This paper considers the problem of robust stability for a class of uncertain linear quantum systems subject to unknown perturbations in the system Hamiltonian. The case of a nominal linear quantum system is considered with quadratic perturbations to the system Hamiltonian. A robust stability condition is given in terms of a strict bounded real condition.

Keywords: quantum control; robust stability; frequency domain condition

1. Introduction

An important concept in control theory is the notion of robust stability for systems with an uncertain block that satisfies a sector-bound condition [1]. This enables a frequency domain condition for robust stability to be given. This characterization of robust stability enables robust feedback controller synthesis to be carried out using $H^\infty$ control theory [2]. The aim of this paper is to extend classical results on robust stability to the case of linear quantum systems. This is motivated by a desire to apply quantum $H^\infty$ control such as presented in James et al. [3] and Maalouf & Petersen [4] to uncertain linear quantum systems.

In recent years, a number of papers have considered the feedback control of systems whose dynamics are governed by the laws of quantum mechanics rather than classical mechanics [3–16]. In particular, Gough & James [5] and James & Gough [17] consider a framework for quantum systems defined in terms of a triple $(S, L, H)$, where $S$ is a scattering matrix, $L$ is a vector of coupling operators and $H$ is a Hamiltonian operator. The paper of James & Gough [17] then introduces notions of dissipativity and stability for this class of quantum systems. In this paper, we build on the results of James & Gough [17] to obtain robust stability results for uncertain linear quantum systems in which the quantum system Hamiltonian is decomposed as $H = H_1 + H_2$, where $H_1$ is a known nominal

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Hamiltonian and $H_2$ is a perturbation Hamiltonian, which is contained in a specified set of Hamiltonians $\mathcal{W}$. The set of perturbation Hamiltonians $\mathcal{W}$ corresponds to the set of exosystems considered in James & Gough [17].

The paper then considers the case in which the nominal Hamiltonian $H_1$ is a quadratic function of annihilation and creation operators, and the coupling operator vector is a linear function of annihilation and creation operators. This case corresponds to a nominal linear quantum system [10]. Then, stability results are obtained in terms of a frequency domain condition.

2. Quantum systems

We consider open quantum systems defined by parameters $(S, L, H)$, where $H = H_1 + H_2$ [5,17]. The corresponding generator for this quantum system is given by

$$G(X) = -i[X, H] + \mathcal{L}(X),$$

where $\mathcal{L}(X) = \frac{1}{2}L^\dagger[L, X] + \frac{1}{2}[L^\dagger, X]L$. Here, $[X, H] = XH - HX$ denotes the commutator between two operators and the notation $\dagger$ denotes the adjoint transpose of a vector of operators. Also $H = H_1 + H_2$, where $H_1$ is a self-adjoint operator on the underlying Hilbert space referred to as the nominal Hamiltonian and $H_2$ is a self-adjoint operator on the underlying Hilbert space referred to as the perturbation Hamiltonian. The triple $(S, L, H)$, along with the corresponding generators, defines the Heisenberg evolution $X(t)$ of an operator $X$ according to a quantum stochastic differential equation [17].

The problem considered involves establishing robust stability properties for an uncertain open quantum system for the case in which the perturbation Hamiltonian is contained in a given set $\mathcal{W}$. By using the notation of James & Gough [17], the set $\mathcal{W}$ defines a set of exosystems. This situation is illustrated in the block diagram shown in figure 1. The main robust stability results presented in this paper will build on the following result from James & Gough [17].

**Lemma 2.1 (see lemma 3.4 of James & Gough [17]).** Consider an open quantum system defined by $(S, L, H)$, and suppose there exists a non-negative self-adjoint operator $V$ on the underlying Hilbert space such that

$$G(V) + cV \leq \lambda,$$
where \( c > 0 \) and \( \lambda \) are real numbers. Then for any plant state, we have
\[
\langle V(t) \rangle \leq e^{-ct}\langle V \rangle + \frac{\lambda}{c}, \quad \forall \ t \geq 0.
\]
Here, \( V(t) \) denotes the Heisenberg evolution of the operator \( V \) and \( \langle \cdot \rangle \) denotes quantum expectation [17].

(a) Commutator decomposition

Given a set of non-negative self-adjoint operators \( \mathcal{P} \) and real parameters \( \gamma > 0 \), \( \delta \geq 0 \), we now define a particular set of perturbation Hamiltonians \( \mathcal{W}_1 \). This set \( \mathcal{W}_1 \) is defined in terms of the commutator decomposition
\[
[V, H_2] = [V, z^\dagger]w - w^\dagger[z, V]
\]
for \( V \in \mathcal{P} \), where \( w \) and \( z \) are vectors of operators. Here, the notation \([z, V] \) for a vector of operators \((z)\) and a scalar operator \((V)\) denotes the corresponding vector of commutators. Also, this set will be defined in terms of the sector-bound condition:
\[
w^\dagger w \leq \frac{1}{\gamma^2} z^\dagger z + \delta. \tag{2.4}
\]
Indeed, we define
\[
\mathcal{W}_1 = \{ H_2 : \exists w, \ z \text{ such that (2.4) is satisfied and (2.3) is satisfied } \forall V \in \mathcal{P} \}.
\]
By using this definition, we obtain theorem 2.2.

**Theorem 2.2.** Consider a set of non-negative self-adjoint operators \( \mathcal{P} \) and an open quantum system \((S, L, H)\), where \( H = H_1 + H_2 \) and \( H_2 \in \mathcal{W}_1 \) defined in (2.5). If there exists a \( V \in \mathcal{P} \) and real constants \( c > 0 \), \( \tilde{\lambda} \geq 0 \) such that
\[
-i[V, H_1] + \mathcal{L}(V) + [V, z^\dagger][z, V] + \frac{1}{\gamma^2} z^\dagger z + cV \leq \tilde{\lambda}, \tag{2.6}
\]
then
\[
\langle V(t) \rangle \leq e^{-ct}\langle V \rangle + \frac{\tilde{\lambda} + \delta}{c}, \quad \forall \ t \geq 0. \tag{2.7}
\]

**Proof.** Let \( V \in \mathcal{P} \) be given and consider \( \mathcal{G}(V) \) defined in (2.1). Then
\[
\mathcal{G}(V) = -i[V, H_1] + \mathcal{L}(V) - i[V, z^\dagger]w + iw^\dagger[z, V] \tag{2.8}
\]
using (2.3). Now because \( V \) is self-adjoint \([V, z^\dagger]^\dagger = [z, V] \). Therefore,
\[
0 \leq ([V, z^\dagger] - iw^\dagger)([V, z^\dagger] - iw^\dagger)^\dagger
= [V, z^\dagger][z, V] + i[V, z^\dagger]w - iw^\dagger[z, V] + w^\dagger w.
\]
Substituting this into (2.8) and using (2.4), it follows that
\[
\mathcal{G}(V) \leq -i[V, H_1] + \mathcal{L}(V) + [V, z^\dagger][z, V] + \frac{1}{\gamma^2} z^\dagger z + \delta. \tag{2.9}
\]
Hence, (2.6) implies (2.2) with \( \lambda = \tilde{\lambda} + \delta \). Then, (2.7) follows from lemma 2.1. ■
We consider a set \( \mathcal{W}_2 \) of quadratic perturbation Hamiltonians of the form

\[
H_2 = \frac{1}{2} \begin{bmatrix} \zeta^\dagger & \zeta^T \end{bmatrix} \Delta \begin{bmatrix} \zeta \\ \zeta^# \end{bmatrix},
\]

where \( \Delta \in \mathbb{C}^{2m \times 2m} \) is a Hermitian matrix of the form

\[
\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 \\
\Delta_2^# & \Delta_1^# \end{bmatrix},
\]

and \( \Delta_1 = \Delta_1^\dagger, \Delta_2 = \Delta_2^T \). Also, \( \zeta = E_1 a + E_2 a^# \). Here, \( a \) is a vector of annihilation operators on the underlying Hilbert space and \( a^# \) is the corresponding vector of creation operators. Also, in the case of matrices, the notation \( \dagger \) refers to the complex conjugate transpose of a matrix. In the case vectors of operators, the notation \( # \) refers to the vector of adjoint operators and in the case of complex matrices, this notation refers to the complex conjugate matrix.

We assume \( a \) and \( a^# \) satisfy the canonical commutation relations:

\[
\begin{bmatrix} [a, a^#] & [a^#, a] \end{bmatrix} = \begin{bmatrix} [a, a^#] & [a^#, a] \end{bmatrix}^\dagger - \begin{bmatrix} [a, a^#] & [a^#, a] \end{bmatrix}^T = J,
\]

where \( J = \begin{bmatrix} I & 0 \\
0 & -I \end{bmatrix} \) \([6,7,10]\).

The matrix \( \Delta \) is subject to the norm bound

\[
\|\Delta\| \leq \frac{2}{\gamma},
\]

where \( \| \cdot \| \) denotes the matrix-induced norm (maximum singular value). Then,

\[
\mathcal{W}_2 = \{ H_2 \text{ of the form (3.1), (3.2) such that condition (3.4) is satisfied} \}.
\]

By using this definition, we obtain lemma 3.1.

**Lemma 3.1.** For any set of self-adjoint operators \( \mathcal{P} \),

\[
\mathcal{W}_2 \subset \mathcal{W}_1.
\]

**Proof.** Given any \( H_2 \in \mathcal{W}_2 \), let

\[
w = \frac{1}{2} \begin{bmatrix} \Delta_1 & \Delta_2 \\
\Delta_2^# & \Delta_1^# \end{bmatrix} \begin{bmatrix} \zeta \\ \zeta^# \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Delta_1 \zeta + \Delta_2 \zeta^# \\
\Delta_2^# \zeta + \Delta_1^# \zeta^# \end{bmatrix}
\]

and

\[
\begin{bmatrix} \zeta \\ \zeta^# \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\
E_2^# & E_1^# \end{bmatrix} \begin{bmatrix} a \\ a^# \end{bmatrix} = E \begin{bmatrix} a \\ a^# \end{bmatrix}.
\]

Hence,

\[
H_2 = w^\dagger z = \frac{1}{2} [a^\dagger a^T] E^\dagger E \begin{bmatrix} a \\ a^# \end{bmatrix}.
\]
Then, for any $V \in P$,
\[
[V, z^\dagger]w = \frac{1}{2} \left( V\zeta^\dagger \Delta_1 \zeta + V\zeta^T \Delta_2 \zeta^\# \right) - \frac{1}{2} \left( \zeta^\dagger V\Delta_1 \zeta + \zeta^T V\Delta_2 \zeta^\# \right).
\]

Also,
\[
w^\dagger [z, V] = \frac{1}{2} \left( \zeta^\dagger \Delta_1 \zeta V + \zeta^T \Delta_2 \zeta^\# V \right) - \frac{1}{2} \left( \zeta^\dagger V\Delta_1 \zeta + \zeta^T V\Delta_2 \zeta^\# \right).
\]

Hence,
\[
[V, z^\dagger]w - w^\dagger [z, V] = \frac{1}{2} \left( V\zeta^\dagger \Delta_1 \zeta + V\zeta^T \Delta_2 \zeta^\# + V\zeta^T \Delta_1 \zeta^\# V \right)
- \frac{1}{2} \left( \zeta^\dagger \Delta_1 \zeta V + \zeta^T \Delta_2 \zeta^\# V + \zeta^\dagger \Delta_2 \zeta^\# V + \zeta^T \Delta_1 \zeta^\# V \right)
= VH_2 - H_2 V = [V, H_2]
\]
and thus (2.3) is satisfied. Also, condition (3.4) implies
\[
\frac{1}{4} [\zeta^\dagger, \zeta^T] \Delta \left[ \begin{array}{c} \zeta \\ \zeta^\# \end{array} \right] \leq \frac{1}{4} [\zeta^\dagger, \zeta^T] \left[ \begin{array}{c} \zeta \\ \zeta^\# \end{array} \right],
\]
which implies (2.4) for any $\delta \geq 0$. Hence, $H_2 \in W_1$. Therefore, $W_2 \subset W_1$.  

4. Linear nominal quantum systems

We now consider the case in which the nominal quantum system is a linear quantum system [10]. In this case, $H_1$ is of the form
\[
H_1 = \frac{1}{2} [a^\dagger, a^T] M \left[ \begin{array}{c} a \\ a^\# \end{array} \right],
\]
where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form $M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_1^T \end{bmatrix}$ and $M_1 = M_1^\dagger$, $M_2 = M_2^T$. In addition, we assume $L$ is of the form
\[
L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \left[ \begin{array}{c} a \\ a^\# \end{array} \right],
\]
where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. Also, we write
\[
\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \left[ \begin{array}{c} a \\ a^\# \end{array} \right] = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \left[ \begin{array}{c} a \\ a^\# \end{array} \right].
\]

In addition, we assume that $V$ is of the form
\[
V = [a^\dagger, a^T] P \left[ \begin{array}{c} a \\ a^\# \end{array} \right],
\]
where $P \in \mathbb{C}^{2n \times 2n}$ is a positive-definite Hermitian matrix of the form
\[
P = \begin{bmatrix} P_1 & P_2 \\ P_2^\# & P_1^\# \end{bmatrix}.
\]
Hence, we consider the set of non-negative self-adjoint operators $\mathcal{P}_1$ defined as
\[
\mathcal{P}_1 = \{V \text{ of the form (4.3) such that } P > 0 \text{ is a Hermitian matrix of the form (4.4)}\}.
\]

In the linear case, we will consider a notion of robust mean square stability.

**Definition 4.1.** An uncertain open quantum system defined by $(S, L, H)$, where $H = H_1 + H_2$ with $H_1$ of the form (4.1), $H_2 \in \mathcal{W}$, and $L$ of the form (4.2) is said to be robustly mean square stable if for any $H_2 \in \mathcal{W}$, there exist constants $c_1 > 0$, $c_2 > 0$ and $c_3 \geq 0$ such that
\[
\left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}, \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}^\dagger \right\rangle \leq c_1 e^{-c_2 t} \left\langle \begin{bmatrix} a \\ a^\# \end{bmatrix}, \begin{bmatrix} a \\ a^\# \end{bmatrix} \right\rangle + c_3 \quad \forall t \geq 0.
\]

Here, $\begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix}$ denotes the Heisenberg evolution of the vector of operators $[ a \ a^\# ]$ [17].

In order to address the issue of robust mean square stability, we first require some algebraic identities.

**Lemma 4.2.** Given $V \in \mathcal{P}_1$, $H_1$ defined as in (4.1) and $L$ defined as in (4.2), then
\[
[V, H_1] = \left[ a^\dagger a^T \right] P \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix}, \frac{1}{2} \begin{bmatrix} a^\dagger \ a^T \end{bmatrix} M \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix} \right] \right]
\]
\[
= \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix} \right] [PJM - MJP] \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix} \right].
\]

Also,
\[
\mathcal{L}(V) = \frac{1}{2} L^\dagger[V, L] + \frac{1}{2} [L^\dagger, V] L
\]
\[
= \text{Tr} \left( PJN^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} NJ \right)
\]
\[
- \frac{1}{2} \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix} \right]^\dagger (N^\dagger JNJP + PJN^\dagger JN) \left[ \begin{bmatrix} a \\ a^\# \end{bmatrix} \right].
\]

**Proof.** The proof of this result follows via a straightforward but tedious calculation using (3.3).

We now specialize the results of §3 to the case of a linear nominal system in order to obtain concrete conditions for robust mean square stability. In this case,
we use the relationship (3.6),
\[
z = \begin{bmatrix} \zeta \\ \zeta^\# \end{bmatrix} = \begin{bmatrix} E_1 & E_2^\# \\ E_2 & E_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} = E \begin{bmatrix} a \\ a^\# \end{bmatrix},
\] (4.10)
to show that the following strict bounded real condition provides a sufficient condition for robust mean square stability when \( H_2 \in \mathcal{W}_2 \):

1. The matrix
\[
F = -iJM - \frac{1}{2}JN^\dagger JN \text{ is Hurwitz};
\] (4.11)
2. \[
\|E(sI - F)^{-1}D\|_\infty < \frac{\gamma}{2},
\] (4.12)
where \( D = iJE^\dagger \).

This leads to theorem 4.3.

**Theorem 4.3.** Consider an uncertain open quantum system defined by \((S, L, H)\) such that \( H = H_1 + H_2 \), where \( H_1 \) is of the form (4.1), \( L \) is of the form (4.2) and \( H_2 \in \mathcal{W}_2 \). Furthermore, assume that the strict bounded real condition (4.11) and (4.12) is satisfied. Then, the uncertain quantum system is robustly mean square stable.

**Proof.** If the conditions of the theorem are satisfied, then it follows from the strict bounded real lemma that the matrix inequality
\[
F^\dagger P + PF + 4PJE^\dagger EJP + \frac{E^\dagger E}{\gamma^2} < 0
\] (4.13)
will have a solution \( P > 0 \) of the form (4.4) [2,4]. This matrix \( P \) defines a corresponding operator \( V \in \mathcal{P}_1 \) as in (4.3). Now it follows from lemma 4.2 that
\[
[z, V] = 2EJP \begin{bmatrix} a \\ a^\# \end{bmatrix},
\] (4.14)
where \( z \) is defined as in (3.6) and (4.10). Hence,
\[
[V, z^\dagger][z, V] = 4 \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger PJE^\dagger EJP \begin{bmatrix} a \\ a^\# \end{bmatrix}.
\]
Also,
\[
z^\dagger z = \begin{bmatrix} a \\ a^\# \end{bmatrix}^\dagger E^\dagger E \begin{bmatrix} a \\ a^\# \end{bmatrix}.
\]
Hence, by using lemma 4.2, we obtain

$$-i[V, H_1] + \mathcal{L}(V) + [V, z][z, V] + \frac{z^\dagger z}{\gamma^2} + cV$$

$$= \begin{bmatrix} a & \# \end{bmatrix}^\dagger \begin{pmatrix} F^\dagger P + PF & +4PJE^\dagger EJP + E^\dagger E \gamma^2 \\ \end{pmatrix} \begin{bmatrix} a & \# \end{bmatrix}$$

$$+ \text{Tr} \left( P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right), \quad (4.15)$$

where $F = -iJM - \frac{1}{2} JN^\dagger JN$. Therefore, it follows from (4.13) that there exists a constant $c > 0$ such that condition (2.6) will be satisfied with

$$\tilde{\lambda} = \text{Tr} \left( P J N^\dagger \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N J \right) \geq 0.$$

Hence, choosing $\delta = 0$, it follows from lemma 3.1, theorem 2.2 and $P > 0$ that

$$\left\langle \begin{bmatrix} a(t) \\ a^\#(t) \end{bmatrix} \right\rangle \leq \frac{\langle V(t) \rangle}{\lambda_{\text{min}}[P]}$$

$$\leq e^{-ct} \frac{\langle V \rangle}{\lambda_{\text{min}}[P]} + \frac{\tilde{\lambda}}{c\lambda_{\text{min}}[P]}$$

$$\leq e^{-ct} \left( \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix} \right)^\dagger \begin{bmatrix} a(0) \\ a^\#(0) \end{bmatrix} \frac{\lambda_{\text{max}}[P]}{\lambda_{\text{min}}[P]}$$

$$+ \frac{\tilde{\lambda}}{c\lambda_{\text{min}}[P]} \quad \forall t \geq 0.$$

Hence, the condition (4.6) is satisfied with $c_1 = \lambda_{\text{max}}[P]/\lambda_{\text{min}}[P] > 0$, $c_2 = c > 0$ and $c_3 = \tilde{\lambda}/c\lambda_{\text{min}}[P] \geq 0$.

**Example.** We consider an example of an open quantum system with

$$S = I, \quad H_1 = 0, \quad H_2 = \frac{1}{2}i((a^\dagger)^2 - a^2), \quad L = \sqrt{\kappa}a,$$

which corresponds to an optical parametric amplifier [18]. This defines a linear quantum system of the form considered in theorem 4.3 with $M_1 = 0$, $M_2 = 0$, $N_1 = \sqrt{\kappa}$, $N_2 = 0$, $E_1 = 1$, $E_2 = 0$, $\Delta_1 = 0$, $\Delta_2 = i$. Hence, $M = 0$, $N = \begin{bmatrix} \sqrt{\kappa} & 0 \\ 0 & \sqrt{\kappa} \end{bmatrix}$, $F = \begin{bmatrix} -\frac{\gamma}{2} & 0 \\ 0 & -\frac{\gamma}{2} \end{bmatrix}$, which is Hurwitz, $E = I$, and $D = iJ$. In this case, $\Delta^2 = I$. Hence, we can choose $\gamma = 1$ to ensure that (3.4) is satisfied and $H_2 \in \mathcal{W}_2$. Also,

$$\|E(sI - F)^{-1}D\|_\infty = \left\| \begin{bmatrix} 1 & 0 \\ s + \kappa/2 & 1 \end{bmatrix} \right\|_\infty = \frac{2}{\kappa}.$$
Hence, it follows from theorem 4.3 that this system will be mean square stable if \(2/\kappa < \frac{1}{2}\); i.e. \(\kappa > 4\).

5. Conclusions

In this paper, we have considered the problem of robust stability for uncertain quantum systems with quadratic perturbations to the system Hamiltonian. The final stability result obtained is expressed in terms of a strict bounded real condition. Future research will be directed towards considering the robust stability of uncertain linear quantum systems with uncertainty in the coupling operator as well as in the Hamiltonian operator.

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