instantaneous moments

(a) Global second-order moments

The aggregate second-order structure of $x_+(t)$ is captured by the deterministic joint analytic spectrum [13] of

$$S_x(\omega) \equiv E_x^{-1}\|X_+(\omega)\|^2,$$  

which is simply the average of the spectra of the $N$ analytic signals, normalized to unit energy with

$$E_x \equiv \frac{1}{2\pi} \int_0^\infty \|X_+(\omega)\|^2 d\omega = \int_{-\infty}^{\infty} \|x_+(t)\|^2 dt.$$  

Global characteristics in terms of aggregate properties of a Fourier-domain multiplier operator $F(\omega)$ with time-domain form $f(t)$ can be found by aggregating contributions over all frequencies, e.g.

$$\langle F(\omega) \rangle_x = \int_0^\infty F(\omega)S_x(\omega) d\omega = \frac{\int_{-\infty}^{\infty} x^H(t)f(t)x_+(t) dt}{\int_{-\infty}^{\infty} \|x_+(t)\|^2 dt},$$

where $f(t)$ is the operator corresponding to the multiplier of $F(\omega)$ [42,43]. Equation (3.3) corresponds to scalar summaries of joint structure, or an inner-product moment. Furthermore, equation (3.3) is the generalization of equation (2.5), and normally polynomials of $\omega$ are used for the multiplier $F(\omega)$. Via such arguments the form of the instantaneous bandwidth $\nu(t)$ given in equations (2.3) and (2.11) was derived [13,44]. The instantaneous bandwidth describes how spread in frequency the signal is, where the spread is not due solely to meandering of the instantaneous frequency, but rather due to changes in amplitude, etc.
Figure 7. A trivariate example. This shows the three components in time in solid, with their Hilbert transforms overlaid in dotted line.

Figure 8. The trivariate example, showing the elliptical trajectories mapped out by the real (solid) and imaginary (dotted) parts of the complex multivariate signal plotted in three-dimensional space as the value of the components.

One may also define the matrix-valued representations

$$[F(\omega)]_x = (2\pi \mathcal{E}_x)^{-1} \int_0^\infty F(\omega)X_+(\omega)X_+^H(\omega) \, d\omega = \frac{\int_{-\infty}^\infty (f(t)x_+(t))x_+^H(t) \, dt}{\int_{-\infty}^\infty \|x_+(t)\|^2 \, dt},$$

(3.4)
to be able to characterize the multivariate aspects of the data, defining Fourier multiplier $F(\omega)$, with corresponding time-domain operator $f(t)$. The first moment to characterize is to take $F(\omega) = 1$. Thus, I define the average structure matrix via

$$\bar{\Sigma}_x = [1]_k = \frac{\int_0^\infty 1 \cdot X_+(\omega)X_+^2(\omega) \, d\omega}{\int_0^\infty X_+^2(\omega)X_+^2(\omega) \, d\omega' - \int_0^\infty X_+^2(\omega')X_+(\omega') \, d\omega'}$$

$$= \sum_{k=1}^N \tilde{\gamma}_k \tilde{u}_k \tilde{u}_k^H \tag{3.5}$$

where $\tilde{\gamma}_k$ and $\tilde{u}_k$ are the eigenvalues and eigenvectors, respectively, of $\bar{\Sigma}_x$. We see directly that the average structure matrix is constructed from averaging $u_1(t)u_1^H(t)$ across time using a weighted average. Note that the overall norm of the average structure matrix is constrained as

$$\text{trace}(\bar{\Sigma}_x) = \sum_{k=1}^N \tilde{\gamma}_k = 1, \tag{3.6}$$

as can be seen directly from equation (3.5). The global dimensionality of the signal is encoded by the eigenvalues and

$$1 \geq \tilde{\gamma}_1 \geq \cdots \geq \tilde{\gamma}_N \geq 0.$$

The eigenvalues are very informative, as they globally explain how many directions exist in the data. If the model had been equation (2.12), then $N = I$ and $\gamma_k = \int \tilde{\gamma}_k(t) \, dt$ with the normalization $\tilde{u}_k^H \tilde{u}_k = 1$, where the modes are ordered according to magnitude of $\gamma_k$ instead of using the fixed ordering of $u_k$. In our case, instead, the multiple directions are built up by the evolution of $u_1(t)$. We may arrive at the same average structure matrix from using different models for $x_+(t)$.

To gain some insight into this description of the global properties of the signal, I introduce a purely unidirectional signal given by

$$x_+(t) = x_+(t)u_1, \tag{3.7}$$

where $u_1$ is complex-valued but constant in time and $x_+(t)$ is an analytic signal. This signal will map out an oscillation in a fixed complex direction across all time. Then the structure matrix becomes (with $u_1 = v_1 + i w_1$)

$$\bar{\Sigma}_x = u_1^H u_1 \tag{3.8}$$

$$= v_1 v_1^T + w_1 w_1^T + i(w_1v_1^T - v_1w_1^T) \tag{3.9}$$

and $\bar{\Sigma}_x$ has a very simple form. A purely unidirectional signal (equation (3.7)) is defined analogously to either a pure phase signal (see [27, §3] or [13, eqn 29]) or a constant phase amplitude modulated signal. For the purely unidirectional signal, we can find by direct calculation that we have

$$\tilde{x}_1(t) = \frac{\|x(t)\|'}{\|x(t)\|} x(t). \tag{3.10}$$

Here both the phase and amplitude are modulated, but the multivariate structure is not evolving. I calculate the axis of the signal from

$$\text{Re}(\bar{\Sigma}_x) = v_1 v_1^T + w_1 w_1^T = a^2 q_1 q_1^T + b^2 q_2 q_2^T, \tag{3.11}$$

where $q_1$ and $q_2$ are the eigenvectors of $\text{Re}(\bar{\Sigma}_x)$ and are therefore real-valued vectors. Then we see that the signal has the global dimensionality of either one or two, depending on if $b = 0$ or not,
as the vectors we get out, \( q_1 \) and \( q_2 \), are time constant. The case \( b = 0 \) corresponds to a linearly polarized signal [45], while if \( a = b \) then the signal is considered to be circularly polarized.

To better characterize the multivariate properties of the signal, I calculate the global rank [46] of the signal from the rank of the matrix \( \text{Re}\{\bar{\Sigma}_X\} \). This is

\[
n_X = \text{rank}\{\text{Re}\{\bar{\Sigma}_X\}\}.
\]

(3.12)

The rank determines the dimensionality of the globally averaged signal. It is apparent that the local rank of a signal is one or two, and so this global rank is built up from the local contributions.

For general signals, we end up with a time-varying \( u(t) \) that over time visits many directions. Then we need the description of (cf. equation (3.7))

\[
x_+(t) = x_+(t)u_1(t) \quad \text{and} \quad x_+(t) = \|x_+(t)\|e^{i\phi(t)},
\]

(3.13)

where the variability of \( u_1(t) \) and \( \|x_+(t)\| \) is small, similarly to the constraints of Bedrosian’s theorem [47] for the amplitude of a univariate signal. Note that for a non-constant \( u_1(t), x_+(t) \) defined above need not itself be an analytic signal. In this case,

\[
\bar{\Sigma}_X = \frac{\int_{-\infty}^{\infty} x_+(t)x_+^H(t)\,dt}{\int_{-\infty}^{\infty} \|x_+(t)\|^2\,dt} = \frac{\int_{-\infty}^{\infty} |x_+(t)|^2u_1(t)u_1^H(t)\,dt}{\int_{-\infty}^{\infty} |x_+(t)|^2\,dt},
\]

(3.14)

so that the matrix now has all dimensionality possible (e.g. \( \bar{\Sigma}_X \) may be rank \( N \)), as we consider all the possible structure over \( t \), weighted by \( |x_+(t)|^2 \). Here many eigenvalues may be non-zero, and as we traverse over the time axis, we visit many different vectors in the linear space of

\[
V = \{v = a_1\bar{u}_1 + a_2\bar{u}_2 + \cdots + a_K\bar{u}_K, \, a_k \in \mathbb{C}\}.
\]

Depending on the form of \( a_k \) and the nature of the complex vectors \( \bar{u}_k \), the resulting space may have different dimensionality, \( 2K \leq N \). Note here the peculiarity that complex vector spaces such as \( V \) have the same dimensionality as real vector spaces, but the dimensionality of \( \bar{\Sigma}_X \) is not the same as \( \text{Re}\{\bar{\Sigma}_X\} \). Despite this, the maximum number of degrees of freedom that we have is still \( N \) and so, no matter how many elliptical signals we put into the data, we can never get back more than \( N \) degrees of freedom from the \( N \) signals.

To give an example of the difference between the global structure matrix, let us revisit the example plotted in figures 8 and 9. Figure 10 describes the magnitude square of the combined signal in marginal energy in time and frequency (figure 10a–c). The final three plots describe the signal in terms of its multivariate structure (figure 10d–f). One can see the time-frequency characteristics of the components and the complete signal. The signal is localized to a small set of times and frequencies. In terms of the multivariate structure, there is a big difference between the instantaneous and average structure matrix (figure 9d–f). The global structure matrix has two eigenvalues corresponding to 0.2522 and 0.7478, respectively, with the third eigenvalue being exactly zero. If on the other hand we calculate the eigenvalues of the local structure matrix, e.g. calculate the eigenvalues of \( x_+(t)x_+^H(t) \), then we find that there is only one eigenvalue, respectively, which is non-zero at any one time: if real parts are taken we get either three or two non-zero eigenvalues. Finally, we can also investigate how the instantaneous moments change across time (figure 10). In figure 10, we see that the instantaneous frequency (solid black line, equation (2.17)) dominates the instantaneous bandwidth (dash-dotted, equations (2.3)) and the chirp rate \( \omega'(t) \) (dashed) over the entire length of time. We also see the eigenvalues normalized by the total sum in figure 10b; it is clear that for part of the signal there are two equal non-zero eigenvalues, while in the latter section, there are two unequal eigenvalues, this characterizing the signal as elliptical (two non-zero, unequal eigenvalues) rather than circular (two equal eigenvalues).
Figure 9. A trivariate example. The original signal corresponds to two different time-frequency components, localized to different frequencies and times. The time-domain signal magnitude is displayed in (a) for each component (solid, dotted, dashed), (b) the total energy in time of the multivariate signal, and (c) the total energy in frequency. The local multivariate structure $x_+^i(t)x_+^{j}(t)$ is shown at two time instances (d) $t = 1000$ and (e) $t = 3000$, and the global structure matrix is shown in (f). The third eigenvalue of the real part of the structure matrix is zero in the first two instances, but this is not true in the final (f). The instantaneous matrices have been normalized so their trace is one.

(b) Minimizing the deviation from a unidirectional signal

The big difference between the two globally multidirectional signals (viz. equations (2.12) and (2.15)) is that locally in time the model of (2.15) only has one single direction. It is therefore interesting to characterize how far a signal is from a purely unidirectional signal. To define such a measure, we recall that the bandwidth \[13\] of the signal is defined by

$$
\sigma_x^2 = \frac{1}{E_x} \int_{-\infty}^{\infty} \|x_+(t)\|^2 \sigma_x^2(t) \, dt = \frac{1}{2\pi E_x} \int_{0}^{\infty} (\omega - \bar{\omega}_x)^2 |X_+(\omega)|^2 \, d\omega.
$$

This can be decomposed into two contributions of

$$
\sigma_x^2(t) = (\omega_x(t) - \bar{\omega}_x)^2 + \upsilon_x^2(t),
$$

where $\upsilon_x^2(t)$ is given in equation (2.11). This equation describes the spread in frequency of $x_+(t)$ around its mean frequency $\bar{\omega}_x$ in terms of instantaneous frequency deviations $\omega_x(t) - \bar{\omega}_x$ and other changes in the time structure of the signal quantified by $\upsilon_x(t)$. In turn, $\upsilon_x(t)$ describes how well the
model of the modulated oscillation fits the signal. The square direction deviation from fixed vector $u_o$ to measure how unidirectional a signal is can be defined analogously as

$$
\begin{align*}
\varsigma_X^2(u_o) &= \frac{\int_{-\infty}^{\infty} (x_+^\prime(t) - \|x_+^\prime(t)\|e^{i\phi(t)}u_o)^H(x_+^\prime(t) - \|x_+^\prime(t)\|e^{i\phi(t)}u_o) \, dt}{\int_{-\infty}^{\infty} \|x_+^\prime(t')\|^2 \, dt'} \\
&= 1 + u_o^H u_o - 2 \frac{\int \|x_+^\prime(t)\|^2 \text{Re}(u^H(t)u_o) \, dt}{\int_{-\infty}^{\infty} \|x_+^\prime(t')\|^2 \, dt'}.
\end{align*}
$$

To measure how far $x_+^\prime(t)$ is from a purely unidirectional signal, I first find the best-fitting single-direction signal by solving

$$
\mathbf{u}_{\text{min}} = \arg \min_{u \in \mathbb{C}^N} \varsigma_X^2(u),
$$

analogously to how we defined the instantaneous bandwidth [13]. Elementary usage of calculus then yields the optimal ‘direction’ to be

$$
\mathbf{u}_{\text{min}} = \frac{\int \|x_+^\prime(t)\|^2 \mathbf{u}(t) \, dt}{\int_{-\infty}^{\infty} \|x_+^\prime(t')\|^2 \, dt'}.
$$

This is the average direction over the whole signal, if not itself normalized to be a direction; hence our usage of the quotation marks. Define the ‘angle’ between the complex vector at two different
times to be $\exp(i\alpha(t,s)) = \mathbf{u}^\text{H}(t)\mathbf{u}(s)$, and note that $\mathbf{u}^\text{H}(s)\mathbf{u}(t) = (\mathbf{u}^\text{H}(s)\mathbf{u}(t))^\text{H}$; this assures us of the reality of

$$\mathbf{u}_\text{min}^\text{H}\mathbf{u}_\text{min} = \frac{\int \int \|\mathbf{x}_+(t)\|^2 \|\mathbf{x}_+(s)\|^2 \exp(i\alpha(t,s)) \, dt \, ds}{(\int_{-\infty}^{\infty} \|\mathbf{x}_+(t')\|^2 \, dt')^2} = \varepsilon_\text{min}^2. \quad (3.18)$$

Thus $\mathbf{u}_\text{min}$ is only a unitary vector if $\alpha(t,s) = 0$ for all $t$ and $s$. In general, the value of $\zeta_x^2(\mathbf{u}_\text{min})$ is

$$\zeta_x^2(\mathbf{u}_\text{min}) = 1 + \varepsilon_\text{min}^2 - 2\varepsilon_\text{min}^2 = 1 - \varepsilon_\text{min}^2. \quad (3.19)$$

A truly univariate signal will reach the absolute minimum of this quantity, which is zero; any deviation from zero tells us that the signal is not truly globally univariate; and so $\zeta_x^2(\mathbf{u}_\text{min})$ is a measure of how unidirectional a given signal is. If we observe the signal in noise, clearly $\zeta_x^2(\mathbf{u}_\text{min})$ is to be preferred as a measure to $n_x$, as the latter will always be $N$.

4. Oceanographic example

A large source of data for the analysis of ocean currents are the so-called float and drifter datasets. These measurement devices are released into the ocean and are tracked, where the change of their position is the relevant information in the data (http://www.whoi.edu/page.do?pid=10320).

We normally model the position of a float (see [13, eqn 7]) as

$$\mathbf{x}^{(o)}(t) = \mathbf{x}^{(b)}(t) + \mathbf{x}(t) + \mathbf{x}^{(e)}(t). \quad (4.1)$$

Thus, the observed position is an aggregation of three contributions, namely the background variability due to general currents, the contribution from any eddy present, $\mathbf{x}(t)$, and any measurement noise, $\mathbf{x}^{(e)}(t)$. I focus here on the eddy component $\mathbf{x}(t)$, which is not assumed to be zero mean at any time, but of course, being oscillatory, its time average will be zero. It is easier to study $d\mathbf{x}^{(o)}(t)/dt$ than the original position measurements. I pick two trajectories, the seventh and eighth time series in the set, and plot their velocities in figure 11, both in space–time (figure 11a) and as a Lissajous plot (figure 11b), plotting $(x_1(t), x_2(t))$. It is clear from the Lissajous figure that the model as an evolving oscillation is appropriate and that the amplitude is also changing in time. To complement the time-domain description of the signal I also show
the Fourier-transform magnitude square of the seventh time series in figure 12. From the Fourier transform we can see that a range of frequencies in the band $\omega \in [-2, -1]$ are present in the signal. This gives no time specificity at all. I therefore plot the northward and eastward velocities with their Hilbert transform in figure 13. I also calculate the instantaneous frequency and bandwidth and plot these in figure 13c. We see that the start of the record has a strongly oscillatory structure where the change of the common component remains moderate. Towards the latter part of the record the oscillatory component is less stable—as we can see, the joint instantaneous bandwidth is close in magnitude to the instantaneous frequency. These two functions characterize the signal, and from their ratio we can tell which portions of the signal are naturally treated as one modulated oscillation. Finally, let us take a look at the multivariate structure of the dataset. The average structure matrix is shown in figure 14a, and three instantaneous structure matrices are shown in

Figure 12. The magnitude square of the Fourier transform of the seventh float velocity signal on a decibel scale.

Figure 13. Velocities of the two components of the seventh float. (a) and (b) Plots of the north and east velocity (solid lines), as well as the Hilbert transforms of the two components (dotted lines). (c) An estimate of the instantaneous frequency (solid), and the instantaneous bandwidth (dotted).
Figure 14. The multivariate structure of the data. (a) Shows the absolute value entry-by-entry of the average structure matrix $\Re\{\bar{\Sigma}_x\}$ over the entire time course, while (b)–(d), show the entry-by-entry absolute values of the instantaneous multivariate structure $x(t)\times^H(t)$ at three different time points, $t = 70$, $t = 210$ and $t = 510$. This is in standardized time units, rather than physical units.

We can see directly from these matrices that the multivariate structure appears to be changing over the time steps.

5. Discussion

This paper has reviewed newly developed theory for representing multivariate signals as coupled modulated oscillations. Recent years have seen considerable extension of existing models for multivariate oscillations [13–15]. The main point of these papers is that they describe the single phenomenon observed in several channels as one object rather than as $N$ separate oscillations. This yields one instantaneous frequency of the entire signal that is the local frequency of all the signals observed together. In addition, a measure of how well the multivariate model fits all channels is given by the instantaneous bandwidth. I note that the modulated multivariate oscillation locally in time is very simple, corresponding to a single unidirectional signal, but because its multivariate structure is allowed to evolve, it can globally contain multiple directions.

To quantify the degree of multidirectionality of the signal, I defined the average structure matrix. This produces a link between the local multivariate structure of the signal and the globally aggregated structure. The dimensionality of the average structure matrix (like its rank) produces a discrete measure of signal complexity. In the presence of noise, unless the structure matrix is thresholded, such a measure is meaningless. As the modulated multivariate oscillation is locally unidirectional, it makes more sense to compare the signal to a globally unidirectional signal. To do so, a measure analogous with the instantaneous bandwidth was introduced—this being the square direction deviation. I show how to minimize this quantity to come up with a globally best-fitting direction to the entire signal, in close similarity to the average frequency. The value of the square direction deviation at this vector quantifies how far the signal is from being globally unidirectional.
Multivariate signals are complicated—very few reasonable models describe their full structure. Most of the time very simplistic models are posited that model the multivariate structure channel by channel and in terms of correlation. The multivariate modulated oscillation is evolving in time, and so simple linear methods such as principal components analysis cannot be applied globally across the data to uncover their structure. Developing models that can simply capture the properties of observed data, that are flexible enough to go beyond the simplest forms of model structure, is important. The modulated multivariate oscillation is one such model, and its utility for application in oceanography has already been demonstrated [48]. However, there remain many aspects of its properties that are still to be uncovered, and understanding of its parametrization must be developed for its best usage.

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