The quantum trajectory approach to quantum feedback control of an oscillator revisited

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We revisit the stochastic master equation approach to feedback cooling of a quantum mechanical oscillator undergoing position measurement. By introducing a rotating wave approximation for the measurement and bath coupling, we can provide a more intuitive analysis of the achievable cooling in various regimes of measurement sensitivity and temperature. We also discuss explicitly the effect of backaction noise on the characteristics of the optimal feedback. The resulting rotating wave master equation has found application in our recent work on squeezing the oscillator motion using parametric driving and may have wider interest.

Keywords: quantum control; quantum measurement; nanomechanics

1. Introduction

Recent years have seen an explosion of interest in mechanisms of cooling of nanomechanical oscillators to close to their ground state of motion; recent results include [1–3]. One technique that has been investigated is the use of feedback cooling or ‘cold damping’, and it is in this sort of experiment that a theory of quantum feedback control should be relevant. Several experiments have been conducted [4–8], although they have yet to approach the limit of feedback cooling to the ground state. There is also some hope that feedback control protocols may assist in preparing more exotic quantum states of motion of oscillators. For example, one goal has been to use feedback control to induce squeezed states of mechanical motion [9]. In recent work we showed how to combine position measurement and feedback control with a detuned parametric drive in order to prepare squeezed states of mechanical motion in the steady state [10].

During the 1980s Belavkin developed a theory of optimal feedback control of continuously measured quantum systems [11–14]. This theory models the system of interest fully quantum mechanically and includes the effects of backaction noise. This theory is based on what are now called quantum trajectory descriptions of quantum measurements, and the direct application of these to physical systems was greatly stimulated by the independent development of a theory of quantum feedback by Wiseman & Milburn [15]. One of the simplest examples of the
application of optimal control theory for quantum systems is to feedback cooling
of an oscillator undergoing position measurement as discussed in [9,16,17]. In
this paper, we hope to provide a more straightforward description of the effects
of a thermal bath, more intuition about the optimal control protocols and
and a comparison with the literature on controlled oscillators that makes use of
Langevin equations (e.g. [6–8,18–20]). Because our approach will make extensive
use of stochastic differential equations and the Itô calculus and of the quantum
trajectory description of continuous quantum measurements, good introductory
references on these are [21,22] for stochastic differential equations and [15,23,24]
for quantum trajectories.

In this paper, we revisit the model of a controlled quantum oscillator used
in [16], with a view to providing a more physical description of the quantum limits
on control that arise. In order to facilitate this, we develop a simplified model that
employs a rotating wave approximation that is accurate when the measurement
device is not too sensitive, and that greatly simplifies the analysis, without
changing much of the physics when the oscillator is high-Q. This rotating wave
approximation to the stochastic master equation describing position measurement
on a high-Q oscillator was introduced in Szorkovszky et al. [10] without a detailed
justification; so the derivation we outline here may be of independent interest.

The paper is structured as follows. In §2, we describe the usual model of
continuous position measurement of a quantum mechanical oscillator. In §3, we
derive simplified equations of motion using a rotating wave approximation. In §4,
we solve these equations of motion for their steady states and in §5, we use the
techniques of optimal control theory to analyse feedback cooling of a quantum
oscillator in this simplified model.

2. Traditional continuous position measurement model

We will discuss a standard model for continuous position measurement of an
oscillator using quantum trajectories [25–28]. This model can be realized as
an approximate description of more realistic models, for example optical or
microwave interferometric detection (e.g. [16]). For a textbook introduction
to quantum trajectories and feedback, we refer the reader to Wiseman &
Milburn [15]. The oscillator is of mass \( m \), has position and momentum operators
\( \hat{x} \) and \( \hat{p} \), respectively, and has angular frequency \( \omega_m \). The Hamiltonian for the
system is

\[
H = \frac{p^2}{2m} + \frac{1}{2}m\omega_m^2 \hat{x}^2 + f(t) \hat{x}.
\] (2.1)

The feedback force is \( f(t) \) and will be chosen in a way that depends on
earlier measurement results, according to some feedback protocol. The oscillator
is assumed to damp at rate \( \gamma \) owing to coupling to a bath of temperature
\( T \). The thermal phonon number corresponding to this temperature is \( N = [\exp(h\omega_m/k_B T) - 1]^{-1} \). Finally, the oscillator is coupled to a measurement device.

Ignoring, for the moment, the reading on the measurement device, we wish to
describe the system using a master equation for the density matrix \( \rho \), as follows:

\[
\dot{\rho} = \mathcal{L}\rho = -i[H, \rho] + \mathcal{L}_{\text{env}}\rho + \mathcal{L}_{\text{meas}}\rho,
\] (2.2)

where \( \mathcal{L}_{\text{env}} \) and \( \mathcal{L}_{\text{meas}} \) describe the effect of the bath and the measuring device,
respectively.

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In a master equation approach, it is straightforward to model the thermal bath in the rotating wave and Markov approximations, as follows [15,29]:

\[ \mathcal{L}_{\text{env}} \rho = 2\gamma(N + 1)\mathcal{D}[\hat{a}]\rho + 2\gamma N\mathcal{D}[\hat{a}^\dagger]\rho, \] 

where we have introduced the usual oscillator lowering operators \( \hat{a} \) such that \( \hat{x} = \sqrt{\hbar/2m\omega_m}(\hat{a} + \hat{a}^\dagger) \) as well as the compact notation for Lindblad master equations

\[ \mathcal{D}[\hat{c}]\rho = c\rho c^\dagger - \frac{c^\dagger c\rho}{2} - \frac{\rho c^\dagger c}{2}. \] 

Making a rotating wave approximation for the coupling to the bath, we avoid some of the vexed issues related to finding a Markovian master equation for Brownian motion without making the rotating wave approximation [29]. This approximation will be valid in the regime where \( \gamma \ll \omega_m \), that is to say for a high-Q oscillator. If the feedback signal is strong it can result in much stronger damping of the oscillator motion; in this case, it will be the effective damping rate of the oscillator that must be smaller than \( \omega_m \) for the Markov and rotating wave approximations to be valid. The coupling of the oscillator to the measurement device does not usually involve an analogous rotating wave approximation. The effect of the measuring device is modelled as follows:

\[ \mathcal{L}_{\text{meas}} \rho = \chi\mathcal{D}[\hat{x}]\rho. \] 

This term contributes to heating the oscillator by adding a term to \( d\langle \hat{p}^2 \rangle/dt \) that is equal to \( \hbar^2\chi \). This is known as the backaction heating. Notice that the parameter \( \chi \) has units of \( \text{[frequency]} \text{[length]}^{-2} \). It is related to the sensitivity of the measurement in a fashion that will soon be clear.

The model of a measurement that we have adopted corresponds to position detection in a background of white noise. If the measured current is \( I(t) = dQ(t)/dt \), then this current is modelled by the following equation [15,16,28]:

\[ dQ(t) = 2\sqrt{\hbar \chi}(\hat{x}(t))dt + dW(t). \] 

In this expression, \( dQ \) should be thought of as the measurement result at time \( t \). The conditioned mean value \( \langle \hat{x} \rangle \) is the expectation value of \( \hat{x} \), given the results of earlier measurements. The innovation \( dW(t) \) describes the extent to which the measured value departs from this expectation. It is a mean zero Gaussian-distributed increment that, in the continuum limit, satisfies \( dW^2 = dt \). Speaking loosely, the singular quantity \( dW/dt \) is a white noise signal with unit power. Positive values of \( dW \) correspond to a measurement result \( dQ \) that was larger than we expected, and negative values to a smaller measurement result. The parameter \( \eta \) is the quantum efficiency of the measurement and ranges from 0 to 1, with 1 corresponding to the best possible measurement, having the least backaction noise for a given measurement sensitivity. Position measurements are often described in terms of the transduction noise power, called \( S_N \) in Lee et al. [8] and having units of \( \text{[length]}^2 \text{[frequency]}^{-1} \). In terms of the parameters of our model, \( S_N = 1/4\eta\chi \).
Throughout this paper, we will distinguish different regimes of measurement by considering the resolution that would be available with a given measurement device if it were possible to ignore the oscillator motion. By comparing this with length scales characteristic of the oscillator motion, we will be able to determine qualitatively the effectiveness of feedback cooling of the oscillator. Consider integrating over a time $\Delta t$

$$\Delta Q = \int_t^{t+\Delta t} [2\sqrt{\eta X}(\hat{x}(t'))dt' + dW(t')] \simeq 2\sqrt{\eta X}(\hat{x})\Delta t + \int_t^{t+\Delta t} dW(t'). \quad (2.7)$$

In making the approximation, we have assumed that the oscillator motion is negligible over the time $\Delta t$. Unless $\Delta t \ll 1/\omega_m$, this will be a bad approximation, but nevertheless it will be a useful indication of the effectiveness of the measurement device to consider this sort of hypothetical estimate where $\Delta t \simeq 1/\omega_m$ or $\Delta t \simeq 1/\gamma$. The stochastic integral that appears here, $\int_t^{t+\Delta t} dW(t')$, is simply a Gaussian-distributed random number with mean zero and variance $\Delta t$ [21]. As a result, $\Delta Q$ is a Gaussian random variable with mean $2\sqrt{\eta X}(\hat{x})\Delta t$ and variance $\Delta t$. So, if we could ignore oscillator motion, the measurement could resolve a distance $\delta x(\Delta t) = \sqrt{1/4\eta X\Delta t}$ with a signal-to-noise of one in an averaging time $\Delta t$. In order to determine the effectiveness of the measurement we will choose an averaging time of the order of the decay rate $\Delta t \simeq 1/4\gamma$ and then compare the length $\delta x \equiv \delta x(1/4\gamma) = \sqrt{\gamma/\eta X}$ either with $\delta x_T = \sqrt{\hbar B T/m\omega_m^2}$ (the r.m.s. displacement of a classical thermal oscillator at temperature $T$) or with $\delta x_Q = \sqrt{\hbar/2m\omega_m}$ (the r.m.s. displacement of a quantum harmonic oscillator in its ground state). We will find that in order to cool the oscillator significantly below its thermal temperature, we will need $\delta x \ll \delta x_T$, while to have feedback cooling close to the harmonic oscillator ground state, it will be necessary to have $\delta x \ll \delta x_Q/\sqrt{N}$. We will see later that another way to express this condition is that it is necessary for the backaction heating $\hbar^2\chi$ to exceed the thermal heating $2\gamma k_B T$. We will also sometimes be interested in a shorter averaging time, of the order of the oscillation period $\delta x_{\omega_m} \equiv \delta x(1/4\omega_m)$.

The theory of continuous quantum measurements allows us to update the density matrix of the system in light of the measured results [15]. The resulting density matrix is usually known as the conditioned state because it is conditioned on earlier measurement results. As we will see the properties of the conditioned state place limits on the performance of any feedback system using the measurement. Intuitively, this update following a measurement should be non-zero only when the innovation $dW$ is non-zero, and when $dW$ is positive, it should shift the state so as to increase the mean position, in line with the larger than expected measurement result $dQ$. The size of this correction should increase as the signal-to-noise of the measurement increases. The following stochastic master equation can be derived in many models for the measuring device [15,16,28] and satisfies all of these intuitive expectations:

$$d\rho = \mathcal{L}\rho \, dt + \sqrt{\eta X}\mathcal{H}[\hat{c}]\rho \, dW,$$

where we have used the shorthand notation

$$\mathcal{H}[\hat{c}]\rho = c\rho + \rho c^\dagger - (c + c^\dagger)\rho.$$
Some intuition for this model can be gained by looking at the effect of measurement terms on the expectation value of the position. We can find an update for the position from the master equation as follows:

$$d\langle \hat{x} \rangle = \text{Tr}[\hat{x} \, d\rho] = (\ldots) dt + 2\sqrt{\eta \chi}(\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2) dW. \quad (2.9)$$

The ellipsis is to indicate the contribution to $d\langle \hat{x} \rangle$ that is deterministic and independent of the measurement result. By focusing just on the update due to the measurement (which is proportional to $dW = dQ - 2\sqrt{\eta \chi}(\hat{x}) dt$), we see that the mean value of position is indeed increased when $dW > 0$ and vice versa. The size of the increase depends on the strength of the measurement $\chi$, with better measurements resulting in a larger correction, and on the variance of position, so that the update is larger when the position is less well determined by earlier measurements.

### 3. Simplified measurement model

It is possible to simplify this model by making a rotating wave approximation for the measurement that is analogous to the rotating wave approximation used to derive the master equation for the coupling to the thermal bath. The resulting master equation was introduced, but not explicitly derived, in our recent paper [10]. The main ideas for deriving rotating wave approximation stochastic master equations are to be found in the derivation of the heterodyne detection master equation by Wiseman & Milburn [30]. A very similar implementation to that which we describe here is made in a related measurement model in Doherty et al. [31]. There is a good reason to make this approximation when the oscillator is sufficiently high-Q and the measurement is not too strong. In this limit, the measured current $I(t) = dQ/dt$ has a spectrum that is strongly peaked around $\omega_m$. In an experimental setting, it is natural to demodulate the measured current to obtain signal with only low-frequency time variations. This demodulation roughly corresponds to multiplying the signal by either $\cos \omega_m t$ or $\sin \omega_m t$. This is not quite accurate because the resulting signal would have peaks in its spectrum at $\pm2\omega_m t$ and not only close to zero frequency. As a result, in a real demodulation scheme, this higher frequency signal is filtered out. This can be done for example as follows:

$$I_X \simeq \sqrt{2} \int_{-\infty}^{t} dt' e^{-\tilde{\kappa}(t-t')} \cos[\omega_m(t-t')] I(t') \quad (3.1)$$

and

$$I_Y \simeq \sqrt{2} \int_{-\infty}^{t} dt' e^{-\tilde{\kappa}(t-t')} \sin[\omega_m(t-t')] I(t'), \quad (3.2)$$

where $\gamma \ll \tilde{\kappa} \ll 2\omega_m$. These two demodulated signals will depend on the two quadrature observables of the oscillator. Our goal in the following is to write an approximate stochastic master equation explicitly in terms of $I_X$ and $I_Y$. The odd-looking factor of $\sqrt{2}$ is there to compensate for the fact that otherwise the demodulation would reduce the noise power by one half. This normalization is not necessary but simplifies some of the other factors in later equations.
We may define (time-dependent) quadrature observables as follows:

\[
\hat{X}(t) = \frac{\hat{a}e^{iut} + \hat{a}^{\dagger}e^{-iut}}{\sqrt{2}} \quad (3.3)
\]

and

\[
\hat{Y}(t) = -\frac{i(\hat{a}e^{iut} - \hat{a}^{\dagger}e^{-iut})}{\sqrt{2}}. \quad (3.4)
\]

The quadrature observables are chosen such that they are time-independent in the interaction picture. For an ideal oscillator, they correspond to the initial \( t = 0 \) position and momentum. With these definitions, we have

\[
[\hat{X}, \hat{Y}] = i. \quad (3.5)
\]

Also, we find that the ground state of the oscillator satisfies \( \langle \hat{X}^2 \rangle = \langle \hat{Y}^2 \rangle = \frac{1}{2} \).

Our approach is now to move to the interaction picture with respect to the oscillator Hamiltonian and then make the usual rotating wave approximation. When we do this, we will now need to imagine that our feedback force \( f(t) \) has a spectrum centred at frequencies \( \pm \omega_m \) and is therefore slowly varying in the interaction picture. There are going to be two quadratures of such a driving signal, so we now imagine that we have the following interaction picture Hamiltonian, written in terms of \( \hat{X} \) and \( \hat{Y} \) such that

\[
H_I = \hbar u_Y(t) \hat{X} + \hbar u_X(t) \hat{Y}. \quad (3.6)
\]

The two quadratures of the driving force are given by the rates \( u_X \) and \( u_Y \), which are required to be slowly varying in the interaction picture. Here, the swapped notation is to simplify later calculations; this reflects the fact that the driving force \( u_X \) will appear in the differential equation for the \( \hat{X} \) quadrature. It also reflects the phase shift of the feedback relative to the measured signal that we intuitively realize is necessary to cool the oscillator. We are measuring position but in order to damp the motion, we wish to exert a force \( f(t) \propto -\langle \hat{p} \rangle \). This means that we expect the feedback protocol will delay the measurement signal by a quarter of an oscillation cycle, at least for frequency components close to \( \omega_m \).

The usual rotating wave approximation amounts to disregarding the terms in the master equation that have time-dependence in the interaction picture. A slightly more rigorous way to justify the approximation is to use perturbation theory to integrate the full master equation over a time \( \Delta t \) that is short compared with \( u_X, \gamma \) and other rates in the interaction picture master equation. Choosing \( \Delta t \) equal to some integer number of periods of the mechanical oscillation, so that \( \Delta t = n2\pi/\omega_m \) for some \( n \) makes it possible to simplify the usual Dyson expansion. Finally, one takes the continuous time limit \( \Delta t \to dt \), finding a time-independent master equation that should approximately agree with the true time evolution once every period. At lowest non-trivial order in perturbation theory, this procedure has precisely the effect of disregarding the time-dependent terms in the master equation. This discussion should make clear that the rotating wave master equation describes a kind of time-averaged evolution that is valid in the interaction picture for long times compared with the oscillation period, and that
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will give an accurate picture of dynamics with frequencies close to the oscillation frequency. A straightforward calculation gives the following form for the rotating wave master equation:

$$\dot{\rho}_I = \mathcal{L}_{RW} \rho_I = -i[H_I, \rho_I] + [2\gamma(N + 1) + \mu]D[\hat{a}]\rho_I + [2\gamma N + \mu]D[\hat{a}^\dagger]\rho_I,$$

(3.7)

where we have defined the rate $\mu \equiv \chi \hbar/2 m \omega_m$.

Returning to the measured signal itself, we may write it in terms of the quadratures as follows:

$$dQ = 2 \sqrt{\eta \mu} (\cos \omega_m t \langle \hat{X} \rangle + \sin \omega_m t \langle \hat{Y} \rangle) dt + dW(t).$$

(3.8)

The form of the demodulated measurement currents can be seen by considering the following:

$$\sqrt{2} \cos(\omega_m t) dQ = 2 \sqrt{\eta \mu} [(1 + \cos 2\omega_m t) \langle \hat{X} \rangle + \sin 2\omega_m t \langle \hat{Y} \rangle] dt$$

$$+ \sqrt{2} \cos(\omega_m t) dW(t).$$

(3.9)

Because the quadrature expectation values will have time dependence close to zero frequency, the signal is clearly split up into components at frequencies $\pm 2\omega_m$ and components near zero frequency. The high-frequency components will be filtered out in the demodulation and we will be left with a signal proportional to the $\hat{X}$ quadrature, as expected.

One way to imagine implementing the demodulation, that is consistent with the derivation of the rotating wave master equation, is to average the measured current over a time $\Delta t$ that is some number of periods of the oscillator but remains short compared with the other time scales in the problem. This gives us

$$\Delta Q_X = \sqrt{2} \int_t^{t+\Delta t} \cos(\omega_m t) dQ \simeq 2 \sqrt{\eta \mu} \langle \hat{X} \rangle \Delta t + \Delta W_X,$$

(3.10)

where we have assumed that $\Delta t$ is sufficiently short that the factor of $\langle \hat{X} \rangle$ is approximately constant and can be removed from the integral. This will hold approximately in the regime in which the rotating wave approximation is good. We have defined the Gaussian noise increment $\Delta W_X \equiv \sqrt{2} \int_t^{t+\Delta t} \cos \omega_m t' dW(t')$.

We find from this definition and the usual properties of stochastic integrals that

$$\langle \Delta W_X^2 \rangle = \int_t^{t+\Delta t} \cos^2 \omega_m t' dt' = \Delta t.$$

(3.11)

Thus, $\Delta W_X$ has the same statistics as the integral of some Ito increment $dW_X$. By making an analogous definition for $\Delta W_Y$, we find the following:

$$\langle \Delta W_Y^2 \rangle = \Delta t \quad \langle \Delta W_X \Delta W_Y \rangle = 0.$$

(3.12)

This clearly suggests that in the limit where we return to a continuous time description and $\Delta t \to dt$, the Gaussian noises $\Delta W_X$ and $\Delta W_Y$ should approach uncorrelated Ito increments. And so we find that the two demodulated signals

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should be modelled as follows:

\[ dQ_X = 2\sqrt{\eta\mu}\langle \hat{X} \rangle dt + dW_X \] (3.13)

and

\[ dQ_Y = 2\sqrt{\eta\mu}\langle \hat{Y} \rangle dt + dW_Y. \] (3.14)

These same definitions and calculations can then be applied to the stochastic term in the original stochastic master equation (2.8) to find the rotating wave stochastic master equation as follows:

\[ d\rho_t = L_{RW}\rho_t dt + \sqrt{\eta\mu\mathcal{H}[\hat{X}]\rho} dW_X + \sqrt{\eta\mu\mathcal{H}[\hat{Y}]\rho} dW_Y. \] (3.15)

This is the stochastic master equation that was introduced, but not explicitly derived, in Szorkovszky et al. [10].

The rotating wave approximation used in this section will be valid whenever \( \mu \ll \omega_m \). There is a straightforward interpretation of this parameter regime in terms of the resolution of the measurement given earlier. We defined the lengthscale \( \delta x_\omega \), which would be resolved by the measurement device in a time of the order of the oscillation period if we could ignore the oscillator motion in that time. We will call the regime in which \( \delta x_\omega \) is less than the width of the ground state wave function of the oscillator \( \delta x_0 \) the regime of strong measurement.

By rearranging the condition \( \delta x_\omega \leq \delta x_Q \), we see that strong measurement occurs whenever \( \omega_m \leq \eta\mu \). As a result, we find that if \( \mu \geq \omega_m \) such that the rotating wave approximation is not valid, the measurement is so good that when \( \eta = 1 \), it can determine displacement with a precision equal to the width of the ground state wave function in a time of the order of a harmonic oscillator period. Of course, in this regime, the naive discussion of signal-to-noise here is insufficient because it does not account for oscillator motion during the measurement; this can be done using the original stochastic master equation (2.8). In this strong measurement regime, the system will be placed in a squeezed state with a narrow position distribution owing to the measurement. However, this regime can be very difficult to reach in experimental conditions and the exact solutions are harder to get an intuitive grasp of; so we will not consider it further here, and the reader is referred to the work of Doherty & Jacobs [16].

4. Solutions of the stochastic master equation

We can now convert this stochastic master equation into a set of equations that describe the evolution of mean values and variances of the conditioned state. We can define variances of the quadratures as \( V_X = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 \) and likewise for \( \hat{Y} \).

The covariance can be defined with a symmetrization so that it is real, as follows:

\[ C = \langle \hat{X} \hat{Y} + \hat{Y} \hat{X} \rangle / 2 - \langle \hat{X} \rangle \langle \hat{Y} \rangle. \]

We may use the stochastic master equation (3.15) to find time evolution equations for all these quantities using \( d\langle \hat{c} \rangle = \text{Tr}\hat{c} d\rho \).

Then the mean values of the quadratures satisfy stochastic differential equations as follows:

\[ d\langle \hat{X} \rangle = -\gamma\langle \hat{X} \rangle dt + u_X dt + 2\sqrt{\eta\mu}(V_X dW_X + C dW_Y) \] (4.1)

and

\[ d\langle \hat{Y} \rangle = -\gamma\langle \hat{Y} \rangle dt + u_Y dt + 2\sqrt{\eta\mu}(V_Y dW_Y + C dW_X). \] (4.2)
Notice in these equations that if there is some correlation between $\hat{X}$ and $\hat{Y}$, then the results of the measurement of $\hat{X}$ lead to some adjustment to the value of $\langle \hat{Y} \rangle$ and vice versa. This is consistent with our interpretation of these terms as an update of the conditioned state using the measurement results and the information from previous measurements. The variances and covariance have dynamics that prove to be independent of the measurement results and the control signal; so, for example, we find

$$\dot{V}_X = -2\gamma V_X + \gamma(2N+1) + \mu - 4\eta \mu(V_X^2 + C^2). \quad (4.3)$$

The first term describes the relaxation of the oscillator, the second term is the heating arising from the coupling to the thermal bath and the third term is the heating due to the measurement device, often known as the backaction noise. The final term in this equation tends to reduce the variances and corresponds to the fact that the quadratures become better known as time goes on because of the information gained through the measurement.

It is straightforward to find the steady-state variances. This steady state is the result of a competition between the thermal and backaction heating, which tends to increase the quadrature variances, and the information gained through the measurement, which tends to reduce the variances. It is most straightforward to assume at first that in the steady state $C = 0$, in which case the various equations decouple and the steady-state value of $V_X$ can be found by setting the right-hand side of equation (4.3) to zero. The solution that results can be chosen to have positive variances and, since $C = 0$, a positive covariance matrix. There are general properties of the Kalman filter equations that guarantee that this is the unique physically relevant solution to these nonlinear algebraic equations. We find

$$V_X = V_Y = \frac{1}{2} + f(r, N), \quad (4.4)$$

where we have characterized the measurement strength in terms of the unitless parameter

$$r = \frac{2\eta \mu}{\gamma} = 2\frac{\eta \chi}{\gamma} \frac{\hbar}{2m\omega_m} = 2\frac{\delta x_Q^2}{\delta x_\gamma^2}. \quad (4.5)$$

Recalling our simple discussion of the measurement above, the parameter $r$ is large ($r \gg 1$) when $\delta r_\gamma \ll \delta x_\gamma$, and the measurement is sufficiently sensitive that it would resolve the width of the ground state wave function during a time of the order of the decay time of the oscillator if we could disregard oscillator motion during that time. We have also defined the function

$$f(r, N) = \frac{1}{2r} \left( -1 + \sqrt{1 + 2r(2N+1) + \frac{r^2}{\eta}} \right) - \frac{1}{2}. \quad (4.6)$$

Recall that the oscillator ground state $V_X = V_Y = \frac{1}{2}$ and so we see that, since $f > 0$, the conditioned steady state never has quadrature variances below those of the ground state. This means that the conditioned state has no mechanical squeezing in the regime in which the rotating wave description of the measurement is valid. This is in contrast to the strong measurement regime discussed above, where $\eta \mu \geq \omega_m$. 

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We will now consider various regimes of the model. We first check that the conditioned state is always a valid quantum state by looking at the ideal case of ideal quantum efficiency \( \eta = 1 \) and zero temperature \( N = 0 \). In this case, the argument of the square root in (4.6) is a perfect square and we find

\[
V_X = V_Y = \frac{1}{\sqrt{2}}. 
\]

(4.7)

This shows that at long times, the conditioned states of the model are coherent states completely independent of the quality of the measurement as given by \( r \).

We should compare this behaviour with the equivalent classical model that has the same position measurement sensitivity but no backaction noise. Removing the backaction noise term from (4.3), we find that steady-state covariances in a classical model are

\[
V_C^X = V_C^Y = \frac{1}{2} \frac{1}{\sqrt{2}} \left( -1 + \sqrt{1 + 2r(2N + 1)} \right). 
\]

(4.8)

In the limit of strong measurement \( 2r(2N + 1) \gg 1 \), we find that \( V_C^X \simeq \sqrt{(2N + 1)/2r} \) approaches zero, consistent with our classical expectation. Because this is a classical limit, it makes sense to convert the quadrature variance back into a position variance in the limit of high temperature, where \( N \to kT / \hbar \omega_m \) to find

\[
\langle (\Delta \hat{x})^2 \rangle_C = \frac{\hbar}{m \omega_m} V_C^X \simeq \delta x_T \delta x_T. 
\]

(4.9)

So we see that the position variance depends on the r.m.s. displacement of the oscillator at equilibrium, and on the strength of the measurement. It is clear that this cannot be the quantum mechanical result because this variance does indeed decrease to zero, as the measurement is improved and \( \chi \) grows (recall that \( \delta x_T = \sqrt{\gamma / \eta \mu} \) and \( \mu = \chi \sqrt{\hbar / 2m \omega_m} \)). However, we will see that this is a good approximation to the true conditioned variance over a wide parameter regime.

We now consider three regimes of measurement, corresponding to the situation where each of the three terms in the square root in equation (4.6) are in turn dominant. The first is bad measurement where the measuring device does not extract enough information to significantly reduce the conditioned variance below its thermal value. In this limit, we require that \( 2r(2N + 1), r^2 / \eta \ll 1 \) and expand the square root in (4.6) using the binomial expansion. In the limit of high temperature, we can use the fact that \( N \simeq kT / \hbar \omega_m \) to see that this condition holds when

\[
\delta x_T \ll \delta x_T. 
\]

(4.10)

As a result, this bad measurement condition has the interpretation that \( \chi \) is so small that, even neglecting the oscillator motion, it is not possible to resolve a displacement equivalent to the r.m.s. displacement of a Brownian oscillator during a time of order \( 1 / \gamma \). In this limit, we find

\[
V_X = V_Y \simeq \frac{1}{2} (2N + 1) \simeq \frac{kT}{\hbar \omega_m}, 
\]

(4.11)

which is just the thermal equilibrium variance of \( \hat{X} \) corresponding to an equilibrium position variance \( \langle (\Delta \hat{x})^2 \rangle \simeq kT / m \omega_m^2 \). The conditioned and unconditioned mean values are almost equal.

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The second regime of interest is the regime of classical measurement for which \(2r(2N + 1) \gg 1, \frac{r^2}{\eta}\). We can manipulate these conditions to see that they hold when
\[
\delta x_T \gg \delta x_\gamma \gg \frac{\delta x_Q}{\sqrt{\eta N}}.
\] (4.12)

This is the regime in which the measurement extracts enough information to reduce the variances well below their equilibrium values, but the backaction noise has no significant effect on the steady state. At low temperature, the second condition has the interpretation that, even neglecting oscillator motion, it is not possible to resolve the width of the ground state wave function during a time of order \(1/\gamma\). However, as the temperature rises it is necessary to have better measurement sensitivity to break out of this regime, because of the more rapid heating of the oscillator. In this limit, we find
\[
V_X = V_Y \simeq \sqrt{\frac{2N + 1}{2r}}.
\] (4.13)

Note that in this case, the steady state is roughly equal to \(V^C_X\), the steady-state covariance of a classical measurement with the same sensitivity but no backaction. Hence the term classical measurement.

The final regime is backaction-dominated measurement in which the backaction heating sets the steady-state quadrature variance and from inspection of (4.6), we need \(r^2/\eta \gg 1, 2r(2N + 1)\). The condition \(r^2/\eta \gg 1\) is only the limiting one in the limit of very low temperature and very low quantum efficiency measurement; so we will neglect it here. A simple calculation shows that the measurement is backaction-dominated if
\[
\frac{\delta x_Q}{\sqrt{\eta N}} \gg \delta x_\gamma.
\] (4.14)

The second version of the first condition is valid in the high temperature limit and has the intuitive interpretation that the heating due to the backaction noise (\(\hbar^2 x\)) must be greater than heating due to the coupling to the thermal bath (\(2\gamma mkT\)). In the backaction-dominated regime, we find
\[
V_X = V_Y \simeq \frac{1}{2\sqrt{\eta}} \left(1 + \frac{\eta(2N + 1) - \sqrt{\eta}}{r}\right).
\] (4.15)

Note that the second term in brackets is required to be a small correction in the backaction-dominated regime we are considering here. So in this limit, we see that for sufficiently high quantum efficiency the conditioned quadrature variance approaches its ground state value of \(\frac{1}{2}\).

## 5. Feedback control

We now consider the problem of optimal feedback cooling along the lines of Doherty & Jacobs [16] and of Gough [32] for the discussion of linear Gaussian models of the kind we are studying here. Suppose that we wish to minimize the energy using the feedback signals \(u\). Given a particular set of measurement results

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Table 1. Optimal energy of a quantum mechanical oscillator undergoing feedback cooling in the three regimes of position measurement discussed in the text.

<table>
<thead>
<tr>
<th>regime</th>
<th>optimal feedback cooling</th>
</tr>
</thead>
<tbody>
<tr>
<td>bad measurement</td>
<td>$\delta x_r \gg \delta x_T$</td>
</tr>
<tr>
<td>classical measurement</td>
<td>$\delta x_T \gg \delta x_r \gg \frac{\delta x_Q}{\sqrt{\eta N}}$</td>
</tr>
<tr>
<td>backaction-dominated measurement</td>
<td>$\frac{\delta x_Q}{\sqrt{\eta N}} \gg \delta x_T$</td>
</tr>
</tbody>
</table>

\[
\{I_X(t), I_Y(t)\} \text{ and feedback signals } \{u_X(t), u_Y(t)\}, \text{ the energy of the oscillator is}
\]

\[
E = \frac{1}{2} \hbar \omega_m (\langle \hat{X} \rangle^2 + V_X + \langle \hat{Y} \rangle^2 + V_Y) \leq \frac{1}{2} \hbar \omega_m (V_X + V_Y). \tag{5.1}
\]

So the first observation is that the energy of the steady state under feedback cannot be less than the energy associated with the variances of the conditioned state. As we will see later, it is reasonable to presume that the feedback can be chosen such that the contribution from the conditioned mean values is negligible and so these lower bounds are reasonable estimates of the performance of the feedback. This results in the estimates of the energy of the oscillator given in table 1. We note that in the backaction-dominated regime, feedback cooling can approach the ground state energy when the quantum efficiency $\eta \simeq 1$.

(a) Optimal control

The methods of linear quadratic Gaussian control \cite{16,32,33} allow us to choose the feedback signals $u_X(t)$ and $u_Y(t)$ optimally. Suppose that we wish to minimize the average energy of the oscillator and the average power in the control signals over some time interval $[t_i, t_f]$ as follows:

\[
J(\{I_X(t), I_Y(t), u_X(t), u_Y(t)\}, \rho(t_i)) = \int_{t_i}^{t_f} dt [E(t) + q(u_X^2(t) + u_Y^2(t))]. \tag{5.2}
\]

The notation with braces is to indicate that the cost function is a functional of the measurement results over the control period and also of the control inputs over that period, as well as of the initial density matrix. The parameter $q > 0$ weights the cost of control or the importance of minimizing the average power of the control inputs relative to the energy of the oscillator. For any given measurement results, the control protocol will choose optimal control signal $u_{X,Y}^*(\{I_X(t), I_Y(t)\}, \rho(t_i))$ that minimizes $J$, given those measurement results. Finally, we wish to average this over measurement results to find the optimal cost

\[
S(\rho(t_i)) = E_{\{I_X(t), I_Y(t)\}} \inf_{\{u_X(t), u_Y(t)\}} J(\{I_X(t), I_Y(t), u_X(t), u_Y(t)\}, \rho(t_i)). \tag{5.3}
\]

In this equation, the symbol $E$ indicates an average over measurement results. Although it appears very challenging, this optimization can be solved.

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If we choose to consider the control systems for times $t_i \ll t \ll t_f$, then the gain of the optimal servo is constant and going through the optimal control calculation will result in an optimal feedback as follows:

$$u_X^* = -g\gamma(\hat{X}),$$

(5.4)

for some unitless number $g(q, \omega_m, \gamma)$ that can be calculated using the standard techniques of linear quadratic control theory [16,32,33]. It is notable that the optimal value of $g$ does not depend on the backaction noise in any way and as a result, this part of the problem is exactly the same as for the analogous classical problem. $g$ is a monotonically decreasing function of $q$ so that as the cost of control $q$ becomes small, $g$ becomes large. As a result, we can think of $g$ as an independent parameter describing the strength of the feedback control and dispense with $q$ in the following. Since $g$ relates directly to the damping of the controlled oscillator and also to the width of the mechanical resonance when the feedback is on, it has a very clear physical significance. Our parameter $g$ is essentially the same as the parameter $g$ in the experimental work of [6–8], where it was termed the ‘gain’. (They are not exactly comparable because they refer to different feedback protocols.) Because $g$ is not the gain from the measured current $I_X$ to the feedback rate $u_X$, we will not use this terminology here so as to avoid confusion. An exactly similar expression, with the same value for $g$, holds for the $Y$ quadrature.

We can interpret this control protocol as a filter of the measurement signal by reference to equation (4.1). We find that we may substitute in our choice for $u_X$ to find

$$d\langle \hat{X} \rangle = -\gamma(1 + g)\langle X \rangle \, dt - 4\eta\mu V_X \langle \hat{X} \rangle \, dt + 2\sqrt{\eta\mu} V_X dQ_X. \quad (5.5)$$

We notice that this is a low-pass filter with bandwidth $(1 + g)\gamma + 4\eta\mu V_X$. We may solve this linear stochastic differential equation and write the optimal control signal as a filtered version of the measured current. In order to do this, we should redefine the measured current so that, like $u_X$, it is a rate. So by defining $I'_X \equiv \sqrt{\eta\mu} I_X$, we find

$$u_X(t) = -2 \frac{\gamma}{1 + g} V_X \int_{t'}^t e^{-(1 + g)\gamma + 4\eta\mu V_X[(t-t')]} I'_X(t') \, dt'. \quad (5.6)$$

This is a simple low-pass filter with low-frequency gain $G \equiv -2 g V_X/(1 + g + 4\eta\mu V_X/\gamma)$ and bandwidth $(1 + g)\gamma + 4\eta\mu V_X$. We note that the gain $G$ depends on our chosen normalization of $u_X$ and $I'_X$, and may be hard to evaluate in a given experimental setup. In terms of the original measurement, this means that we have a band pass filter centred at $\omega_m$ with the appropriate $\pi/2$ phase shift to provide damping, and the stated gain and bandwidth.

The first thing to note about these formulae is that in the classical measurement regime, the gain and bandwidth of the optimal filter that we apply to the measurement signal to obtain the feedback signal $u_X$ are exactly the same as in the classical case where there is no backaction noise. As the cost of control is reduced and the optimal bandwidth $g$ increases, the gain $G$ tends to increase, and in the limit where $g \gg 1, 4\eta\mu V_X/\gamma$, we have $G \approx -2 V_X$. In the classical case, this gain is very large. In the backaction-limited regime, the conditioned quadrature variances differ from the analogous classical model without backaction.
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noise and $V_X \to 1/2\sqrt{\eta}$ rather than zero and $G \to -1/\sqrt{\eta}$. Likewise in this regime, the optimal filter has a bandwidth that is larger by $2\sqrt{\eta\mu}$ than if there were no backaction noise. In contrast to the feedback protocols usually analysed theoretically [6–8,18] in the frequency domain Langevin equation approach, this protocol is physically realizable in that it is a finite bandwidth causal filter on the measurement signal. Notice that the parameter $g$ modifies the gain and bandwidth of the feedback simultaneously; in the earlier studies [6–8], it modifies the gain in a limit of infinite bandwidth.

Finally, we wish to average over measurement results to determine the steady-state energy of the oscillator under feedback. To do this, we should solve the equation for $\langle \hat{X} \rangle$

$$d\langle \hat{X} \rangle = -\gamma(1 + g)\langle \hat{X} \rangle + 2\sqrt{\eta\mu} V_X dW_X$$

(5.7)

using standard techniques [21] to find the steady-state mean of $\langle \hat{X} \rangle^2$, which is as follows:

$$E_{(I_X(t), I_Y(t))}[(\hat{X})^2] = \frac{2\eta\mu V_X}{\gamma(1 + g)} V_X;$$

(5.8)

so we find that the optimal energy achieved by the feedback cooling is

$$E_{opt} = \hbar\omega_m V_X \left(1 + \frac{2\eta\mu V_X}{\gamma(1 + g)}\right).$$

(5.9)

The second term is the effect of the fluctuating mean value, while the first term is the lower bound resulting from the quadrature variance of the conditioned state. In the backaction-dominated regime, and when $\eta \simeq 1$, the conditioned state variance dominates this expression when

$$\mu \ll g\gamma \ll \omega_m.\quad (5.10)$$

The first condition can be read off from equation (5.9), while the second is necessary so that the rotating wave approximation we have used here is still accurate, which implies that the oscillator remains high-Q despite the extra damping introduced by the feedback. For the sake of completeness, this broadband limit where the conditioned variance dominates the feedback performance can also be reached in principle in the classical measurement regime where it requires

$$\gamma \frac{\delta x_T}{\delta x_\gamma} \ll g\gamma \ll \omega_m.\quad (5.11)$$

Again this condition requires a very high-Q oscillator because in the classical measurement regime $\delta x_T/\delta x_\gamma$ is a large number. In the classical measurement regime, we can compare the optimal energy we have obtained here to existing expressions valid in that regime [7,8]. Looking at the regime of small $g$, we find $E_{opt} \simeq k_B T/(1 + g)$ in precise agreement with [7,8]. As $g$ increases, the minimum energy given by equation (5.9) approaches a minimum value $\hbar\omega_m V_X$, and again this agrees precisely with [7,8].

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