A new view of flow topology and conditional statistics in turbulence

Lipo Wang$^1$ and Norbert Peters$^2$

$^1$University of Michigan-Shanghai Jiaotong University Joint Institute, 800 Dongchuan Road, 200240 Shanghai, Republic of China
$^2$Institut für Technische Verbrennung, RWTH-Aachen, Templergraben 64, 52056 Aachen, Germany

By partitioning a turbulent flow field into relative simple units, the original complex system may be better understood from studying decomposed structures. In this paper, some general principles for identifying geometrical decomposition are discussed. Logically, to make analysis more objective and quantitative, the decomposed units need to be non-arbitrarily defined and space filling. Following this vein, we introduced two topological approaches satisfying these prerequisites and the relevant work is reviewed. For a given scalar variable, dissipation elements are defined as the spatial regions that the gradient trajectories of this scalar can share the same pair of scalar extremums (one maximum and one minimum), whereas for the general vector variables, vector tube segments are the part of vector tubes bounded by adjacent extremums of the magnitude of the given vector. Both structures can be characterized by representative shape parameters: the length scale and the extremum difference. On the basis of direct numerical simulation data, the statistics of the shape parameters have been studied. Physically, those structures reveal the ‘nature’ topology of turbulence, and thus their characteristic parameters reflect the flow properties. For instance, when the vector tube segment approach is applied to the velocity case, the negative skewness of the velocity derivative can be explained by the asymmetry of the joint probability density function of the shape parameters of streamtube segments. Conditional statistics based on these newly defined structures identify finer flow physics and are believed helpful for modelling improvement. Application examples illustrate that, in principle, these methods can generally be applied to different flow cases under different situations.
1. Introduction

Over more than a century, the understanding of turbulence has remained a big challenge with great importance in both science and engineering. As a continuum field phenomenon, turbulence, in principle, is infinite dimensional and strongly non-local and nonlinear. Although the Navier–Stokes equations formally are deterministic, turbulence dynamics is by no means the same. These features make mathematics alone inadequate to depict the complex system. Generally speaking, most successful triumphs from mathematics can be attributed to the following properties of an object of study: symmetry and locality [1], both of which are obviously violated in turbulence. Additionally, non-locality and perturbations from boundary conditions make any mathematical descriptions unclosed. Meanwhile, no appropriate small parameter has yet been identified for perturbation analysis in turbulence.

Therefore, it is necessary to attack turbulence from different points of view than mathematics alone. One possible idea for solution is to partition the flow field spatially into relatively simple units. The following ‘principle’ may be taken for granted that, by decomposing an entire field into sub-units, the complexity may be reduced, and detailed structures can better be understood. If these decomposed units are well defined, it will then be possible to reproduce at least statistical properties of the original flow field. Specifically, if the topological features of the decomposed units can be described by a set of parameters \(p_1, p_2, \ldots\), then, in principle, it is much easier and more accurate to represent some statistical property \(X\) by the parameter set \(p_i\) within individual units than to parametrize \(X\) in the entire field. Once the joint probability density function (PDF) of \(p_i, P(p_1, p_2, \ldots)\), has been modelled, the ensemble average of \(X\), denoted by \(\langle X \rangle = \langle X(p_1, \ldots, p_n) \rangle\), is then determined by

\[
\langle X \rangle = \int \ldots \int X(p_1, \ldots, p_n) P(p_1, \ldots, p_n) \, dp_1 \ldots dp_n. \tag{1.1}
\]

Because the ensemble average is based on all material points in a given flow field, equation (1.1) can be applied only if the decomposed units are space filling. This property will be addressed in more detail later. In some sense, the phenomenological K41 theory by Kolmogorov, a great milestone in turbulence research, shares a similar spirit. In the K41 theory, the flow field is viewed as a set of eddies of different sizes, among which the smallest ones follow a scaling principle of their lower-order moments. For higher-order moments, unknown features related to geometry seem to limit the use of this concept. The scaling relations in K41 theory are considered in the Fourier space, where geometries are presented by different base functions. Those base functions in nature are local in wavenumber space but non-local in physical space. Essentially, the Fourier representation cannot appropriately serve to present geometrical structures. As pointed out by Farge [2], a localized expansion should be preferred over unbounded trigonometric functions used in Fourier analysis. Farge believes that trigonometric functions are at risk of misinterpreting the characters of field phenomena. To overcome the weakness of Fourier analysis, different mathematical tools have been suggested, emphasizing more the local properties of geometrical objects directly in physical space other than in transformed spaces. Typical examples include wavelet analysis [2], curvelet transform [3], Hilbert spectrum analysis [4], etc.

In principle, different representations essentially should convey the same information, i.e. the real physical and natural laws. However, in different representations, physical systems may assume different complexity and interpretability. In nature, mechanical laws are defined with respect to material points. Under the action of space transformation, material points in a given system will inhomogeneously and distortedly be reimaged. Taking Fourier transforms, for instance, material points of fluids will disperse in Fourier space and their physical parameters then contribute to different wavenumber components; inversely, one Fourier space point when imaged to physical space will also be scattered unboundedly. The only possibility to preserve geometry is to represent material points directly in the physical space without any transformation.
In the following, the paper reviews the work focusing on geometrical features of turbulence in physical space. Although many efforts and discussions have been made in the relevant areas [5–7], the present review mainly addresses some unique points. Firstly, the geometrical objects are newly constructed based on the primary flowing natures. Secondly, more quantitative relations between geometry and flow physics are explored under some theoretical framework, rather than mere qualitative and illustrative explanation. To make the understanding of turbulence more insightful, it is necessary to step further beyond tentative illustration of a small number of specific structures in the flow and their interaction.

2. Flow topology

The notion of flow topology is somewhat related to coherent structures in flows. Different definitions for ‘coherent structures’ are available in the literature [7]. Intuitively, a coherent structure is a three-dimensional region in the flow over which some properties of the flow exhibit significant correlation over a range of space/time that is significantly larger than the smallest random scales of the flow. Directly speaking, flow topology is a kind of spatial characterization of the flow field.

There have been many attempts to define the topological structures in turbulence. As mentioned earlier, it is helpful to partition a whole field into relatively simple geometrical units. For three-dimensional flows, geometries can be classified as point, line, surface or volume. In turbulence, it is interesting to study the kinematics and dynamics of points, especially critical points, which are those with degenerated gradient matrices, for instance, nodes, saddles, foci, etc. [8]. Critical points are basically landmarks of the flow structures. For instance, stagnation points are located where the orientation of streamlines is indeterminate; extremal points in a scalar field correspond to places where the scalar gradient trajectories converge. The work by Poincaré is an important contribution on the critical point theory by investigating the singular points of differential equation systems [6]. Perry & Chong [5] discussed the kinematic properties of critical points and found that the geometrical features of the flow field can be recovered to some extent by those isolated points. Dynamical properties of stagnation points are believed to be capable of revealing the flow topology as well [9]. Gibson [10] analysed the mechanism by which extremal points are generated under the action of both convection and diffusion. Physically, it is argued that the dominant physical mechanism to produce the small-scale features of scalar fields is the local stretching. On the hypothesis that the local convective velocity is of the order of the Kolmogorov velocity, Gibson concluded that extremal points are generated at scales of the Obukhov–Corrsin length.

In principle, it is more informative to study higher-dimensional objects as lines and surfaces. For the passive scalar case, isosurfaces have been found, both theoretically and experimentally [11,12], to be fragmental in the inertial range. In the case of the isosurfaces of more intermittent parameters such as scalar dissipation, the concept of multi-fractal dimension is more appropriate for statistics at different scales [12]. The pressure isosurface is a meaningful geometrical object for understanding turbulent dynamics [13]. In turbulent combustion, flame surfaces are fractal as well [14].

For three-dimensional objects, topological features become more complex, and at the same time, more substantial to understand the turbulence physics. One well-studied three-dimensional example is the vortex tube structure, which corresponds to spatial regions of high vorticity. Because of stability, regions of significant vorticity are inclined to be organized into tubes [15], named as vortex tubes. Moffatt et al. [16] described vortex tubes as the sinews of turbulence because of their important role in energy dissipation. Townsend [15] suggested that a random distribution of vortex tubes and sheets would represent the turbulent flow field. On average, vortex tubes have radii of the Kolmogorov scale $\eta$, while it is generally believed that the tube length is comparable to the Taylor microscale or other smaller scales. Sarpkaya [17] described different types of connections, disconnections and reconnections of vortex filaments, especially in the near-wall and near-free-surfaces regions. Nevertheless, some basic questions such as the
characteristic scales of vortex tubes are still controversial. It needs to be noted that because their volume and shape depend on a vorticity threshold, which is arbitrarily chosen, in a strict sense, vortex tubes are not well defined. A remedy by Haller suggests that the vortex definition should be invariant under a general coordinate transform, based on which the new vortex definition is frame independent [18]. Although characterized differently, vortex tubes physically address the regions of intense events, which take only a very small volume portion (basically < 10%) of the entire flow field, i.e. not space filling. Therefore, the entire flow field cannot be represented by vortex tubes only. In other words, properties of vortex tubes may not always be relevant to properties of turbulence. Wray & Hunt [19] tried to find a systematic way to subdivide the whole space, and their interesting suggestion is to define four types of space-filling regions, according to the characteristic values of the second invariant of the velocity derivative tensor, as well as the pressure. The definition of these regions contains, however, some arbitrariness introduced from setting the threshold values.

It is evident that the aforementioned structures are more illustrative and descriptive, rather than objectively well defined. Relatively early in turbulence research, Corrsin [20] asked the following questions. (i) What types (of geometry) are ‘naturally’ identifiable in turbulent flows? (ii) What roles do they play or what properties do they have? (iii) What stochastic games can we invent that share some of the difficulties of the turbulent case, but are more treatable? To answer these questions and make the study of geometry more quantitative than qualitative, one needs to construct methods that can identify geometrical objects unambiguously and find a clear definition of the quantities to be sampled. Which method to choose is by no means evident. Logically, sound methods of identifying geometries need to have the following properties: completeness and uniqueness, which mean that each material point should be included once and only once in a geometrical element. In this paper, some newly developed approaches are introduced, i.e. dissipation-element analysis and vector tube segment analysis, aimed to study the scalar and vector field variables, respectively. The unique characters of these descriptions are that shapes are unambiguously defined and space filling, which makes it possible to quantify turbulence physics rather than mere illustrative explanation.

It is indispensable to mention that the study of turbulence typology is based on the flow-field data analysis. The detailed field information can be obtained from either experiments or numerical simulations. In many aspects, experiment and simulation have their respective advantages and disadvantages. Some primary properties, basically of lower dimensionality in space or time, can be obtained from experimental measurements and flow visualization techniques. The high quality of visualization techniques in modern experiments is a remarkable demonstration that helps to grasp reality of flows [21]. However, to extract reliable and complex information of three-dimensional structures, the data to be analysed must be highly resolved with large enough capacity, which seems still to be a serious drawback of experimentation, even in the foreseeable future. Alternatively, the numerical generation of three-dimensional data has been developing rapidly with the help of high-performance computing. Relevant work in turbulence-related areas is attracting increasing attention [22,23]. Computer experiments help to understand many details of the flow physics and model validation. In the following, analyses have been performed based on data from the direct numerical simulation (DNS) of three-dimensional homogeneous shear turbulence with various flowing parameters. Although the existing results do improve our knowledge, work in this area is far from complete.

(a) Dissipation-element analysis

Turbulence is often viewed as the system consisting of eddies of different scales at different motions. It remains a challenge to describe these eddies in a quantitative way for subsequent statistics. A tentative solution hereof is dissipation-element analysis. Consider an instantaneous scalar field that is locally smooth because of the dominant diffusion effect at small scales. Schematically shown in figure 1 for a two-dimensional case, starting from any spatial point,
its corresponding trajectory can be determined by proceeding along the scalar gradient in the descending (ascending) direction, until the local maximum (minimum) of the given scalar is reached. The region of spatial points whose trajectories share the same pair of minimum and maximum points defines a dissipation element [24]. Via DNS data analysis of a two-dimensional turbulent passive scalar fields, the real structure of dissipation elements is plotted in figure 2, where yellow lines show the element boundaries and red and blue points are the local maximum and minimum of the scalar value, respectively. Geometrically, dissipation elements have quite varied sizes or areas, and their shapes are very different. Typically, extremal points are scattered, whereas some regions see clustering extremal points as well. Physically, the extrema clustering phenomenon may be related to the so-called secondary splitting mechanism owing to large local strain [10]. From this geometry definition, it is obvious that the dissipation-element structure
Figures 3. Examples of dissipation elements from three-dimensional DNS data: (a) an isolated element and (b) interaction with a vortex tube. (Online version in colour.)

satisfies the aforementioned completeness and uniqueness conditions, i.e. each spatial point belongs to one, and only one, dissipation element because spatial points are space filling, and so are the dissipation elements.

From two- to three-dimensional space, it is natural to expect more complex element geometry. Figure 3 shows examples of dissipation elements from the three-dimensional DNS passive scalar field in homogeneous shear turbulence. DNS data analysis suggests that in three-dimensional space, on average, dissipation elements are lengthy: the arc length is about the Taylor microscale and the diameter is about the Kolmogorov scale [24]. This property can be explained from the secondary splitting mechanism. Gradient trajectories, thus dissipation elements, are subject to or interact with turbulent motions, and consequently their topological features are believed to reflect the imprint of the flow field.

Compared with the original entire flow field, dissipation elements are relatively simply structured. According to different analyses, the scalar variables can be chosen differently to identify different dissipation-element structures. The primary scalar variables in turbulence, for instance, the passive scalar, kinetic energy, spatial projections of vector variables, etc., are all diffusion controlled at small scales, which ensures theoretically that gradient trajectories and extremal points can be fully determined. In principle, there is no restriction for the application of this approach because it is generally valid once the scalar fields to be considered are smooth and diffusion controlled at small scales. The passive scalar is a typical choice, and other scalar field variables have been discussed as well [25].

(b) Streamtube segment analysis

The vector variables such as velocity and vorticity have structures different from the scalar case, and it is not straightforward to apply the dissipation-element approach to investigate vectors in a similar vein. In a given vector field snapshot, the vector lines are locally tangent to the vector everywhere. One may expect that, similar to gradient trajectories for scalar variables, the vector lines show the inherent geometry of vector variables as well. However, gradient
Figure 4. (a) Illustration of the streamline segments, each of which is the part of its corresponding streamline bounded by a local maximum (in red) and minimum (in blue) of the velocity magnitude. (b) Vectorline segments and extremal points in an arbitrary two-dimensional vector field, backgrounded with the vector magnitude. (Online version in colour.)

trajectories are typically of finite length, which makes it possible to define the length scale, whereas vectorlines are mostly unclosed and thus infinitely long in space, which may not be helpful in identifying field geometry. For this reason, the vectorline segment concept has been developed as a reasonable solution [26].

For the velocity vector case, for instance, as shown in figure 4a, from any grid point, along its streamline a local maximum and minimum of the velocity magnitude can be reached. The part of the streamline bounded by the two adjacent extremal points (one maximal and one minimal) is defined as the streamline segment with respect to the given grid point. It may be flexible to divide streamlines into segments based on whatever rules. However, the most relevant parameter for a general vector is its magnitude and thus it is pertinent to partition segments according to the extrema of the vectors magnitude. The set of extremal points from neighbouring streamlines constitutes extremal surfaces. For illustration purposes, figure 4b plots examples of vectorline segments in an arbitrary two-dimensional vector field. The blue and red points, representing the local minimal and maximal points of the vector’s magnitude, join together as extremal lines.

The streamline segment structure, although instructive to address the field topology and flow kinematics, is not volumetric and thus irrelevant to mass-related dynamics. A remedy hereof is the tube segment concept [26]. As shown in figure 5, a streamtube segment is the part of an infinitely thin (but still with non-zero volume) streamtube cross-cut by two adjacent extremal surfaces. All of the streamtube segments are organized in a non-overlapping and space-filling manner. Differently from dissipation elements that end up at single points, the cross-sectional area of streamtube segments does not taper off to zero. Although the demarcation of the tube boundary is indefinite, statistical properties weighted by the volume of streamtube segments are independent of the boundary definition. In theory, for a uniformly distributed grid point array, the volume of each tube segment is proportional to the number of grid points it can encompass; thus, the volume weighted statistics is equivalent to that based on streamline segments from every grid point, by which data analysis can simply be implemented. Examples of streamtube segments in three-dimensional turbulence are shown in figure 6.

Similar to dissipation-element analysis, which is generally applicable for various scalar fields, the vectortube segment approach will also be valid to analyse different vector fields. For instance, in addition to the work with respect to the velocity field [26], the turbulent vorticity vector has also been studied recently [27].
Figure 5. In a uniform grid point array, the volume of each streamtube segment is proportional to the number of grid points it can encompass.

Figure 6. Streamtube segments from a three-dimensional turbulent velocity field (colour represents the velocity magnitude). (Online version in colour.)

(c) Parametric description

In geometrical studies of flow dynamics, most existing efforts focus on conceptual illustration without deeper quantitative investigation, which makes theoretical development difficult to attain. The lack of relevant work is probably due to the difficulty of clearly defining geometry in turbulence. Some attempts to overcome this problem have been made. For instance, Miyauchi & Tanahashi [28] visualized contour surfaces of the second invariant of the velocity gradient tensor to identify vortex filaments from DNS data, where the tube length was determined by finding the local minima of the second invariant. They claimed that the mean length is of the order of magnitude of the Taylor microscale. Jimenez & Wray [29] studied the radius of vortex filaments and found that the PDF of large length scales decreases exponentially.

As stated in §1, it is necessary to study geometry beyond visualization and illustration, for quantitative theories and flow physics, if possible. To realize equation (1.1), the quantitative characterization of geometrical objects is indispensable. For the passive scalar case in
homogeneous shear turbulence, dissipation elements are determined based on $\phi'$, the fluctuating scalar that is defined as $\phi - \langle \phi \rangle$. For such kinds of three-dimensional complex structures, there are many possibilities to choose describing parameters, among which we have taken the linear length $l$, which is defined as the straight line connecting two extremal points, and $\Delta \phi'$, the scalar difference of the extremal points. Figure 7 shows a typical result from DNS of the joint PDF of $\Delta \phi'$ and $l$, normalized by its mean $l_m$. It can be observed that the most probable region appears apart from the origin because at very small scales, extremal points may rapidly be annihilated owing to molecular diffusion to prohibit the existence of very small $\Delta \phi'$ and very small $l$.

An important feature extracted from this joint PDF is that the conditional mean of $\Delta \phi'$ with respect to $l$ satisfies a $\frac{1}{3}$ scaling, shown in the compensated form in figure 8. This scaling agrees with the Kolmogorov dimensional analysis, even at relatively low Reynolds number ranges. If considered in the conventional Cartesian frame, the $\frac{1}{3}$ scaling is hardly observed because of the insufficient Reynolds number effect. Physically, correlation around extremal points is stronger than that in other regions because the zero gradient around extremal points makes the function difference very small in a certain spatial range. For relatively small Reynolds numbers, in the Cartesian frame, the classical structure function analysis is considerably contaminated because of the mixing of the strong and weak correlation regions, which can then be avoided in the dissipation-element structure to manifest this favourable behaviour [24].

Besides the conditional mean of $\Delta \phi'$, another informative property from figure 7 is the marginal PDF of $l$. In principle, the length of dissipation elements is primarily controlled by two mechanisms: one is the random perturbation from the underlying field by turbulent motions to generate new extremal points and thus smaller scales; the other is the diffusion process that smooths the profile to remove extremal points and generate large scales. At the stationary
state, the length PDF remains invariant. For any given \( l \), its dynamical balance is due to the following actions: the generation (of \( l \)) from longer lengths by perturbation; the removal (of \( l \)) by perturbation; the generation (of \( l \)) by the reconnection of the smaller lengths; the removal (of \( l \)) by reconnection with other lengths. In addition, a slow process due to compressive and extensive strain will induce a drift velocity. A length-distribution model based on the above physical picture is posed as follows [24]:

\[
\frac{\partial \tilde{P}(\tilde{l}, t)}{\partial t} + \frac{\partial}{\partial \tilde{l}} (\tilde{v}(\tilde{l}) \tilde{P}(\tilde{l}, t)) = \Lambda \left[ \int_0^\infty 2\tilde{l} \tilde{l} \tilde{P}(\tilde{l} + \tilde{z}, t) \, d\tilde{l} - \tilde{P}(\tilde{l}, t) \right] + 8 \frac{\partial \tilde{P}(\tilde{l}, t)}{\partial \tilde{l}} \bigg|_{\tilde{l}=0} \left[ \int_0^\infty \tilde{z} \tilde{l} \tilde{P}(\tilde{l} - \tilde{z}, t) \tilde{P}(\tilde{z}, t) \, d\tilde{l} - \tilde{P}(\tilde{l}, t) \right]. \tag{2.1}
\]

In equation (2.1), \( \tilde{l} = l/l_m \), where \( l_m \) is the mean of \( l \), \( \tilde{v}(\tilde{l}) \) is the drift velocity describing the motion of extremal points relative to each other and \( \Lambda \) is an eigenvalue that can be determined from the PDF’s normalization condition. The stationary solution of equation (2.1) is plotted in figure 9, with a satisfactory agreement with DNS results. In particular, this solution presents an exponential decay at large scales that can be explained from a Poisson-like behaviour of the random perturbation process. Moreover, it suggests that the PDF of the normalized length \( l/l_m \) is not or very weakly Reynolds number dependent. Both DNS and theoretical results [24,31] suggest that \( l_m \) is of the turbulent Taylor scale \( \lambda \); more details will be discussed later. Modelling of the joint PDF of \( \tilde{l} \) and \( \Delta \phi' \) has been discussed in Wang & Peters [32], and the predicted results can satisfactorily reproduce the \( \frac{1}{3} \) scaling.

For the vector variables such as the velocity case, for each streamtube segment, the velocity magnitude \( v \) varies monotonously from \( v_{\text{start}} \) at the start point to \( v_{\text{end}} \) at the end point along the velocity direction. Similar to dissipation elements, streamtube segments are characterized by the segment arc length \( l_s \) and velocity difference \( \Delta v \equiv v_{\text{end}} - v_{\text{start}} \). Typically, the joint PDF \( P(l_s, \Delta v) \) is

\[\text{Figure 9.}\] The marginal PDF of the normalized length of dissipation elements from DNS data, compared with the solution from equation (2.1) (adapted from Wang & Peters [25]).
Figure 10. DNS result of the joint PDF of $l_s$ and $\Delta v$.

is shown in figure 10. From the sign of $\Delta v$, segments can be either positive or negative, if $\Delta v$ is positive and negative, respectively. The strong asymmetry of this joint PDF comes from the different kinematic properties of positive and negative steamtube segments. During the segment evolution process, for positive segments, the velocity magnitude along the flowing direction increases, and thus positive segments are inclined to be elongated, whereas negative segments are likely compressed because of the decrease of $v$ along the velocity direction. Therefore, the positive and negative $\Delta v$ branches in figure 10 evolve differently to make the joint PDF asymmetric [26]. Consequently, one may expect that on average, the positive segments are longer and the negative segments are shorter. By the same argument, it can also be concluded that the volume occupied by the positive segments must be larger than that by the negative ones, which has been verified numerically. For instance, such a volume ratio is about 1.3:1 at $Re_\lambda \sim 255$, where $Re_\lambda$ is the Taylor-scale-based Reynolds number [26].

This asymmetry also explains the negativeness of the longitudinal velocity derivative skewness, which is, for instance, along the x-direction, measured by $S = \langle (\partial v_1/\partial x)^3 \rangle \langle (\partial v_1/\partial x)^2 \rangle^{3/2}$. In isotropic turbulence, it is plausible to expect that streamtube segments are non-preferentially orientated. The contributions to skewness from the positive and negative segments are positive and negative, respectively, and the signs remain unchanged under coordinate reflection. Because on averages, the (absolute value of) slope of the negative segments are larger than that of the positive segments, the negative part is significantly amplified to lead to the overall negative skewness. For flows with higher Reynolds numbers, more intensive turbulent motion can strengthen the difference between the positive and negative segments to make them more distinct. Numerically, the relative length difference between positive and negative parts, i.e. $(l_{m,\text{positive}} - l_{m,\text{negative}})/l_{m,\text{positive}}$, increases from 0.20 to 0.29 when $Re_\lambda$ increases from 125 to 255 [26]. This tendency also explains the Reynolds number dependence of the velocity derivative skewness, as indicated in the literature [33].

The solution from equation (2.1) fits the PDF of the length of streamtube segments [26] and vorticity tube segments [27] because the perturbation and reconnection picture integrated in equation (2.1) are, in principle, generally valid for the vector cases as well.
3. Conditional statistics

As a system with complex structures, turbulence reveals different statistics when conditioned on different properties. Because of the lack of clear definition of geometry in the literature, relevant work on conditional statistics basically focuses on specific quantities, for instance, the estimation of conditional terms in turbulence model closure. In the present context, conditional statistics are studied differently based on geometrical features. Several representative results are introduced in this section, and more other properties can be explored in a similar way.

(a) Two-point structure function along the scalar gradient trajectory

The most prominent results derived from the Navier–Stokes equations are the Kolmogorov equation for the velocity structure function and the Yaglom equation for velocity-passive scalar-coupled structure function [34], based on the assumption of isotropy. These structure functions are basically studied in the Cartesian frame. In the inertial range, from the isotropy condition, \( \langle |\Delta u| \rangle \propto r^{1/3} \), where \( \Delta u \) is the velocity difference in the longitudinal direction and \( r \) is the separation length between two points.

From a different viewpoint, structure functions can also be defined with respect to flow geometry. Conventionally, length is an independent variable that can change freely in space. Differently to geometrical analysis of the scalar and vector fields, length is understood as the characteristic shape parameter intrinsically determined by the flowing status, such as the scale of scalar gradient trajectories or vector tube segments. In a passive scalar field, denote \( u_{\text{grad}} \) as the projection of the velocity vector along the scalar gradient direction. Wang studied a new structure function, \( \Delta u_{\text{grad}} \), the difference of \( u_{\text{grad}} \) between two points along gradient trajectories [31]. The two-point correlation of the scalar gradient along the same gradient trajectory can be derived from the exact governing equation. Physically, for a large separation distances \( s \) between two points along a same trajectory, the two-point scalar gradient product becomes uncorrelated with the velocity difference. On this basis of observation, some theoretical results can be derived, and it can be concluded that in the inertial range along the scalar gradient trajectories, \( \Delta u_{\text{grad}} \) is proportional to the separation arc length inbetween, i.e.

\[
\langle \Delta u_{\text{grad}} \rangle \sim \frac{1}{r} s_r \quad (3.1)
\]

where \( r \) is of the order of magnitude of the integral time. Figure 11 shows, from the homogeneous shear DNS data, this linear increase tendency of \( \langle \Delta u_{\text{grad}} \rangle \) with an increase of the normalized \( s \) [31].

Compared with the classical \( \frac{1}{r} \) scaling, this surprising result can be explained as follows. By conditioning the statistics on gradient trajectories, regions of large extensive strain are preferentially extracted because the strain has acted on the scalar fields in a way to allow gradient trajectories to proceed over large distances. Therefore, on average, the strain is extensive. The negative velocity or compressive strain for small scales is an effect of diffusivity on the direction preference of scalar gradients. In the limit \( s \to 0 \), this behaviour agrees with the finding that the scalar gradient is aligned with compressive strain [35]. When applied to study the temporal evolution of dissipation elements, this effect tends to elongate the elements such that the mean linear distance between their extremal points becomes of the order of the Taylor scale [24]. These results may also play an important role in studying the scalar mixing process and scaling analysis in turbulence.

(b) Statistics along the vectorline

The vectorlines of turbulent vector fields are a kind of intrinsic structure in flow physics. Phenomenologically, the arc length of streamline segments can be considered as an index of turbulent eddies. From the energy cascade picture, energy is mostly dissipated at the dissipative scale, which can be estimated from dimensional analysis. It is natural to ask the following interesting question: what is the relation of the local energy dissipation to the characteristic scale?
Figure 11. The two-point velocity difference function along the passive scalar gradient trajectories.

Figure 12. Variation of $\varepsilon$ with respect to the normalized $s$ for different subgroups of positive streamline segments.

According to the arc length, we can categorize streamtube segments into groups with small, intermediate and large arc lengths. Define $l^*$ as the ratio of the arc length between a given grid point and the starting point of its streamline segment to the total segment arc length [26]. Figure 12 presents the dependence of the mean $\varepsilon$ (i.e. $\langle \varepsilon \rangle$) on $l^*$. The counterpart results for the negative segments are almost identical. It is clear that $\langle \varepsilon \rangle$ values of different groups are largely separated: for those with small arc length, $\langle \varepsilon \rangle$ is much larger than other groups and it varies strongly along $l^*$, whereas in the large segment group, $\langle \varepsilon \rangle$ is much smaller and does not change strongly [26].

In fluid dynamics, vorticity $\omega$, defined as the curl of the velocity vector, i.e. $\omega = \nabla \times u$, is another important vector. High vorticity is closely related to turbulence intermittency. In turbulent flows, large compression regions interact with large stretching regions to make vorticity
locally lumped and organized into tubes, named vortex tubes. It is generally believed that vorticity is more stretched than compressed [36], which has been widely used in theoretical analysis and modeling. Vortex stretching dynamically causes velocity fluctuation to spread at different scales [37]. This stretching mechanism is considered to be important to understand, in three-dimensional turbulence, some most basic and essential features [36]. When applying the vector tube segment approach to the vorticity vector case, some interesting statistic properties can be explored. Differently from the streamtube segment case in the velocity field, the vorticity tube segment structure is Galilean invariant.

Express $\omega = \omega t$, where $\omega$ is the magnitude and $t$ is the unit orientation vector. Consider the vorticity energy $\omega^2/2$, or enstrophy, then its production term $P$ is crucial to keep the balance of total enstrophy. Formally, $P$ results from the interaction between vorticity and the local strain (velocity gradient). Physically, at the stationary state, $P$ must be positive to balance the viscous dissipation, which is one of the most distinctive features of three-dimensional turbulence from the two-dimensional case. By observing the PDF of the angle between $\omega$ and the eigenvectors of the strain rate tensor, Tsinober [36] provided a phenomenological explanation of the positiveness of $P$. In the context of vorticity line segment analysis, this problem is considered differently.

Denote $u_t$ as the velocity projection along vorticity lines. For each vorticity line segment, $\Delta u_t$ is the difference of $u_t$ at two extremal points, and the mean strain along a segment of length $l_\omega$ is $\Delta u_t/l_\omega$. It can be shown that statistically $P \sim \omega^2 \Delta u_t/l_\omega$ [27]. Figure 13 shows, for the positive segments (results for negative segments are almost identical), the PDF of $\Delta u_t$ conditioned on different magnitudes of $\Delta \omega$. It is clear that for segments with small $\Delta \omega$, $P(\Delta u_t)$ is quite symmetrical, while for those with larger $\Delta \omega$, the PDF becomes more skewed towards the positive side. In summary, the stretching intensity is strongly dependent on $\Delta \omega$, the difference between two vorticity extrema of segments. Physically, this result is originated from the pressure change. For segments with large $\Delta \omega$, if a constant background pressure is assumed, from the starting point to the ending point, centrifugal force will induce a strong pressure difference; thus, the mean strain becomes significant under this condition. Only the value of $\omega$ itself cannot lead to the pressure change and the local strain. Therefore, back to the aforementioned claim that vorticity is more stretched than compressed, it suggests that more precisely the vorticity tube segments with small $\Delta \omega$ are almost unbiased to stretching or compressing; only segments with large $\Delta \omega$ are likely to be stretched. Thus, the main contribution to the positiveness of enstrophy production is from segments with large $\Delta \omega$ [27].
4. Application and perspective

The present geometrical diagnoses, both for scalar and vector variables, together with the theoretical results thereof derived, have seen insightful applications. As stated already, equation (1.1) functions as an important link between the overall flow properties and those of the decomposed units. It can be applied if and only if the elementary structure ensemble is space filling. Compared with other geometrical analyses, the present approaches are capable of describing turbulent flows in a more quantitative manner.

For instance, in view of the particular importance of energy dissipation $\varepsilon$ in turbulence, we explore how it is distributed over the different classes of dissipation elements defined from the $\varepsilon$ field itself. Similar to the streamtube segment case, the conditional $\varepsilon$ on smaller length scales is larger, while for large scales, on average, $\varepsilon$ is smaller. This effect is found to be closely related to dissipation intermittency. Specifically, dissipation elements having a large linear length will naturally contain a larger portion of the low-activity region where $\varepsilon$ is low. Therefore, the mean $\varepsilon$ for large elements is smaller than that for small elements [25]. Because intermittency is strongly Reynolds-number dependent, the conditional mean of $\varepsilon$ with respect to the element length is also Reynolds-number dependent. When using the finding of the conditional mean of $\varepsilon$, we can apply equation (1.1) to estimate the overall $\langle \varepsilon \rangle$ to shed some light on the $\varepsilon$ equation often used in engineering models and its empirical coefficients [25]. Because of the $Re$ dependence of the conditional mean, it is a natural conclusion that the model coefficients are $Re$ dependent as well, while the present constant ones may be understood as those at the infinite $Re$ case.

In studying shear-layer turbulence, one critical problem necessary to address is the turbulent/non-turbulent interface. In turbulent flows with external intermittency, vorticity in the external laminar region increases rapidly across a thin layer, the so-called turbulent/non-turbulent interface, to the turbulent core region. This interface was first studied by Corrsin & Kistler [38], who introduced the concept of a laminar superlayer and postulated that its thickness scales with the Kolmogorov scale. Some comprehensive reviews of the relevant work can be found both from experimentation [39] and theoretical analysis [40]. As in defining vortex tubes, conventionally this interface is determined based on some preset threshold values [41,42].

Physically, in flow fields, both the scalar and vector variables are under the action of two basic mechanisms: perturbation from convective motions and smoothing from diffusion. To the best knowledge of the authors, turbulence is essentially such that at large enough scales, the convective perturbation is relatively stronger to make field variables fluctuate. In this sense, both dissipation-element analysis and vector tube segment analysis are generally applicable to turbulence flows, and the concept of the extremal point is a crucial geometrical ingredient. Using gradient-trajectory analysis, the entire flow field can be partitioned to different regimes and the turbulent/non-turbulent interface is interpreted from a different viewpoint. As shown in figure 14, from a predefined scalar field (such as passive scalar, scalar dissipation, etc.), each grid point determines its corresponding gradient trajectory. If through a grid point, the gradient trajectory joins one minimum and one maximum, that point is considered to be inside the turbulent zone A, which is depicted by the two regions enclosed by solid lines; if a trajectory B, for instance, connects only one extremum (maximum or minimum), then its corresponding grid point belongs to the turbulent/non-turbulent interface; another different scenario is that if some trajectory such as D connects the upper and lower laminar regions without any intermediate extremal point, its corresponding grid point is then located in the internal quasi-laminar diffusion layer [43].

In principle, dissipation-element analysis and vector tube segment analysis are capable of quantifying the topological structures in various flow configurations, both for the scalar and vector variables. On the one hand, turbulence is disorganized with large fluctuation of the field variables from the perturbation of the turbulent eddy random motion; on the other hand, the smoothing mechanism decays fluctuation owing to molecular diffusion at small scales. The generality of this perturbation-smoothing picture makes the present approaches meaningful. Similarly, the length PDF equation (2.1) is also derived from the interaction of
Figure 14. Different flowing regimes classified by the topology of gradient trajectories: turbulent zone A: grid points with gradient trajectories from minimum to maximum; interface zones B and C: grid points with gradient trajectories from the laminar stream to one extremal; quasi-laminar diffusion layer zone D: grid points with gradient trajectories connecting the upper and lower laminar regions. Solid lines separate turbulent zones from turbulence interfaces. Dashed lines indicate thresholds in the gradient magnitude to define the conventional intermittency function (adapted from Mellado et al. [43]).

These interacting mechanisms and thus it is proved to fit different DNS results. Moreover, a unique feature of dissipation-element and vector tube segment structures is space filling, only by which the quantitative relations between the entire flows and geometrical objects are possibly attainable.

5. Concluding remarks

Turbulence has remained prohibitively difficult as a scientific problem. It still seems to be very challenging to obtain solutions only from mathematics. Geometry analysis plays an important role in turbulence studies to help to understand the flow structure, dynamics and physics. However, it is not yet sufficient to only visualize illustrative and qualitative results. There is a strong need and desire to develop quantitative relations and theories based on the topological properties of the flow fields. Although complex as a whole, turbulent flow fields may be composed of relatively simple structures. Compared with those in transformed spaces, representations of geometrical objects directly in physical space are more inherently related to the flow features. Logical methods of geometrical decomposition need to have the following properties: completeness and uniqueness, which, however, cannot yet be satisfied in most existing work. In these senses, to define geometries is by no means trivial. The present paper reviews some progress and promising approaches developed in this area in recent years.

For the scalar variable case, dissipation elements are defined as the spatial regions that the gradient trajectories of a given scalar can share the same pair of the scalar extremums (one maximum and one minimum), while for the vector variables, for instance, the velocity vector, streamtube segments are the part of streamtubes bounded by the adjacent extremal points of the velocity magnitude. Both approaches from flow physics are capable of defining space-filling geometries unambiguously, which makes more quantitative analysis of turbulence possible. In principle, dissipation-element and vector tube segment analyses are generally valid diagnostic methods and can be applied under different situations. Statistics conditioned on geometry build an important link between turbulence physics and geometry, which may provide us with a broader view and deeper insights into flow dynamics and turbulence modelling.

The authors are grateful for the continuous funding support of the research by Deutsche Forschungsgemeinschaft. The Juelich Supercomputing Centre is also acknowledged for providing the computing resource.
References