Stabilizing entangled states with quasi-local quantum dynamical semigroups

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We provide a solution to the problem of determining whether a target pure state can be asymptotically prepared using dissipative Markovian dynamics under fixed locality constraints. Besides recovering existing results for a large class of physically relevant entangled states, our approach has the advantage of providing an explicit stabilization test solely based on the input state and constraints of the problem. Connections with the formalism of frustration-free parent Hamiltonians are discussed, as well as control implementations in terms of a switching output-feedback law.

Keywords: quantum dynamical semigroups; entanglement generation; environment engineering

1. Introduction

While uncontrolled couplings between a quantum system of interest and its surrounding environment are responsible for unwanted non-unitary evolution and decoherence, it has also been long acknowledged that suitably engineering the action of the environment may prove beneficial in a number of applications across quantum control and quantum information processing [1–3]. It is well known, in particular, that open-system dynamics are instrumental in control tasks such as robust quantum state preparation and rapid purification, and both open-loop and quantum feedback methods have been extensively investigated in this context [4–8], including recent extensions to engineered quantum memories [9] and ‘pointer states’ in the non-Markovian regime [10].

Remarkably, it has also been recently shown that it is, in principle, possible to design dissipative Markovian dynamics so that non-trivial strongly correlated quantum phases of matter are prepared in the steady state [11,12] or the output of a desired quantum algorithm is retrieved as the asymptotic equilibrium [13]. From a practical standpoint, scalability of such protocols for multi-partite systems of increasing size is a key issue, as experimental constraints on the available control operations may, in fact, limit the set of attainable states. Promising results have been obtained by Kraus et al. [14] and Verstraete et al. [13] for

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One contribution of 15 to a Theo Murphy Meeting Issue ‘Principles and applications of quantum control engineering’.
a large class of entangled pure states, showing that Markovian dissipation acting non-trivially only on a finite maximum number of subsystems is, under generic conditions, sufficient to generate the desired state as the unique ground state of the resulting evolution. As proof-of-principle methodologies for engineering dissipation are becoming an experimental reality [15,16], it is important to obtain a more complete theoretical characterization of the set of attainable states under constrained control resources, as well as to explore schemes for synthesizing the required dissipative evolution.

Building on our previous analysis [6,17,18], in this work, we address the problem of determining whether a target pure state of a finite-dimensional quantum system can be prepared using ‘quasi-local’ dissipative resources with respect to a fixed locality notion (see also Yamamoto [19] for related results on infinite-dimensional Markovian–Gaussian dissipation). We provide a stabilizability analysis under locality-constrained Markovian control, including a direct test to verify whether a desired entangled pure state can be asymptotically prepared. In addition to recovering existing results within a system-theoretic framework, our approach has the important advantage of using only two inputs: the desired state (control task) and a specified locality notion (control constraints), without requiring a representation of the state in the stabilizer, graph or matrix-product formalisms.

2. Problem definition and preliminary results

(a) Multi-partite systems and locality of quantum dynamical semigroups

We focus on quantum dynamical semigroups generated by a (time-independent) Markovian master equation (MME) [20–22] in Lindblad form \( \dot{\rho}(t) = L(\rho(t)) = -i[H, \rho(t)] + \sum_k \left( L_k \rho(t) L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho(t) \} \right) \), specified in terms of the Hamiltonian \( H = H^\dagger \) and a finite set of noise (or Lindblad) operators \( \{ L_k \} \). We are interested in the asymptotic behaviour of MMEs in which the operators \( H, \{ L_k \} \) satisfy locality constraints. More precisely, let us consider a multi-partite system \( \mathcal{Q} \), composed of \( n \) (distinguishable) subsystems, labelled with index \( a = 1, \ldots, n \), with associated \( d_a \)-dimensional Hilbert spaces \( \mathcal{H}_a \). Thus, \( \mathcal{H}_Q = \bigotimes_{a=1}^n \mathcal{H}_a \). Let \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{D}(\mathcal{H}) \) denote the sets of linear operators and density operators on \( \mathcal{H} \), respectively. It is easy to show ([17], proof of theorem 2) that the semigroup generated by equation (2.1) is factorized with respect to the multi-partite structure, i.e. the dynamical propagator

\[ T_t := e^{L t} = \bigotimes_{a=1}^n T_{a,t} \quad \forall t \geq 0, \]

with \( T_{a,t} \) a completely positive, trace-preserving map on \( \mathcal{B}(\mathcal{H}_a) \), if and only if the following conditions hold:

- each \( L_k \) acts as the identity on all subsystems except (at most) one and
- \( H = \sum_a H_a \), where each \( H_a \) acts as the identity on all subsystems except (at most) one.

Phil. Trans. R. Soc. A (2012)
This motivates the following definitions: we say that a noise operator $L_k$ is local, if it acts as the identity on all subsystems except (at most) one, and that a Hamiltonian $H$ is local, if it can be written as a sum of terms with the same property. Note that in the language of quantum networks [23], this corresponds to the concatenation product $(I, L, H) = \boxplus_{a=1}^n (I_a, L_a, H_a)$, where $I$ and $I_a$ denote identity operations. However, it is easy to verify that if a semigroup associated to local operators admits a unique stationary pure state, the latter must be a product state. Thus, in order for the MME (2.1) to admit stationary entangled states, it is necessary to weaken the locality constraints.

We shall allow the semigroup dynamics to act in a non-local way only on certain subsets of subsystems, which we call neighbourhoods. These can be generally specified as subsets of the set of indexes labelling the subsystems

$$\mathcal{N}_j \subseteq \{1, \ldots, n\}, \quad j = 1, \ldots, M.$$ 

In analogy with the strictly local case, we say that a noise operator $L$ is quasi-local (QL) if there exists a neighbourhood $\mathcal{N}_j$ such that

$$L = L_{\mathcal{N}_j} \otimes I_{\mathcal{N}_j}^c,$$

where $L_{\mathcal{N}_j}$ accounts for the action of $L$ on the subsystems included in $\mathcal{N}_j$, and $I_{\mathcal{N}_j}^c := \bigotimes_{a \notin \mathcal{N}_j} I_a$ is the identity on the remaining subsystems. Similarly, a Hamiltonian is QL, if it admits a decomposition into a sum of QL terms,

$$H = \sum_j H_j, \quad H_j = H_{\mathcal{N}_j} \otimes I_{\mathcal{N}_j}^c.$$  

An MME will be called QL if both its Hamiltonian and noise operators are QL. It is well known that the decomposition into Hamiltonian and dissipative parts of (2.1) is not unique: nevertheless, the QL property remains well defined because the freedom in the representation does not affect the tensor structure of $H$ and $\{L_k\}$. The above way of introducing locality constraints is very general and encompasses a number of specific notions that have been used in the physical literature, notably in situations where the neighbourhoods are associated with sets of nearest-neighbour sites on a graph or lattice, and/or one is forced to consider Hamiltonian and noise generators with a weight no larger than $t$ (so-called $t$-body interactions), see also Kraus et al. [14] and Verstraete et al. [13].

We are interested in states that can be prepared (or, more precisely, stabilized) by means of MME dynamics with QL operators. Recall that an invariant state $\rho$ for a system driven by (2.1) is said to be globally asymptotically stable (GAS) if for every initial condition $\rho_0$, we have

$$\lim_{t \to \infty} e^{Lt}[\rho_0] = \rho.$$ 

In particular, following Kraus et al. [14], the aim of this paper is to characterize pure states that can be rendered GAS by purely dissipative dynamics, for which the state is ‘dark’. More precisely, we have the following.

**Definition 2.1.** A pure state $\rho_d = |\Psi\rangle\langle\Psi| \in \mathcal{D}(\mathcal{H}_Q)$, is dissipatively quasi-locally stabilizable (DQLS) if there exist QL operators $\{D_k\}_{k=1,\ldots,K}$ on $\mathcal{H}_Q$, with
D_k|\Psi\rangle = 0$, for all $k$ and $D_k$ acting non-trivially on (at most) one neighbourhood, such that $\rho_d$ is GAS for
\[ \dot{\rho} = \mathcal{L}_D[\rho] = \sum_k \left( D_k \rho D_k^\dagger - \frac{1}{2} \{ D_k^\dagger D_k, \rho \} \right). \quad (2.2) \]

We will provide a criterion for determining whether a state is DQLS, and in doing so, we will also show how assuming a single QL noise operator for each neighbourhood does not restrict the class of stabilizable states. From now on, we thus let $K \equiv M$ and $D_k \equiv D_{N_k} \otimes I_{\bar{N}_k}$.

We begin by noting that if a pure state is factorized, then we can realize its tensor components ‘locally’ with respect to its subsystems (see Ticozzi and co-workers [17,18] for stabilization of arbitrary quantum states in a given system with ‘simple’ generators, involving a single noise term). Thus, we can iteratively reduce the problem to subproblems on disjoint subsets of subsystems, until the states to be stabilized are either entangled, or completely factorized. A preliminary result is that the DQLS property is preserved by arbitrary local unitary (LU) transformations, of the form $U = \bigotimes_{a=1}^n U_a$. In order to show this, lemma 2.2 is needed.

**Lemma 2.2.** Let $\mathcal{L}$ denote the Lindblad generator associated to operators $H, \{L_k\}$. Then, for every unitary operator $U$, we have
\[ U \mathcal{L}[U^\dagger \rho U] U^\dagger = \mathcal{L}'[\rho], \quad (2.3) \]
where $\mathcal{L}'$ is the generator associated to $H' = UHU^\dagger$, $L'_k = UL_k U^\dagger$, and
\[ U e^{\mathcal{L}t}[U^\dagger \rho_0 U] U^\dagger = e^{\mathcal{L}'t}[\rho_0], \quad \forall t \geq 0. \quad (2.4) \]

Identity (2.3) is easily proved by direct computation, while (2.4) follows directly from the properties of the (matrix) exponential. The desired invariance of the QL stabilizable set under LU transformation follows.

**Proposition 2.3.** If $\rho$ is DQLS and $U$ is LU, then $\rho' = U \rho U^\dagger$ is also DQLS.

**Proof.** Assume that the generator $\mathcal{L}$ associated to QL operators $\{D_k\}$ stabilizes $\rho$. Because $\rho$ is GAS, for any initial condition $\rho_0$, we may write
\[ \rho' = \lim_{t \to +\infty} U e^{\mathcal{L}t}[\rho_0] U^\dagger = \lim_{t \to +\infty} U (e^{\mathcal{L}'t}[U^\dagger \rho_0 U]) U^\dagger. \quad (2.5) \]

By applying lemma 2.2, it suffices to show that each $D'_k = UD_k U^\dagger$ is QL. Because $D_k = D_{N_k} \otimes I_{\bar{N}_k}$, we have $D'_k = U(D_{N_k} \otimes I_{\bar{N}_k}) U^\dagger = (U_{N_k} D_{N_k} U_{N_k}^\dagger) \otimes I_{\bar{N}_k}$, where $U_{N_k} := \bigotimes_{\ell \in \bar{N}_k} U_{\ell}$. Hence, $D'_k$ is QL.

(b) **Quantum dynamical semigroups for unconstrained stabilization**

We next collect some stabilization results that do not directly incorporate any locality constraint, but will prove instrumental to our aim. Let $\mathcal{H}_S := \text{span}\{|\Psi\rangle\}$. Given corollary 1 in Ticozzi & Viola [17], $\rho_d$ is invariant if and only if
\[ L_k = \begin{bmatrix} L_{S,k} & L_{P,k} \\ 0 & L_{R,k} \end{bmatrix} \quad \text{and} \quad iH_P = -\frac{1}{2} \sum_k L^\dagger_{S,k} L_{P,k} = 0, \quad (2.6) \]

Phil. Trans. R. Soc. A (2012)
where we have used the natural block representation induced by the partition \( \mathcal{H}_Q = \mathcal{H}_S \oplus \mathcal{H}_\Sigma^I \) and labelled the blocks as

\[
X = \left[ \begin{array}{c|c}
X_S & X_P \\
X_Q & X_R \\
\end{array} \right].
\]

Assume \( \rho_d \) to be invariant. Then, \( |\Psi\rangle \) must be a common eigenvector of each \( L_k \). Call the corresponding eigenvalue \( \ell_k \equiv L_{S,k} \). By lemma 2.4 in Ticozzi & Viola [17], the MME is invariant upon substituting \( L_k \) with \( \tilde{L}_k = L_k - \ell_k I \), and \( H \) with \( \tilde{H} = H + (i/2) \sum_k (\ell_k^* L_k - \ell_k L_k^*) \). Using the series product of Gough & James [23], this means \((I, \tilde{L}_k, \tilde{H}) = (I, [L_k], 0) \triangleright (I, [L_k], H) \). In this way, we have \( L_{S,k} = 0 \) for all \( k \), so that \( \tilde{H}_P \) must be zero in order to fulfill the above condition. Thus, \( \tilde{H} \) is block-diagonal, with \(|\Psi\rangle\) being an eigenvector with eigenvalue \( h \equiv \tilde{H}_S \). Using this representation for the generator, we can let \( L_{S,k} = \ell_k = 0 \) and \( H_P = 0 = H_Q \). This further motivates the use of noise operators \( D_k \) such that \( D_k |\Psi\rangle = 0 \) in the DQLS definition.

**Lemma 2.4.** An invariant \( \rho_d = |\Psi\rangle \langle \Psi| \) is GAS for the MME (2.1), if there are no invariant common (proper) subspaces for \( \{L_k\} \) other than \( \mathcal{H}_S = \text{span}(|\Psi\rangle) \).

**Proof.** By lemma 8 and theorem 9 in Ticozzi & Viola [6], \( \rho_d \) is GAS if and only if there are no other invariant subspaces for the dynamics. Given the conditions on \( L_k \) for the invariance of a subspace, \( \rho_d \) is GAS as long as there are no other invariant common subspaces for the matrices \( L_k \).

On the basis of the above characterization, in order to ensure the DQLS property, it suffices to find operators \( \{D_k\}_{k=1}^K \subset \mathcal{B}(\mathcal{H}_Q) \) such that \( \mathcal{H}_S \) is the unique common (proper) invariant subspace for the \( \{D_k\} \).

### 3. Characterization of dissipatively quasi-locally stabilizable states

**Main result**

The key elements in our approach are the reduced states that the target state \( \rho_d \) induces with respect to the given locality structure. Let us define

\[
\rho_{N_k} = \text{trace}_{\tilde{N}_k}(\rho_d),
\]

where \( \text{trace}_{\tilde{N}_k} \) indicates the partial trace over the tensor complement of the neighbourhood \( \tilde{N}_k \), namely \( \mathcal{H}_{\tilde{N}_k} = \bigotimes_{a \notin \tilde{N}_k} \mathcal{H}_a \). Lemma 3.1 follows from the properties of the partial trace.

**Lemma 3.1.** \( \text{supp}(\rho_d) \subseteq \bigcap_k \text{supp}(\rho_{\tilde{N}_k} \otimes I_{\tilde{N}_k}) \).

**Proof.** From the spectral decomposition \( \rho_{\tilde{N}_k} = \sum_q p_q \Pi_q \), we can construct a resolution of the identity \( \{\Pi_q\} \) such that \( \rho_{\tilde{N}_k} \otimes I_{\tilde{N}_k} = \sum_q p_q \Pi_q \otimes I_{\tilde{N}_k} \), where, by definition of the partial trace, \( p_q = \text{trace}(\rho_d \Pi_q \otimes I_{\tilde{N}_k}) \). If \( p_{\hat{q}} = 0 \) for some \( \hat{q} \), then it must be \( \rho_d (\Pi_{\hat{q}} \otimes I_{\tilde{N}_k}) = 0 \) and therefore \( \text{supp}(\rho_d) \perp \text{supp}(\Pi_{\hat{q}} \otimes I_{\tilde{N}_k}) \). Thus, \( \text{supp}(\rho_d) \subseteq \bigcup_q \text{supp}(p_q \Pi_q \otimes I_{\tilde{N}_k}) = \text{supp}(\rho_{\tilde{N}_k}) \), for all \( k \).

Let us now focus on QL noise operators \( D_k = \rho_{\tilde{N}_k} \otimes I_{\tilde{N}_k} \) such that \( D_k |\Psi\rangle = 0 \).
Lemma 3.2. Assume that a set \( \{D_k\} \) makes \( \rho_d = |\Psi\rangle\langle\Psi| \) DQLS. Then, for each \( k \), we have \( \text{supp}(\rho_{N_k}) \subseteq \ker(\hat{D}_{N_k}) \).

Proof. Because \( |\Psi\rangle \) is by hypothesis in the kernel of each \( D_k \), with respect to the decomposition \( \mathcal{H}_Q = \mathcal{H}_S \oplus \mathcal{H}_S^\perp \) every \( D_k \) must be of block form,

\[
D_k = \begin{bmatrix} 0 & D_{p,k} \\ D_{R,k} & 0 \end{bmatrix},
\]

which immediately implies \( D_k \rho_d D_k^\dagger = 0 \). It then follows that \( \text{trace}_{\tilde{N}_k}(D_k \rho_d D_k^\dagger) = 0 = \text{trace}_{\tilde{N}_k}(D_{N_k} \otimes I_{\tilde{N}_k} \rho_d D_{N_k}^\dagger \otimes I_{\tilde{N}_k}) \). Therefore, it also follows that \( D_{N_k} \rho_{N_k} D_{N_k}^\dagger = 0 \). If we consider the spectral decomposition \( \rho_{N_k} = \sum q_j |\phi_j\rangle\langle\phi_j| \), with \( q_j > 0 \), the latter condition implies that, for each \( j \), \( \hat{D}_{N_k} |\phi_j\rangle\langle\phi_j| \hat{D}_{N_k}^\dagger = 0 \). Thus, it must be \( \text{supp}(\rho_{N_k}) \subseteq \ker(\hat{D}_{N_k}) \), as stated. ■

Theorem 3.3. A pure state \( \rho_d = |\Psi\rangle\langle\Psi| \) is DQLS if and only if

\[
\text{supp}(\rho_d) = \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k}) = \bigcap_k \ker(D_{N_k} \otimes I_{\tilde{N}_k}). \tag{3.2}
\]

Proof. Given lemmas 3.1 and 3.2, for any set \( \{D_k\} \) that make \( \rho_d \) DQLS, we have

\[
\text{supp}(\rho_d) \subseteq \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k}) \subseteq \bigcap_k \ker(D_{N_k} \otimes I_{\tilde{N}_k}).
\]

By negation, assume that \( \text{supp}(\rho_d) \not\subseteq \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k}) \). Then, there would be (at least) another invariant state in the intersection of the kernels of the noise operators, contradicting the fact that \( \rho_d \) is DQLS. Thus, a necessary condition for \( \rho_d \) to be GAS is that \( \text{supp}(\rho_d) = \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k}) \). Conversely, if the latter condition is satisfied, then for each \( k \) we can construct operators \( \hat{D}_{N_k} \) that render each \( \text{supp}(\rho_{N_k}) \) GAS on \( \mathcal{H}_{N_k} \) (see [6,17,18] for explicit constructions). Then, \( \bigcap_k \ker(\hat{D}_{N_k} \otimes I_{\tilde{N}_k}) = \text{supp}(\rho_{N_k}) \), and there cannot be any other invariant subspace. By lemma 2.4, \( \rho_d \) is hence rendered GAS by QL noise operators. ■

(b) An equivalent characterization: quasi-local parent Hamiltonians

Consider a QL Hamiltonian \( H = \sum_k H_k \), \( H_k = H_{N_k} \otimes I_{\tilde{N}_k} \). A pure state \( \rho_d = |\Psi\rangle\langle\Psi| \) is called a frustration-free ground state if

\[
\langle\Psi| H_k |\Psi\rangle = \min \lambda(H_k), \quad \forall k,
\]

where \( \lambda(\cdot) \) denotes the spectrum of a matrix. A QL Hamiltonian is called a parent Hamiltonian if it admits a unique frustration-free ground state [24].

Suppose that a pure state admits a QL parent Hamiltonian \( \hat{H} \). Then, the QL structure of \( \hat{H} \) may be naturally used to derive a stabilizing semigroup: it suffices to implement QL operators \( L_k \) that stabilize the eigenspace associated to the minimum eigenvalue of each \( H_k \). In view of theorem 3.3, it is easy to show that the following condition is also necessary.

Corollary 3.4. A state \( \rho_d = |\Psi\rangle\langle\Psi| \) is DQLS if and only if it is the ground state of a QL parent Hamiltonian.
Proof. Without loss of generality, we can consider QL Hamiltonians $H = \sum_k H_k$, where each $H_k$ is a projection. Let $\rho_d$ be DQLS, and define $H_k := \Pi_{N_k}^{\perp} \otimes I_{\tilde{N}_k}$, with $\Pi_{N_k}^{\perp}$ being the orthogonal projector onto the orthogonal complement of the support of $\rho_{N_k}$, i.e. $H_{N_k} \ominus \text{supp}(\rho_{N_k})$. Given theorem 3.3, $|\Psi\rangle$ is the unique pure state in $\bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k})$, and thus the unique state in the kernel of all the $H_k$. Conversely, if a QL parent Hamiltonian exists, to each $H_k$ we can associate an $L_k$ that asymptotically stabilizes its kernel. A single operator per neighbourhood is, in principle, always sufficient—see Ticozzi & Viola [6,17] for explicit constructions and examples of $L_k$ stabilizing a desired subspace. ■

The earlier-mentioned result directly relates our approach to the one pursued in Kraus et al. [14] and Verstraete et al. [13] and a few remarks are in order. In these works, it has been shown that matrix product states (MPSs) are QL stabilizable, up to a condition (the so-called injectivity) that is believed to be generic [24]. MPSs that allow for a compact representation (i.e. in the corresponding ‘valence-bond picture’, those with a small bond dimension) are of key interest in condensed matter as well as quantum information processing [24–27]. However, any pure state admits a (canonical) MPS representation if sufficiently large bond dimensions are allowed, suggesting that arbitrary pure states would be DQLS. The problem with this reasoning is that the locality notion that is needed in order to allow stabilization of a certain MPS is in general induced by the state itself. The number of elements to be included in each neighbourhood is finite but need not be small: while this is both adequate and sufficient for addressing many relevant questions in many-body physics (where typically a thermodynamically large number of subsystems is considered), engineering the required dissipative process may entail interactions that are not easily available in experimental settings. For this reason, our approach may be more suitable for control-oriented applications. It is also worth noting that the injectivity property is sufficient but not necessary for the target state to admit a QL parent Hamiltonian (an example on a two-dimensional lattice is provided in Perez-Garcia et al. [26]). Once the locality notion is fixed, our test for DQLS can be performed irrespective of the details of the MPS representation, and it is thus not affected by whether the latter is injective or not (rather, our DQLS test may be used to output a QL parent Hamiltonian if so desired).

(c) Examples

(i) Greenberger–Horne–Zeilinger states and W states

Consider an $n$-qubit system and a target Greenberger–Horne–Zeilinger (GHZ) state $\rho_{GHZ} = |\Psi\rangle\langle\Psi|$, with $|\Psi\rangle \equiv |\Psi_{GHZ}\rangle = (|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$. Any reduced state on any (non-trivial) neighbourhood is an equiprobable mixture of $|000\ldots0\rangle$ and $|111\ldots1\rangle$. It is then immediate to see that

$$\text{span}\{|000\ldots0\rangle, |111\ldots1\rangle\} \subseteq \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\tilde{N}_k}),$$

and hence $\rho_{GHZ}$ is not DQLS. In a similar way, for any $n$, the W state $\rho_W = |\Psi\rangle\langle\Psi|$, with $|\Psi\rangle \equiv |\Psi_W\rangle = (|101\ldots0\rangle + |010\ldots0\rangle + \ldots + |000\ldots1\rangle)/\sqrt{n}$
has reduced states that are statistical mixtures of \(|000\ldots0\rangle\) and a smaller W state \(|\Psi_W\rangle\), of the dimension of the neighbourhood. Thus,

$$\text{span}(|000\ldots0\rangle, |\Psi_W\rangle) \subseteq \bigcap_k \text{supp}(\rho_{N_k} \otimes I_{\bar{N}_k}),$$

and \(\rho_W\) is not DQLS (except in trivial limits, see also below). Note that for arbitrary \(n\), both \(\rho_{\text{GHZ}}\) and \(\rho_W\) are known to be (non-injective) MPSs with (optimal) bond dimension equal to two.

(ii) Stabilizer and graph states

A large class of states does admit a QL description, and in turn they are DQLS. Among these are stabilizer states, and general graph states. Here, the relevant neighbourhoods are those that include all the nodes connected to a given one by an edge of the graph. The details are worked out in Kraus et al. [14]. Notice that GHZ states are indeed graph states, but associated only with ‘star’ (or completely connected) graphs. In such cases, relative to the locality notion naturally induced by the graph, any central node has a neighbourhood that encompasses the whole graph, rendering the constraints effectively trivial.

(iii) Dissipatively quasi-locally stabilizable states beyond graph states

Consider a four-qubit system arranged on a linear graph, with (up to) three-body interactions. The two neighbourhoods \(N_1 = \{1, 2, 3\}, N_2 = \{2, 3, 4\}\) are sufficient to cover all the subsystems, and contain all the smaller ones. The state \(\rho_T = |\Psi\rangle\langle\Psi|\), with

$$|\Psi\rangle \equiv |\Psi_T\rangle = \frac{|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle}{\sqrt{6}},$$

is not a graph state because if we measure any qubit in the standard basis, we are left with W states on the remaining subsystems, which are known not to be graph states. In contrast, proposition 9 of Hein et al. [28] ensures that the conditional reduced states for a graph state would have to be graph states as well. Nonetheless, by constructing the reduced states and intersecting their supports, one can establish directly that \(|\Psi_T\rangle\) is indeed DQLS.

4. Switched feedback implementation

From theorem 3.3, it follows that a DQLS state can be asymptotically prepared provided we can engineer QL noise operators \(D_k = D_{N_k} \otimes I_{\bar{N}_k}\) that stabilize the support of each reduced state \(\rho_{N_k}\) on each neighbourhood. Restricting to \(\mathcal{H}_{N_k}\), we must have \(D_{N_k} = \begin{bmatrix} 0 & D_{P,k} \\ 0 & D_{R,k} \end{bmatrix}\), with the blocks \(D_{P,k}, D_{R,k}\) such that the support of \(\rho_{N_k}\) is attractive, i.e. such that no invariant subspace is contained in its complement. Following the ideas of Ticozzi & Viola [6] and Ticozzi et al. [18], a natural explicit choice is to consider noise operators with the following structure:

$$D_{P,k} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ \ell_1 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad D_{R,k} = \begin{bmatrix} 0 & \ell_2 & 0 & 0 \\ 0 & 0 & \ell_3 & \cdots \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix}. \quad (4.1)$$
If the above QL Lindblad operators are not directly available for open-loop implementation, a well-studied strategy for synthesizing attractive Markovian dynamics is provided by continuous measurements and output feedback. In the absence of additional dissipative channels, and assuming perfect detection, the relevant feedback master equation takes the form [5]

\[
\dot{\rho}(t) = -i[H + H_c + \frac{1}{2}(FM + M^\dagger F), \rho(t)] + L_f \rho(t)L_f^\dagger - \frac{1}{2}[L_f^\dagger L_f, \rho(t)],
\]

where \( H_c \) is a time-independent control Hamiltonian, \( F = F^\dagger \) and \( M \) denote, respectively, the feedback Hamiltonian and the measurement operator, and \( L_f := M - iF \). Necessary and sufficient conditions for the existence of open- and closed-loop Hamiltonian control that stabilizes a desired subspace have been provided by Ticozzi and co-workers [6,17].

In order to exploit the existing techniques in the current multi-partite setting, it would be necessary to implement measurements and feedback in each neighbourhood. If the measurement operators do not commute, however, one would have to carefully scrutinize the validity of the model and the consequences of ‘conflicting’ stochastic back-actions when acting simultaneously on overlapping neighbourhoods. These difficulties can be bypassed by resorting to a cyclic switching of the control laws. Consider a DQLS state \( \rho_d \) and the family of generators \( \{L_k\}_{k=1}^M, L_k[\rho] = D_k \rho D_k^\dagger - \frac{1}{2}(D_k^\dagger D_k, \rho) \), with \( D_k \) such that \( \text{supp}(\rho_{N_k} \otimes \mathbb{I}_{\hat{N}_k}) \) is the unique invariant subspace for \( L_k \). Define a switching interval \( \tau > 0 \) and the cyclic switching law \( j(t) = \lfloor t/\tau M \rfloor + 1 \). We can then establish the following.

**Theorem 4.1.** There exists QL \( \{D_k\} \) such that \( \rho_d \) is GAS for the switched evolution \( L_{j(t)} \).

**Proof.** Consider the trace-preserving, completely positive maps \( \mathcal{T}_j(\rho) = e^{L_j \tau}[\rho] \). It is easy to see that \( \rho_d \) is invariant for each \( \mathcal{T}_j \) as a corollary of theorem 1 in Bolognani & Ticozzi [29], it follows that \( \rho_d \) is GAS if it is the only invariant state for \( \mathcal{T} = \mathcal{T}_M \circ \cdots \circ \mathcal{T}_1 \). Assume that \( \rho_d \) is GAS for \( \mathcal{T} \) then either it is fixed for all \( \mathcal{T}_k \), which means that necessarily \( \rho = \rho_d \), or there exists a periodic cycle. Because each \( \mathcal{T}_j \) is a trace-distance contraction [22], this means that each map preserves the trace distance, i.e. \( \|\mathcal{T}_j(\rho_d - \rho)\|_1 = \|\rho_d - \rho\|_1 \). This would in turn imply that each \( \mathcal{T}_j \) admits eigenvalues on the unit circle, and hence each \( L_k \) would have imaginary ones. However, if we choose \( D_k \) as in equation (4.1), in vectorized form, the Liouvillian generator reads

\[
\hat{L}_k = (D_k^\dagger)^T \otimes D_k - \frac{1}{2} I \otimes D_k^\dagger D_k - \frac{1}{2} (D_k^\dagger D_k)^T \otimes I,
\]

which is an upper triangular matrix with eigenvalues either equal to zero, \(-|\ell_j|^2/2\), or \(-(|\ell_j|^2 + |\ell_i|^2)/2\). Therefore, for this choice, \( \rho_d \) is the only invariant pure state for \( \mathcal{T} \) and hence it is GAS.

5. Concluding remarks

We have presented a characterization of DQLS pure states for fixed locality constraints, from a system-theoretic perspective. As a by-product of our analysis, an easily automated algorithm for checking DQLS states is obtained. The
necessary steps entail: (i) calculating the reduced states on all the neighbourhoods specifying the relevant QL notion; (ii) computing their tensor product with the identity on the remaining subsystems, and the corresponding supports; and (iii) finding the intersection of these subspaces. If such intersection coincides with the support of the target state alone, the latter is DQLS. If so, we have additionally showed that the required Markovian dynamics can, in principle, be realized by switching output-feedback control. While we considered homodyne-type continuous-time feedback MME, the study of discrete-time strategies is also possible along similar lines, see also Bolognani & Ticozzi [29] and Barreiro et al. [15].

Our present results have been derived under two main assumptions: the absence of underlying free dynamics, and the use of purely dissipative control (no Hamiltonian control involved). In case a drift internal dynamics is present, a similar approach can, in principle, be adapted to determine what can be attained by dissipative control. When we additionally allow for Hamiltonian control, one may use the algorithm described in §III.B of Ticozzi et al. [30] to search for a viable QL Hamiltonian when dissipation alone fails. Nonetheless, in the presence of locality constraints, a more efficient design strategy may be available: an in-depth analysis of the role of additional internal dynamics as well as of combined coherent and dissipative control will be presented elsewhere.

It is also worth noting that in various experimental situations, the available dissipative state preparation procedures involve two steps: first, enact local noise operators that prepare a known pure state that is factorized; next, use open-loop coherent control to steer the system on the desired entangled target. The approach we discussed here is expected to have an intrinsic advantage in terms of the overall robustness against initialization errors and finite-time perturbations of the dynamics [13,16]. While establishing rigorous robustness results require further study, the actual answer will crucially depend on the physical implementation and its characteristic time scales. Lastly, the estimation of the speed of convergence still presents numerous challenges, most importantly its optimization and a characterization of its scaling with the number of subsystems involved.

F.T. acknowledges support by the QFuture project of the University of Padova.

References

Quasi-local stabilization