The mechanically based non-local elasticity: an overview of main results and future challenges

Mario Di Paola¹, Giuseppe Failla², Antonina Pirrotta¹, Alba Sofi² and Massimiliano Zingales¹,³,⁴

¹Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM), Viale delle Scienze, ed. 8, 90128 Palermo, Italy
²Dipartimento di Ingegneria Civile, dell’Energia, dell’Ambiente, e dei Materiali (DICEAM), Via Graziella, Loc. Feo di Vito, Reggio Calabria, Italy
³Laboratorio di BioMeccanica e BioMateriali (BM)²-Lab, Centro Mediterraneo di Biotecnologie Avanzate e Salute Umana, Viale delle Scienze, ed. 18, Palermo, Italy
⁴Sezione di BioMeccanica e BioMateriali, Istituto Euro-mediterraneo di Scienza e Tecnologia (I.E.Me.S.T.), Via Emerico Amari n. 123, 90128 Palermo, Italy

The mechanically based non-local elasticity has been used, recently, in wider and wider engineering applications involving small-size devices and/or materials with marked microstructures. The key feature of the model involves the presence of non-local effects as additional body forces acting on material masses and depending on their relative displacements. An overview of the main results of the theory is reported in this paper.

1. Introduction

The common description of matter aggregates in terms of a continuum distribution of mass particles has been introduced extensively in physics and engineering contexts starting from some pioneering papers by French mathematicians of the nineteenth century [1,2]. In these studies, the discrete nature of the matter in terms of mass particles connected by elastic springs has been replaced by a continuous elastic model whose elastic constants depend on the physical parameters of the spring elements [3]. Continuum mechanics models
are well-established tools to describe, mathematically, the mechanical and thermodynamical behaviour of several, large-scale, engineering structures such as bridges, dams and buildings, with some beautiful closed-form solutions of the governing equations. However, it is nowadays accepted that the presence of multi-scale material aggregates is the main source of several unpredicted phenomena observed in mechanical tests. Indeed, dispersion of elastic waves, edge effects, stress-tip concentrations, shear bands as well as void nucleation and other macroscopically observed aspects come from nano/microscale phenomena that, progressively, propagate in material bulk. Some of them, such as the dispersion of elastic waves in unbounded linear elastic solids as well as the stress concentration at apexes of cracks, may not be predicted with the aid of classical local elasticity. Indeed, dispersion of elastic waves involves nonlinear frequency–wavenumber relations, whereas local elastic prediction involves a linear relation with inherent constant speed of travelling waves; similarly, local elasticity predicts unbounded stress at crack tips that do not correspond to the physical presence of cohesive zones around the crack and, therefore, to finite value of the stress. These considerations led several mathematicians, at the beginning of the twentieth century, towards the introduction of additional kinematic degrees of freedom of rigid bodies to describe the arrangement of material microstructures \[4,5\]. A more sophisticated model accounting for the strain field was provided in the mid-twentieth century \[6,7\]. In other words, more recent theories of small-scale inhomogeneity have been accounted for introducing structured deformations \[8,9\] accounting for slips and disorders at molecular level \[10\]. These approaches may be classified in the so-called enriched continuum models that are different, from a mechanical perspective, from the ‘non-local’ continuum theories. Indeed, the introduction of kinematically consistent degrees of freedom of the inner microstructures such as rotation of the axis triedron \[4\] and strain and shear of the microstructures \[6\] requires a proper definition of the conjugated static variables that may be obtained, consistently, by means of the virtual work principle. Non-local continuum theories rely, instead, on the introduction of additional contributions in terms of gradients or integrals of the strain field, to account for the material inner microstructure in stress–strain relations. In the former case, the so-called weak non-local models are defined \[11–13\]; in the latter case, the so-called strong non-local models have been reported in the literature \[14–16\]. These models present several drawbacks to fulfil the Neumann boundary conditions that involve (i) gradients of the strain field without an evident mechanical description (weak non-local models) and (ii) geometrical constraints on the functional class of the decaying function (strong non-local models).

A different contribution, bridging the views of enriched continuum and non-local models, has recently been provided by the so-defined mechanically based (MB) model of non-local elasticity. The MB model does not involve any particular care to deal with Neumann boundary conditions, and kinematic and static variables introduced by the MB model are related by the virtual work principle. The theory, basically, aims to introduce non-local interactions among different locations of the body, in terms of elastic, central long-range body forces proportional to the interacting volumes or masses for non-homogeneous solids \[17\]. Under these circumstances, non-local effects are captured, in the equilibrium equations, by an integral term that is the resultant of all long-range interactions. The integral contribution depends on the relative displacements among the interacting locations and, hence, the corresponding elastic equilibrium problem is ruled by a set of integro-differential equations in terms of the displacement field that may be also framed in the context of fractional calculus \[17,18\]. MB theory has proved to be thermodynamically consistent \[19,20\] predicting, correctly, increments of the strain field at a border of a tensile bar \[21\] as well as the dispersion of travelling waves predicted by the Born–Von Karman lattice theory \[22\], and several other phenomena that will be reported in the course of the paper. In §2, the fundamentals of the MB theory of non-local elasticity will be introduced assessing thermodynamic consistency and related variational principles. Section 3 is devoted to a particularization of the theory to a Timoshenko beam model, whereas the use of fractional differential calculus in the analysis of non-local problems is reported in §4.
2. The mechanically based non-local elasticity: fundamental remarks

In this section, the fundamental concepts at the basis of the so-called MB non-local elasticity will be shortly outlined. The main idea behind the theory may be expressed by introducing a three-dimensional, linearly elastic, homogeneous and isotropic solid of volume $V$ that is referred to via external coordinate systems, so that the placement vector of a generic location reads $O_\text{X} \equiv (x_1, x_2, x_3)$. The boundary surface of volume $V$ is dubbed $S = S_f \cup S_c$, where $S_f$ is the free surface of the body that undergoes external tractions $p_n(x)$, and $S_c$ is the constrained surface where prescribed displacement field is assigned $u(x) = \bar{u}(x)$ (figure 1a). The key idea of the MB non-local model is that non-local effects are represented by long-range body forces mutually exerted by elementary volume elements that are modelled as central interactions. Long-range central interactions are proportional to the product of interacting volumes (or masses), to a distance-decaying function as well as to their relative displacement along the directional vector $x - \xi$. The mathematical representation of the $k$th component of the long-range interaction, namely $d f_k(x)$ exerted at location $x$ by mass at location $\xi$, reads

$$d f_k = q_k(x, \xi) dV = g_{kj}(x, \xi) \eta_j(x, \xi) dV$$

with $g_{kj}(x, \xi) = g(x, \xi) r_k(x, \xi) r_j(x, \xi)$ and $r_k$ the $k$th component of the directional vector associated to vector $x - \xi$ and $\eta_j(x, \xi) = u_j(\xi) - u_j(x)$ is the $j$th component of the relative displacement vector among the centroids of volume elements at locations $x$ and $\xi$, respectively. Thermodynamic consistency of the MB model requires that $g(x, \xi) \geq 0$ and $g(x, \xi) = g(\xi, x)$. At this stage, it is necessary to make some further comments about the functional class of the distance-decaying function $g(x, \xi)$. Such a function has been introduced, on a mechanical basis, assuming that the long-range central forces counteract the relative displacements between non-adjacent elements. On this basis, therefore, $g(x, \xi)$ has been taken as a symmetric and strictly positive function of the distance between two interacting volumes. Further, if the material is isotropic, then $g(x, \xi)$

Figure 1. (a) Homogeneous three-dimensional solid with long-range interactions. (b) Volume element with long-range resultant. (Online version in colour.)
depends on the topological distance among the interacting volumes, only. The requirement $g(x, \xi) \geq 0$ in the whole solid domain is mandatory as it is related to the material stability criterion in the presence of long-range interactions as has been assessed in previous studies [19,20]. It should be remarked that the definition of the long-range interactions is provided by the first equality in equation (2.1), $q_k(x, \xi)$ denoting the specific long-range interactions among locations $x$ and $\xi$ of the body. The last equality in equation (2.1) specifies the constitutive prescriptions of the long-range interactions that depend on the relative displacements along the direction of the unitary vector $r(x, \xi)$. In this framework, the constitutive prescriptions of the long-range central interactions do not depend on spatial gradients of the displacement field but only on the relative displacement along the vector $r(x, \xi)$. Therefore, according to Noll’s theory of simple material [23], the constitutive prescription of elastic long-range interactions shall be reported in the form $q = r f(\eta_1, \eta_2, \eta_3)$ with $f$ a proper $C^0$ continuous function relating the long-range central interactions to the relative displacements of the interacting volumes.

The long-range interaction force [24,25] at location $x$ is obtained integrating equation (2.1) over the volume of the body yielding

$$f_k(x) = \int_V q_k(x, \xi) \, dV = \int_V g_k(x, \xi) \eta_j(x, \xi) \, dV. \tag{2.2}$$

The MB model of non-local elasticity relies on the introduction of long-range interactions that are, in couple, self-equilibrated and central interactions and, as a consequence, the overall resultant of the long-range interactions, in the whole body, vanishes. We may conclude that long-range interactions may be used to describe the state of residual stress in the whole body arising from the presence of self-equilibrated internal long-range interactions. Additionally, second-order, distance-decaying tensor $g_{jk}(x, \xi)$ may be related with a mesoscopic parameter that describes the internal length scale of the material. Such a parameter is used to describe the maximum distance of significant long-range interactions, and, as far as it vanishes, the long-range interactions in equation (2.2) vanish in the whole body recovering the local elastic solution as reported in §3 for a Timoshenko-based elastic beam.

The governing equations of the elastic problem of the MB non-local elasticity are expressed, as customary in mechanics, in terms of the balance, constitutive and kinematic relations that read as follows.

(i) Balance of linear momentum among Cauchy contact stress $\sigma_{kij}^{(l)}$, external body forces and unbalanced resultant of the long-range interactions that read (figure 1b)

$$\sigma_{kij}^{(l)}(x) = - \bar{b}_k(x) - f_k(x). \tag{2.3}$$

(ii) Kinematic restrictions among the strain/displacement field variables:

$$\varepsilon_{kj}(x) = \frac{1}{2} [u_{k,j}(x) + u_{j,k}(x)] \tag{2.4a}$$

and

$$\eta_k(x, \xi) = u_k(\xi) - u_k(x). \tag{2.4b}$$

As concerns the kinematic restrictions, it is seen that equation (2.4a) coincides with the classical small strain/small displacement equation of the Cauchy continuum, whereas equation (2.4b) involves a new state descriptor represented by the relative displacement among locations $x$ and $\xi$. It is interesting to note that the neutral change of state with respect to $\eta_k(x, \xi)$ is any rigid-body motion of the three-dimensional continuum. While this is evident for any rigid-body translation ($\eta_k(x) = 0$) it may be proved, with simple arguments, for infinitesimal rigid rotation $\theta$ about any arbitrary axis. Indeed, the absolute rigid displacements of two arbitrary points $P$ and $Q$ of the solid read, for Chasles’ theorem, $\theta \times OP$ and $\theta \times OQ$, where $O$ is the pole of the rotation axis. The relative displacement vector among points $P$ and $Q$ reads $\theta \times (OP - OQ) = \theta \times PQ$. 


It is immediately seen that such a relative displacement has no component along the connection points $P$ and $Q$, thus leading one to conclude that no long-range interactions arise for rigid body motion. It has been proved, indeed, that considering $\eta_k(x, \xi)$ as generalized deformation related to the displacement field by equation (2.4), and dual of the long-range forces $q_k(x, \xi)$, the equilibrium equations (2.3) are recovered by the virtual displacement principle. The kinematic restrictions in equation (2.4) may, instead, be derived by means of the principle of virtual forces, thus establishing a consistent static–kinematic duality between the local terms $\sigma_{kj}^{(l)}(x, \xi)$, $\epsilon_{kj}(x, \xi)$ and the non-local terms $\eta_k(x), q_k(x)$ [24].

(iii) Constitutive equations among the stress/strain and the long-range interactions/relative displacements field:

$$\sigma_{kj}^{(l)}(x) = 2\mu^*\epsilon_{kj}(x) + \delta_{kj}\lambda^*\epsilon_{kk}(x) \quad (2.5a)$$

and

$$q_k(x, \xi) = g_{kj}(x, \xi)\eta_j(x, \xi). \quad (2.5b)$$

The field equations of the elastic problem are supplied by the boundary conditions that involve the displacement and the stress field on the body surface as $u_k(x) = \bar{u}_k(x); x \in S_c$ and $\sigma_{kj}^{(l)}(x)\eta_j = \bar{p}_{nk}(x); x \in S_t$. The non-local elastic constants of the material, $\lambda^*$ and $\mu^*$, are related to the material elastic constants, namely $\lambda$ and $\mu$, by means of a proportionality coefficient as $\lambda^* = \alpha_1\lambda$ and $\mu^* = \alpha_1\mu$, with $0 \leq \alpha_1 \leq 1$. Observation of equation (2.5) shows that the same coefficient $\alpha_1$ is assumed for $\lambda^*$ and $\mu^*$ as related to $\lambda$ and $\mu$, as in the proposed formulation $\alpha_1$ is introduced as a weighting coefficient of the non-local effects, in formal analogy with other non-local theories where the material is conceived as a two-phase material. No substantial restrictions hold, however, in the assumption of two independent weighting coefficients, namely $\alpha_1$ and $\alpha_2$, for $\lambda$ and $\mu$ to achieve a better representation of experimental evidences. It may be observed that the equilibrium equation of the Cauchy tetrahedron at the boundary does not contain the contribution of the body forces $\bar{b}_k(x)$ and the long-range resultants $f_k(x)$ because it is an infinitesimal of the third order and it may be neglected with respect to lower-order infinitesimals as $-\sigma_{kj}^{(l)}(x)\eta_j(x, \xi)\eta_j(x, \xi) - \bar{p}_{nk}(x)\eta_j(x, \xi)$. For this reason, the Neumann boundary conditions are expressed by the same relations of classical local theories. This is a very remarkable consideration, because the MB non-local model is not affected by integrals or differentials of the strain field in the body that, instead, affect weak and strong non-local models.

The solution of the elastic equilibrium problem may be obtained, with the aid of the stiffness method, in terms of the displacement field with $x \in V$ yielding the Navier equations of the problem in integro-differential form:

$$\mu^*\nabla^2 u_k(x) + (\lambda^* + \mu^*)u_{j,k}(x) + \int_V g_{kj}(x, \xi)\eta_j(x, \xi)\,dV = -\bar{b}_k(x), \quad (2.6)$$

where $\nabla^2[\bullet] = [\bullet]_{jj}$ is the Laplace operator, and the relevant boundary conditions read

$$\begin{cases} 
\mu^*(u_{k,j} + u_{j,k})n_j + \lambda^*u_{j,j} = \bar{p}_{nk} & \text{on } S_t \\
u_k(x) = \bar{u}_k(x) & \text{on } S_c.
\end{cases} \quad (2.7)$$

The MB model of non-local elasticity may be also formulated in a variational framework. The starting equation to this purpose is the following work identity:

$$\int_V \bar{b}_k(x)u_k(x)\,dV + \int_V f_k(x)u_k(x)\,dV + \int_{S_t} \bar{p}_{nk}(x)u_k(x)\,dS_t + \int_{S_c} \bar{p}_{nk}(x)\bar{u}_k(x)\,dS_c$$

$$= \int_V \sigma_{kj}^{(l)}(x)\epsilon_{kj}(x)\,dV. \quad (2.8)$$
It may be readily proved that equation (2.8) holds for any arbitrary set of external forces \( \bar{b}_k(x), \bar{p}_{nk}(x) \) in equilibrium with the local stress \( \sigma^{(l)}_{ij}(x) \) and the long-range forces \( f_k(x) \) according to equation (2.3), and any arbitrary set of small strains \( \epsilon_{ij} \) related to the displacements \( u_k(x) \) by equation (2.4) [24]. The expression reported in equation (2.8) may be further reverted to the following identity:

\[
\int_V \bar{b}_k(x)u_k(x) \, dV + \int_{S_1} \bar{p}_{nk}(x)u_k(x) \, dS_t + \int_{S_2} \bar{p}_{nk}(x)\bar{u}_k(x) \, dS_c = \int_V \sigma^{(l)}_{ij}(x)\epsilon_{ij}(x) \, dV + \frac{1}{2} \int_V \int_V q_{kj}(x, \xi)\eta_{kj}(x, \xi) \, dV \, dV
\]  

(2.9)

because the following relation holds true:

\[
\int_V f_k(x)u_k(x) \, dV = -\frac{1}{2} \int_V \int_V q_{kj}(x, \xi)\eta_{kj}(x, \xi) \, dV \, dV
\]  

(2.10)

that may be proved after some straightforward manipulations as

\[
\int_V f_k(x)u_k(x) \, dV = \int_V \int_V q_{kj}(x, \xi)u_k(x) \, dV \, dV
\]  

\[
= -\int_V \int_V g_{kj}(x, \xi)\eta_j(x, \xi)u_k(x) \, dV \, dV
\]  

(2.11)

and, accounting for the symmetry of the double and symmetric tensor \( g_{jk}(x, \xi) \) that reads \( g_{kj}(x, \xi) = g_{jk}(\xi, x) \), the following positions hold true:

\[
\int_V \int_V g_{kj}(x, \xi)\eta_j(x, \xi)u_k(x) \, dV \, dV = -\int_V \int_V g_{kj}(x, \xi)\eta_j(x, \xi)u_k(\xi) \, dV \, dV
\]  

(2.12a)

and

\[
\int_V f_k(x)u_k(x) \, dV = \frac{1}{2} \int_V \int_V g_{kj}(x, \xi)\eta_j(x, \xi)[u_k(x) - u_k(\xi)] \, dV \, dV
\]

\[
= -\frac{1}{2} \int_V \int_V q_{kj}(x, \xi)\eta_{kj}(x, \xi) \, dV \, dV.
\]

(2.12b)

The mechanical and kinematic variables involved in equation (2.10) correspond, indeed, to the principle of virtual work of the non-local three-dimensional continuum, and establish the mechanical consistency of the MB non-local model. It can be concluded that the MB non-local elasticity theory involves, under the assumption of small strain/small displacements, a convex elastic potential energy functional whose Euler–Lagrange (EL) equations are given as equations (2.6) and (2.7). The elastic potential energy of the whole continuum can be expressed, by a volume integration, as

\[
\Phi(\epsilon_{ij}, \eta_k) = \Phi^{(l)}(\epsilon_{ij}) + \Phi^{(nl)}(\eta_k)
\]

\[
= \int_V \Phi^{(l)}(\epsilon_{ij}(x)) \, dV + \frac{1}{2} \int_V \int_V \Phi^{(nl)}(\eta_k(x, \xi)) \, dV \, dV,
\]

(2.13)

where \( \epsilon_{ij} \) and \( \eta_k(x, \xi) = u_k(\xi) - u_k(x) \) define an arbitrary set of kinematically admissible functions satisfying the kinematic restrictions reported in equations (2.4) and (2.5) along with the kinematic boundary conditions reported in equation (2.7). Further kernels of volume integrals in equation (2.13) are the specific elastic potential energies associated with the strain field and the relative displacement field, defined, respectively, as

\[
\Phi^{(l)}(\epsilon_{ij}(x)) = \mu^* \epsilon_{ij} \epsilon_{ij} + \frac{\lambda^*}{2} \delta_{ij} \delta_{ij} \]

(2.14a)

and

\[
\Phi^{(nl)}(\eta_k(x, \xi)) = \frac{1}{2} g_{jk}(x, \xi)\eta_j(x, \xi)\eta_k(x, \xi).
\]

(2.14b)
It can be readily seen that the total elastic potential energy in equation (2.13) can be obtained from the internal work in equation (2.10), as specified for an arbitrary set of kinematically admissible functions \( \epsilon_{kj}(x) \) and \( \eta_k(x, \xi) \) based on Clapeyron’s theorem of linearly elastic continua.

The total potential energy stored in the three-dimensional solid, \( \Pi(u_k, \epsilon_{kj}, \eta_k) = \Phi(\eta_k, \epsilon_{kj}) + P(u_k) \), where \( P(u_k) \) is the potential energy associated with the conservative fields \( b_k(x) \) and \( P_{nk}(x) \), attains its minimum at the solution of the elastic equilibrium problem [25]. Further, the EL equations as well as the mechanical boundary conditions associated with the total potential energy \( \Pi(u_k, \epsilon_{kj}, \eta_k) \) coincide with equation (2.6) in \( V \) and with equation (2.7) on \( S_k \), thus proving the mathematical consistency of the proposed model of non-local interactions. Similar considerations hold true also for the total complementary energy functional as reported in previous papers [24] and they have not been included for brevity. An overview of samples involved in a two-dimensional plane elasticity problem is reported in figure 2 with exponentially decaying long-range interactions as \( g(x, \xi) = C \exp(-|x - \xi|/l_0) \), where \( l_0 \) is a material parameter describing the internal length scale of the material, and \( C \) is a proper force coefficient. Observation of strain fields reported in figure 2a,b shows that for uniform tensile load in a plane elasticity problem a very strong edge effect is predicted by the MB non-local elasticity theory. Such an effect, involving concentrations of strength at the borders of the domain, does not fulfill the prediction of classical, local, elasticity yielding uniform strain field in the solid domain. Prediction of the MB non-local model is, instead, in good qualitative agreement with experimental tests on small dimension specimens in which non-uniform strains are observed as a result of uniform strains [26–28]. Then, it may be observed that, as the internal length \( l_0 \to 0 \), the non-local interactions vanish all over the body, and the local elasticity theory is recovered. Such a consideration is valid as far as the functional class of the distance-decaying function is monotonically decreasing with \( l_0 \) and it possesses a vertical asymptote as \( l_0 \to 0 \). In recent papers, some closed-form solutions for the MB non-local elasticity model obtained for specific functional class of the kernel have been obtained in unbounded solids [29].

The variational formulation of the MB non-local elasticity has also been used to formulate an MB non-local finite-element method (NL-FEM) [30]. In this regard, we observe that the MB NL-FEM involves local and non-local stiffness matrices obtained, respectively, from the local and non-local elastic potential energy of the body and they are symmetric and positive definite as convex potential energy is involved. As a consequence the advantages of the NL-FEM obtained within the MB non-local elasticity rely on the same reliability and robustness of the classical, displacement-based, local FEMs.
3. The mechanically based Timoshenko beam model

The MB non-local elastic model has been used, also, to capture the behaviour of small-size beams, such as nano-beams and nano-rods, as appropriate kinematic assumptions are introduced in the three-dimensional elastic problem discussed in §§2 and 3. In this regard, we consider a straight beam of length $L$ with uniform cross section $A$ as shown in figure 3, referred to a rectangular Cartesian coordinate system $(O, x_1, x_2, x_3)$ whose $x_3$-axis coincides with the centroidal axis of the undeformed beam, whereas the $x_2$- and the $x_3$-axes are principal axes of cross section $A$, respectively. Let us denote with $n(x_3, t) = \tilde{p}_3(x_3, t)$ and $\tilde{p}_2(x_3, t)$, respectively, the time-varying external body forces per unit length acting along the $x_3$-axis of the beam. In the context of the classical Timoshenko beam theory, the displacement field is described as $u_1(x_3, t) = 0$, $u_2(x_3, t) = v(x_3, t)$ and $u_3(x_3, t) = u(x_3, t) - x_2\varphi(x_3, t)$, where $u(x_3, t)$ and $v(x_3, t)$ are the axial and the transverse displacement of the beam axis along the $x_3$ and $x_2$ directions, respectively, whereas $\varphi(x_3, t)$ denotes the rotation of the cross section about the $x_1$-axis (positive if clockwise). The generalized force–strain relations read $N^{(0)}(x_3, t) = E^*Ae(x_3, t)$, $M^{(0)}(x_3, t) = E^*I\chi(x_3, t)$, $T^{(0)}(x_3, t) = K_sG^*A\gamma(x_3, t)$, where $e(x_3, t) = \partial u(x_3, t)/\partial x_3$, $\chi(x_3, t) = -\varphi(x_3, t)/\partial x_3$ and $\gamma(x_3, t) = \partial v(x_3, t)/\partial x_3 - \varphi(x_3, t)$ are the generalized axial strain, bending curvature and shear strain, whereas $N^{(0)}(x_3, t)$, $T^{(0)}(x_3, t)$ and $M^{(0)}(x_3, t)$ are the axial force, the shear force and the bending moment of the classical local beam theory (figure 3). $J$ is the moment of inertia about the $x_1$ axis and $K_s$ is the shear correction factor. The elastic moduli $E^* = \alpha_1 E$ and $G^* = \alpha_1 G$ are the reduced longitudinal and shear elastic moduli of the material measured in the presence of long-range interactions. Based on the displacement field of the Timoshenko kinematic model, the kinetic energy of the beam reads

$$K = \frac{1}{2} \int_V \rho \left[ \left( \frac{\partial u}{\partial t} - x_2 \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 \right] dV, \quad (3.1)$$

where $\rho$ denotes the mass density of the body, herein assumed time-independent. As expressed in §2, the elastic potential energy functional of the three-dimensional body involves, besides the classical local terms, additional contributions owing to the presence of long-range interactions. Upon the introduction of the kinematic model, the elastic potential energy functional reads

$$\Phi(e, \chi, \gamma, \eta_3, \eta_2) = \frac{1}{2} \int_0^L \left[ E^*Ae^2 + E^*I\chi^2 + K_sG^*A\gamma^2 \right] dx + \frac{1}{4} \int_V [\eta_3\eta_3 + \eta_2\eta_2] dV dV, \quad (3.2)$$

where the relative displacements $\eta_2$ and $\eta_3$ are expressed in terms of the kinematic variables of the model as $\eta_2 = [u(x_3, t) - u(x_3, t)] - \xi_2\varphi(x_3, t) - x_2\varphi(x_3, t)$ and $\eta_3 = v(x_3, t) - v(x_3, t)$. The associated, specific, long-range interactions in equation (3.2) may be expressed substituting the kinematic model in equation (2.1). Taking into account equations (3.1) and (3.2) and using Hamilton’s principle, the following EL equations are obtained:

$$E^*A \frac{\partial^2 u(x_3, t)}{\partial x_3^2} + R_3(x_3, t) + F_3(x_3, t) = m \frac{\partial^2 u(x_3, t)}{\partial t^2}, \quad (3.3a)$$

$$K_sG^*A \left[ \frac{\partial^2 v(x_3, t)}{\partial x_3^2} - \frac{\partial \varphi(x_3, t)}{\partial x_3} \right] + R_2(x_3, t) + F_2(x_3, t) = m \frac{\partial^2 v(x_3, t)}{\partial t^2} \quad (3.3b)$$

and

$$E^*I \frac{\partial^2 \varphi(x_3, t)}{\partial x_3^2} + K_sG^*A \left[ \frac{\partial v(x_3, t)}{\partial x_3} - \varphi(x_3, t) \right] + R_1(x_3, t) = I_\rho \frac{\partial^2 \varphi(x_3, t)}{\partial t^2}, \quad (3.3c)$$

where $I_\rho = \int_A \rho x_2^2 dA$ and $R_1(x_3, t)$, $R_2(x_3, t)$, $R_3(x_3, t)$ are the resultants per unit length of the long-range forces (figure 3 and see [31,32] for details). As from previous considerations, the boundary conditions of the model involve, only, the local generalized forces and/or the prescribed displacements at the borders as $N^{(0)}(x_3, t) = \mp N_i$, $T^{(0)}(x_3, t) = \mp T_i$ and $M^{(0)}(x_3, t) = \pm M_i$ or, alternatively, the prescribed kinematic field reads $u(x_3, t) = u_i(t)$, $v(x_3, t) = v_i(t)$ and $\varphi(x_3, t) = \psi_i(t)$ with $u_i(t)$, $v_i(t)$ and $\psi_i(t)$ ($i = 0, L$) denoting prescribed displacements and rotations at the ends of the beam.
The MB Timoshenko beam model, governed by equation (3.3), along with the boundary conditions, exhibits non-local effects for general loading conditions [31,32] and not necessarily for distributed loads only, thus overcoming the paradox encountered in earlier studies [33–38]. The MB Timoshenko beam model is very versatile because it may reproduce a variety of material behaviours, either stiffer or softer than the local one, by proper selection of the physical parameters and the internal length scale of the material $l_0$ governing the maximum distance beyond which the long-range forces become negligible. As an example, figure 4 displays the first three natural frequencies (figure 4a) and the corresponding vibration mode shapes (figure 4b) of a cantilever non-local Timoshenko beam obtained setting $\alpha_1 = 1$ and assuming an exponential attenuation function $g(x, \xi) = C \exp(-|x - \xi|/l_0)$, where $C$ is a material-dependent constant [39]. The presence of long-range interactions involves a stiffer behaviour of the cantilever Timoshenko beam as can be inferred from the ratio of natural frequencies $\omega^{(nl)}/\omega^{(l)} \geq 1$ (figure 4a). Indeed, the long-range interactions introduce additional stiffness being modelled as restoring forces owing to a relative motion between volume elements. Furthermore, as expected, figure 4 shows that the proposed non-local solution becomes progressively stiffer for larger values of the internal length $l_0$ that correspond to an increasing number of mutually interacting volume elements.
and that it tends to the classical local solution as $l_0$ decreases, because the contribution of the non-local terms progressively vanishes. Finally, it can be observed that the mode shapes (figure 4(b)) are influenced to some extent by non-local effects owing to the asymmetry of the boundary conditions.

### 4. The mechanically based fractional non-local elasticity

In principle, the formulation of all strong non-local theories is obtained considering that the kernel function accounting for non-local stress–strain interactions is any symmetric and strictly positive decaying function of the interdistance among interacting elements. Usually, different kernels have been used in the scientific literature such as exponential, Gaussian, Mexican hat and other decaying functions. However, in nature, long-range interactions such as van der Waals, dipole–dipole or Weber electrostatic forces decay with power laws. Such consideration has been used in strong non-local elastic models, introducing a power-law kernel in the convolution integral involving the strain field in the body, yielding the fractional model of Eringen non-local elasticity [40].

In the context of the MB model of non-local elasticity, the power-law decay of the long-range interactions yields, instead, a formulation in terms of the modern mathematical real-order (fractional) differential calculus. Basically, fractional calculus may be considered as the generalization of integer-order integrals and derivatives of differential calculus to the case of differintegration with real order [41]. Fractional-order differential calculus has been used to model the multi-scale features of time-scale hereditariness of complex materials at critical state [42], as well as the anomalous damping in dynamical systems [43, 44]. Fractional calculus has also been introduced in the context of advanced subgrade models [20, 45] and in the field of anomalous heat transfer [46]. Fractional-order operators of real-valued functions $f[\bullet]$ are defined as convolution integrals and/or derivatives of convolutions among functions $f[\bullet]$ and power-law kernels $x^{1-\alpha}$ with $0 \leq \alpha \leq 1$ as [41]

\[
(l^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(\xi) (x - \xi)^{1-\alpha} d\xi \quad (4.1a)
\]

and

\[
(D^\alpha f)(x) = \frac{d}{dx} (l^{1-\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} f(\xi) (x - \xi)^{\alpha} d\xi \quad (4.1b)
\]

for left-side fractional derivatives and for the right-side definitions:

\[
(l^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(\xi) (\xi - x)^{1-\alpha} d\xi \quad (4.2a)
\]

and

\[
(D^\alpha f)(x) = \frac{d}{dx} (l^{1-\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} f(\xi) (\xi - x)^{\alpha} d\xi. \quad (4.2b)
\]

The definition of fractional-order integrals and derivatives introduced in equations (4.1) and (4.2) follows the original formulation traced back to the paper by Riemann and, therefore, defined as Riemann–Liouville fractional integrals and derivatives. Additionally, an alternative form of fractional derivatives, fully equivalent for $C_0$-continuous functions to equation (4.1), is the Marchaud fractional derivatives that are defined, for left- and right-side fractional-order derivatives as

\[
(D^\alpha_{-\infty} f)(x) \overset{\text{def}}{=} \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(\xi) - f(x)}{(x - \xi)^{1+\alpha}} d\xi = (D^\alpha_{-\infty} f)(x) \quad (4.3a)
\]

and

\[
(D^\alpha_f)(x) \overset{\text{def}}{=} \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{\infty} \frac{f(\xi) - f(x)}{(\xi - x)^{1+\alpha}} d\xi = (D^\alpha_f)(x). \quad (4.3b)
\]
The similarities of Marchaud fractional derivatives with the integral operators involved in non-local mechanics led to the formulation of the fractional MB non-local mechanics that involves for instance, in the one-dimensional axial problem, the integral parts of the Marchaud fractional derivatives on finite supports that read

\[
(D_0^\alpha f)(x) = \frac{f(x)}{\Gamma(1-\alpha)x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{f(\xi) - f(x)}{(x-\xi)^{1+\alpha}} d\xi
\]

and

\[
(D_L^\alpha f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(L-x)^\alpha} - \frac{\alpha}{\Gamma(1-\alpha)} \int_x^L \frac{f(\xi) - f(x)}{(\xi-x)^{1+\alpha}} d\xi
\]

As soon as we assume that in the one-dimensional axially loaded elastic solid the kernel is in the form as \(g(|x-\xi|) = (\alpha/\Gamma(1-\alpha))|x-\xi|^{1+\alpha}\) (\(\alpha \geq 0\)), then the equilibrium equation is written as

\[
\frac{d^2 u_3}{dx^2} - \frac{c_0 A}{E} (D_0^\alpha u_3)(x_3) + (D_L^\alpha u_3)(x_3) = -\frac{f_3(x_3)}{EA},
\]

where \(u_3(x_3)\) is the displacement field \(E\) and \(A\) and \(L\) are the modulus of elasticity of the cross section and total length, respectively. It is to be remarked that for an indefinite bar equation (4.4) reverts to the Riemann–Liouville fractional derivative [17]. Additional considerations may be withdrawn from the one-dimensional representation of the MB non-local elastic model as we introduce a discretization grid of the abscissa with step \(\Delta x_3\) yielding a set of \(n+1\) abscissas \(x_{3j} = \Delta x_3(j-1)\) with \(j = 1, 2, \ldots, n+1\) yielding a discrete form of the fractional differential equation as

\[
\frac{E^* A}{\Delta x_3} \sum_{k=1}^{n+1} \frac{(u_{3j} - u_{3k})}{|x_{3j} - x_{3k}|^{\alpha+1}} + n(x_3)\Delta x_3 = 0,
\]

that is, assuming \(K^{(l)} = E^* A/\Delta x_3; K^{(nl)}_{ij} = c_0 A^2 (\Delta x_3)^2 / \Gamma(1-\alpha)|x_{3j} - x_{3k}|^{\alpha+1}\) the equilibrium equation of a one-dimensional lattice model with next-to-the-nearest-next interactions as reported, as an illustration, for \(n = 4\) in figure 5 assuming a power-law decay of long-range interactions. Dirichlet and Neumann boundary conditions involve, respectively, only the axial displacement field of the body as \(u_3(0) = u_{30}\) and \(u_3(L) = u_{3L}\) or, as an alternative, the local Cauchy
stress at the borders that must equate the applied external load, denoted $F_0$ and $F_L$, respectively, as $A\sigma^{(0)}(0) = -F_0$ and $A\sigma^{(0)}(L) = F_L$. Numerical samples of the results provided by the MB non-local elastic models with fractional long-range decay may be observed in figure 6a,b reporting, respectively, the axial displacement and the strain field of an axially loaded elastic bar. The effects introduced with the MB non-local theory may be observed in the non-uniform axial strain field with higher gradients of the strain at the borders with respect to the core domains. Accordingly, the displacement field is no longer linear but it undergoes displacement localization at the boundaries. The mechanical reason for this behaviour must be looked for in the non-uniform
Figure 7. Radial strain $\varepsilon_{rr}(r)$ of the circular domain for different values of the differentiation order. (Online version in colour.)

Figure 8. (a) Contour plot of axial wave propagation in the presence of fractional long-range interactions. (b) Propagation of an elastic wave in a one-dimensional solid with long-range interactions. (Online version in colour.)

arrangements of the elastic connections in the one-dimensional lattice model of the bar as shown in figure 5a.

Similar considerations hold true also for a three-dimensional elastic body that involves analysis with the so-called central Marchaud fractional derivatives (CMFDs) [47] that have been applied to a plane polar case described in figure 5b. The strain field obtained with the CMFD operators is reported in figure 7 depicting the radial non-uniform strain field along the radius for different orders of differentiation $\alpha$. Elastic wave propagation in the one-dimensional lattice described by equation (4.6) and reported in figure 5a may also be investigated introducing the additional inertial contribution at the right-hand side of equation (4.5) as $\rho A (d^2 \dot{u}_3/dt^2)$, and the results in terms of pulse propagations are reported in figure 8a,b [18]. The observation of figure 8a shows that longitudinal elastic waves with different speeds are propagating, away from the loaded edge, in the material bulk as a perturbed region exists beyond the characteristic line of the local theory.
5. Conclusion and discussion

In this paper, the authors discussed the main features of the MB non-local elastic theory presenting some results obtained for a two-dimensional plane elasticity problem, for a Timoshenko beam model and in the context of fractional-order differential calculus. The analyses conducted with the MB models have been related to qualitative comparisons of the proposed theory and some observed phenomena of solids. However, there are several points that may be challenged with the MB theory both in the classical context of solid mechanics and in frontier applications of multi-scale methods in several fields of applied physics. From a classical perspective, an interesting consideration is related to a proper non-local measure to achieve the condition of uniform stress under uniform strain that is a common point of all non-local theories extending a possible non-local measure introduced in the one-dimensional context [17,19]. Moreover, in the context of solid mechanics, several studies are underway to extend MB theory to non-local hereditariness, crack evolution, geometrical nonlinearity and thermoelastic applications. MB non-local models may also be derived on a multi-scale basis involving the presence of non-central interactions and integral fields of smaller-scale kinematics yielding the long-range interactions as a proper unbalance of meta-structured phenomena. Challenging applications of MB theory may be framed in the context of bottom-up multi-scale methods that, starting from experimental measures at nanoscales and prescribed geometrical hierarchy, yield the macroscopic MB theory with proper scaling of material mechanical parameters. Such an approach may be used in several nano- / micro-applications in materials science [48,49] to provide an analytical framework to the so-called bio-inspired materials as well as in medical sciences to predict the efficiency of modern nano-based therapies for solid tumours as in the very recent oncophysics [50].

This work was partially supported by the Italian Ministry of University and Technology (MIUR) in the form of research proposal no. PRIN2008 with national coordinator Prof. A. Carpinteri. This financial support is gratefully acknowledged.

References


