A ‘Dysonization’ scheme for identifying quasi-particles using non-Hermitian quantum mechanics

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Dyson analysed the low-energy excitations of a ferromagnet using a Hamiltonian that was non-Hermitian with respect to the standard inner product. This allowed for a facile rendering of these excitations (known as spin waves) as weakly interacting bosonic quasi-particles. More than 50 years later, we have the full denouement of the non-Hermitian quantum mechanics formalism at our disposal when considering Dyson’s work, both technically and contextually. Here, we recast Dyson’s work on ferromagnets explicitly in terms of two inner products, with respect to which the Hamiltonian is always self-adjoint, if not manifestly ‘Hermitian’. Then we extend his scheme to doped anti-ferromagnets described by the $t$–$J$ model, with hopes of shedding light on the physics of high-temperature superconductivity.

1. Introduction

A major goal in condensed matter physics is to represent the low-energy physics of strongly interacting quantum many-body systems in terms of weakly interacting quasi-particles that are either bosonic or fermionic [1]. In a seminal paper [2], Dyson showed that a Heisenberg ferromagnet could be represented as a theory of weakly interacting bosons called magnons or spin waves; this representation allowed thermodynamic calculations of unprecedented accuracy. The purpose of this article is to merge the recent developments from non-Hermitian quantum mechanics [3,4] with the scheme laid out by Dyson.

Dyson’s formulation had the unorthodox feature that the bosons were governed by a Hamiltonian that was superficially non-Hermitian. More precisely, there were
two inner products at work in Dyson’s representation of a ferromagnet. First, there was what we will call the ‘kinematic inner product’ with regard to which the boson creation and annihilation operators were adjoints of each other. In other words, this was the inner product with regard to which the quasi-particles were bosons. Second there was the ‘dynamical inner product’ with regard to which the Hamiltonian was self-adjoint. Conversely, however, the quasi-particles were not bosonic with respect to the dynamical inner product and the Hamiltonian was not self-adjoint with respect to the kinematic inner product.

In contrast, the conventional approach is far more restrictive in that there is only a single inner product with regard to which the quasi-particles are defined and with regard to which the Hamiltonian and all other physical operators must be self-adjoint. In this paper, we explore whether Dyson’s more flexible concept of non-Hermitian quasi-particles can be more broadly applied, particularly to problems that have so far resisted conventional Hermitian analysis.

The \( t-J \) model is believed to capture the essential physics of the cuprate superconductors, which represent one of the grand unsolved puzzles of theoretical physics [5]. In this article, we apply non-Hermitian quantum mechanics to this model and obtain a representation of its low-energy physics in terms of a Dyson boson and a Dyson fermion. By design, these quasi-particles are defined with respect to a kinematic inner product; the Hamiltonian that governs them is not self-adjoint with respect to the kinematic inner product, but with respect to the dynamical inner product. An outline of the paper is as follows. First, we review Dyson’s work on ferromagnets, highlighting the role of the two inner products. We then adapt the analysis to anti-ferromagnets, a useful prelude to the study of the \( t-J \) model. In the following section, we describe a spin \( s \) generalization of the \( t-J \) Hamiltonian (the physical case relevant to the cuprates is \( s = \frac{1}{2} \)). A natural and convenient way to write the \( t-J \) Hamiltonian is to use a superalgebra that is a supersymmetric generalization of the \( su(2) \) angular momentum algebra [6,7]. After presenting this supersymmetric formulation of the \( t-J \) model, we finally write the problem in terms of non-Hermitian quantum mechanics. The presentation here closely follows that in [8].

2. Magnets

(a) Single spin

A single spin has \( 2s + 1 \) basic states \(|s, m\rangle\), where \( s \) is the total spin and \( m \) is its \( z \)-component. \( s \) is the same for all states of the multiplet and \( m = -s, \ldots, s \). These states are assumed to be orthonormal,

\[
\langle s, m \mid s, m' \rangle = \delta_{mm'}.
\]  

(2.1)

The spin operators \( S_z, S_+ \) and \( S_- \) obey the angular momentum algebra

\[
[S_+, S_-] = 2S_z, \quad [S_+, S_z] = -S_+ \quad \text{and} \quad [S_-, S_z] = S_-,
\]  

(2.2)

where, as usual, the spin-raising operator \( S_+ = S_x + iS_y \) and the spin-lowering operator \( S_- = S_x - iS_y \). As shown in textbooks, the effect of these operators on the basis states \(|s, m\rangle\) is

\[
S_+|s, m\rangle = (s - m)^{1/2}(s + m + 1)^{1/2}|s, m + 1\rangle,
S_-|s, m\rangle = (s - m + 1)^{1/2}(s + m)^{1/2}|s, m - 1\rangle
\]

(2.3)

and

\[
S_z|s, m\rangle = m|s, m\rangle.
\]

Dyson introduced an alternative set of basis states

\[
|u\rangle = F_u|s, -s + u\rangle,
\]  

(2.4)
where \( u = 0, \ldots, 2s \). The state \(|0\rangle\) corresponds to having the \( z \)-component of the spin maximally down; the states \(|1\rangle, |2\rangle, |3\rangle, \ldots \) correspond to raising the \( z \)-component by increments of one. These states are orthogonal but not normalized,

\[
\langle u | v \rangle = F_u^2 \delta_{u,v}.
\]  

The normalization factors are \( F_0 = 1 \) and

\[
F_u = \left( 1 \left[ 1 - \frac{1}{2s} \right] \left[ 1 - \frac{2}{2s} \right] \cdots \left[ 1 - \frac{u - 1}{2s} \right] \right)^{1/2},
\]  

for \( u = 1, 2, \ldots 2s \). \( F_u \) is judiciously chosen to map the spin-raising operator \( S_+ \) to the Bose creation operator \( b^\dagger \), as will be seen below.

Making use of equations (2.3)–(2.5), it is not difficult to show that

\[
S_+ |u\rangle = \sqrt{2s} \sqrt{u + 1} |u + 1\rangle,
\]

\[
S_- |u\rangle = \sqrt{2s} \left[ 1 - \frac{u - 1}{2s} \right] \sqrt{u} |u - 1\rangle
\]

and

\[
S_z |u\rangle = (-s + u) |u\rangle.
\]  

(2.7)

Now consider a different Hilbert space with two operators \( b \) and \( b^\dagger \) that are the adjoints of each other under a certain inner product, the ‘kinematic inner product’. These operators are assumed to satisfy the Bose commutation relations

\[
[b, b^\dagger] = 1.
\]  

(2.8)

Provided the kinematic inner product is positive definite, it follows inexorably by standard textbook arguments that the basic states in this Hilbert space form an infinite ladder \(|u\rangle\) with \( u = 0, 1, 2, \ldots \). The state \(|0\rangle\) has the defining characteristic

\[
b|0\rangle = 0;
\]  

(2.9)

we say this is a state with zero bosons. The state

\[
|u\rangle = \frac{1}{\sqrt{u!}} (b^\dagger)^u |0\rangle
\]  

(2.10)

is said to contain \( u \) bosons. These states are orthonormal under the kinematic inner product

\[
(u | v)_{\text{kin}} = \delta_{u,v},
\]  

(2.11)

and the effect of the Bose creation and annihilation operators on these states is

\[
b^\dagger |u\rangle = \sqrt{u + 1} |u + 1\rangle
\]

\[
b|u\rangle = \sqrt{u} |u - 1\rangle
\]  

(2.12)

and

\[
b^\dagger b |u\rangle = u |u\rangle.
\]

Following Dyson, we now establish a mapping between the space of spins and the Bose oscillator space by identifying the spin state \(|u\rangle\) with the boson state \(|u\rangle\). Thus,

\[
|u\rangle \rightarrow |u\rangle,
\]  

(2.13)

for \( u = 0, \ldots , 2s \). States with more than \( 2s \) bosons have no spin space counterpart.

Dyson’s mapping allows us to export the inner product of the spin space to the Bose space. We call this induced inner product the dynamical inner product. Explicitly,

\[
(u | v)_{\text{dyn}} = F_u^2 \delta_{u,v},
\]  

(2.14)

for \( u = 0, \ldots , 2s \). We take \( F_u = 0 \) for \( u > 2s \). Thus, states with more than \( 2s \) bosons are ‘weightless’. 


Dyson’s mapping equation (2.13) also allows us to establish the following correspondence between spin and Bose operators

\[ S_+ \rightarrow \sqrt{2s} b^\dagger, \]
\[ S_- \rightarrow \sqrt{2s} \left[ 1 - \frac{b^\dagger b}{2s} \right] b \]

and
\[ S_z \rightarrow -s + b^\dagger b. \]

This correspondence follows from comparison of equations (2.7) and (2.12). \( b \) and \( b^\dagger \) are not the adjoints of each other under the dynamical inner product. Because we are denoting the adjoint with respect to the kinematic inner product as \( \dagger \), let us signify the adjoint with respect to the dynamical inner product by \( * \). We can then see, for example, that

\[ (b^\dagger)^* = \left[ 1 - \frac{b^\dagger b}{2s} \right] b \]

and
\[ (b^\dagger b)^* = b^\dagger b. \]

(b) Heisenberg ferromagnet

We now consider a two-dimensional Heisenberg ferromagnet in which the spins occupy the sites of a square lattice. Thus, the lattice sites \((m, n)\) have a position vector \( \mathbf{r}_{mn} = ma\hat{x} + na\hat{y} \), where \( \hat{x} \) and \( \hat{y} \) are unit vectors along the \( x \) and \( y \) axes, \( m \) and \( n \) are integers, and \( a \) is the lattice constant. Each site has four nearest neighbours. The site \((m,n)\) has neighbours located at \( \mathbf{r}_{mn} + \mathbf{d} \), where \( \mathbf{d} = x\hat{x}, y\hat{y}, -x\hat{x}, \) and \(-y\hat{y}, \) respectively, for the four neighbours. We denote the spin operator at position \( \mathbf{r} \) as \( S_+(\mathbf{r}), S_-(\mathbf{r}) \) and \( S_z(\mathbf{r}) \). Operators at a given site are assumed to obey the angular momentum algebra equation (2.2); spin operators at different sites are assumed to commute. We consider a spin \( s \) ferromagnet, so the basic states at each site are a spin multiplet of \( 2s + 1 \) states. The Hamiltonian for a Heisenberg ferromagnet is

\[ H_F = -\frac{J}{2} \sum_{\mathbf{r}} \sum_{\mathbf{d}} [S_+(\mathbf{r})S_+(\mathbf{r} + \mathbf{d}) + S_-(\mathbf{r} + \mathbf{d})S_-(\mathbf{r})] \]

Thus, each spin is coupled to its nearest neighbours. We assume the exchange constant \( J > 0 \).

Now consider a system of bosons \( b(\mathbf{r}) \) and \( b^\dagger(\mathbf{r}) \) that live on a square lattice in two dimensions (lattice constant = \( a \)). The operators \( b(\mathbf{r}) \) and \( b^\dagger(\mathbf{r}) \) are assumed to be adjoints of each other under the kinematic inner product. They are assumed to obey the bosonic commutation relation

\[ [b(\mathbf{r}), b^\dagger(\mathbf{r}')] = \delta_{\mathbf{r}\mathbf{r}'} \]

Thus, \( b^\dagger(\mathbf{r}) \) creates bosons at site \( \mathbf{r} \); \( b(\mathbf{r}) \) annihilates them. We may now represent the ferromagnetic Heisenberg Hamiltonian equation (2.18) in terms of bosonic quasi-particles using Dyson’s mapping. From the correspondence equation (2.15) between spin and Bose operators, we obtain the bosonic form of the Heisenberg Hamiltonian,

\[ \mathcal{H}_F = \frac{J_s}{2} \sum_{\mathbf{r},\mathbf{d}} [2b^\dagger(\mathbf{r})b(\mathbf{r}) - b^\dagger(\mathbf{r})b(\mathbf{r} + \mathbf{d}) - b^\dagger(\mathbf{r} + \mathbf{d})b(\mathbf{r})] \]

\[ + \frac{J}{4} \sum_{\mathbf{r},\mathbf{d}} [b^\dagger(\mathbf{r})b^\dagger(\mathbf{r} + \mathbf{d})b^2(\mathbf{r} + \mathbf{d}) + b^\dagger(\mathbf{r})b^\dagger(\mathbf{r} + \mathbf{d})b^2(\mathbf{r})] \]

\[ - \frac{J}{2} \sum_{\mathbf{r},\mathbf{d}} b^\dagger(\mathbf{r})b(\mathbf{r})b^\dagger(\mathbf{r} + \mathbf{d})b(\mathbf{r} + \mathbf{d}). \]
Note that the boson Hamiltonian $\mathcal{H}_F$ is not self-adjoint under the kinematic inner product $(\mathcal{H}_F^\dagger \neq \mathcal{H}_F)$ owing to the terms in the second line of equation (2.20). However, it is self-adjoint under the dynamical inner product $(\mathcal{H}_F^\dagger = \mathcal{H}_F)$.

(c) Heisenberg anti-ferromagnet

A Heisenberg anti-ferromagnet is simply a ferromagnet with $J < 0$. An equivalent but more convenient description of the Heisenberg anti-ferromagnet on a square lattice is the following. Imagine two interpenetrating square lattices, the site labelled $(m, n)$ on the first lattice is located at $r_1(m, n) = m a \hat{e}_x + n a \hat{e}_y$. Here, $m$ and $n$ are integers. The sites of the second square lattice are displaced from those of the first by $(a/2) \hat{e}_x + (a/2) \hat{e}_y$. Thus, the site labelled $(m, n)$ on the second lattice is located at $r_2 = (m + 1/2) a \hat{e}_x + (n + 1/2) a \hat{e}_y$. Regardless of the lattice on which it sits, each site has four nearest neighbours. The displacements from a given site to its four nearest neighbour sites are $\delta_1 = (a/2) \hat{e}_x + (a/2) \hat{e}_y$, $\delta_2 = (a/2) \hat{e}_x - (a/2) \hat{e}_y$, $\delta_3 = -(a/2) \hat{e}_x + (a/2) \hat{e}_y$ and $\delta_4 = -(a/2) \hat{e}_x - (a/2) \hat{e}_y$. We imagine there is a spin at each site and that the spin at each site is anti-ferromagnetically coupled to its nearest neighbours. Thus, the Hamiltonian for a Heisenberg anti-ferromagnet is

$$H_A = \sum_{r, \delta} [S_z^{(1)}(r) S_z^{(2)}(r + \delta) + \frac{1}{2} S_+^{(1)}(r) S_-^{(2)}(r + \delta) + \frac{1}{2} S_+^{(2)}(r + \delta) S_-^{(1)}(r)].$$

(2.21)

The sum over $r$ in equation (2.21) extends over the sites of the first lattice; the sum over $\delta$ extends over the four nearest neighbour displacements enumerated above. The superscripts ‘(1)’ and ‘(2)’ over the spin operators serve to remind us that the spin is on lattice one or lattice two, respectively.

For the Heisenberg ferromagnet, the exact ground state is that all the spins point maximally down along the $z$-axis. In Dyson’s boson representation, the ferromagnetic ground state is the state in which no bosons are present. Anti-ferromagnets present an altogether more formidable problem. The exact ground state for an anti-ferromagnet is not known, except in one dimension for the case of spin $s = 1/2$. The ideal ‘Néel state’ is one in which the spins on the first lattice are maximally down along the $z$-axis and the spins on the second lattice are maximally up along the $z$-axis. The Néel state is not the exact ground state of the anti-ferromagnet, but it is believed to be qualitatively similar and, therefore, a good starting point from which to obtain a more accurate picture of the ground and excited states of a Heisenberg anti-ferromagnet. Thus, in representing an anti-ferromagnet in terms of Dyson bosons, we shall take the Néel state to be the one with no bosons present.

We therefore establish a second mapping between a single spin and a single Bose oscillator. In this second ‘anti-Dyson’ mapping, a state with spin maximally up is to be identified with the state of zero bosons. Thus, we introduce the anti-Dyson basis for a spin multiplet

$$|u,A\rangle = G_u |s, s-u\rangle,$$

(2.22)

where $u = 0, \ldots, 2s$. The normalization constant $G_0 = 1$ and

$$G_u = \left(1 + \frac{1}{2s} \left[1 - \frac{2}{2s} \right] \cdots \left[1 - \frac{u}{2s} \right] \right)^{-1/2},$$

(2.23)

where $u = 1, \ldots, 2s$. $G_u$ has been judiciously chosen to ensure that the spin-raising operator $S_+$ maps to the Bose annihilation operator $b$, as will be seen below.

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1. Or along any other direction. The ground state of a ferromagnet spontaneously breaks rotational symmetry and thus there is a manifold of equivalent ground states.

2. There are many circumstances where it is known the Néel state is not even qualitatively right: in one dimension, on a triangular lattice in two dimensions or even on a square lattice in two dimensions, if next nearest neighbour interactions act to frustrate Néel ordering.
Making use of equations (2.3), (2.22) and (2.23), it is not difficult to show
\[
\begin{align*}
S_+|u; A\rangle & = \sqrt{2s}\sqrt{u}|u - 1; A\rangle, \\
S_-|u; A\rangle & = \sqrt{2s}\sqrt{u + 1}\left[1 - \frac{u}{2s}\right]|u + 1; A\rangle
\end{align*}
\]
(2.24)
and
\[
S_z|u; A\rangle = (s - u)|u; A\rangle.
\]
(2.25)

We may now establish an anti-Dyson mapping between spins and Bose oscillators by identifying the spin state \(|u; A\rangle\) with the Bose oscillator state \(|u\rangle\). Thus,
\[
|u; A\rangle \rightarrow |u\rangle,
\]
for \(u = 0, \ldots, 2s\). States with more than 2s bosons have no spin space counterpart. The anti-Dyson mapping allows us to export a dynamical inner product to the Bose space as before. The remarks made earlier about this dynamical inner product apply mutatis mutandis (see the paragraph containing equation (2.14)). The anti-Dyson mapping equation (2.25) also allows us to establish a second correspondence between spin and Bose operators,
\[
\begin{align*}
S_+ & \rightarrow \sqrt{2s}b, \\
S_- & \rightarrow \sqrt{2s}b^\dagger\left[1 - \frac{b^\dagger b}{2s}\right], \\
S_z & \rightarrow s - b^\dagger b.
\end{align*}
\]
(2.26)

This correspondence follows from comparison of equations (2.24) and (2.12).

Equipped with the second Dyson mapping we now return to the Heisenberg anti-ferromagnet. We consider two interpenetrating square lattices as above and assume that there are two kinds of lattice bosons. One kind lives on the sites of the first lattice: \(b^\dagger_1(r_1)\) creates this kind of boson at site \(r_1\); \(b_1(r_1)\) annihilates it. The other kind live on the second lattice and are created and annihilated by \(b^\dagger_2(r_2)\) and \(b_2(r_2)\), respectively. These creation and annihilation operators are adjoints of each other with respect to the kinematical inner product and are assumed to obey bosonic commutation relations
\[
[b_i(r), b_j(r')] = \delta_{r,r'} \delta_{ij},
\]
(2.27)
where \(i\) and \(j\) equal 1 or 2.

We may now represent the Hamiltonian for the Heisenberg Hamiltonian equation (2.21) in terms of bosonic quasi-particles using Dyson’s mapping between spins and bosons, equation (2.15) on the sites of the first lattice and using the anti-Dyson mapping equation (2.26) on the sites of the second lattice. This strategy ensures that the Néel state corresponds to the boson vacuum and yields a bosonic form of the Heisenberg Hamiltonian
\[
\mathcal{H}_A = J s \sum_{r, \delta}[b^\dagger_1(r)b_1(r) + b^\dagger_2(r + \delta)b_2(r + \delta)]
\]
\[
+ J s \sum_{r, \delta}[b^\dagger_1(r)b^\dagger_2(r + \delta) + b_2(r + \delta)b_1(r)]
\]
\[
- J \sum_{r, \delta}b^\dagger_1(r)b_1(r)b^\dagger_2(r + \delta)b_2(r + \delta)
\]
\[
- J \frac{1}{2} \sum_{r, \delta}b^\dagger_1(r)b^\dagger_2(r + \delta)b^\dagger_2(r + \delta)b_2(r + \delta)
\]
\[
- J \frac{1}{2} \sum_{r, \delta}b^\dagger_1(r)b_1(r)b_1(r)b_2(r + \delta).
\]
(2.28)

Note that the boson Hamiltonian \(\mathcal{H}_A\) is not self-adjoint under the kinematic inner product (\(\mathcal{H}_A^* \neq \mathcal{H}_A\)) owing to the terms in the last two lines of equation (2.28). However, it is self-adjoint under the
dynamical inner product \((\mathcal{H}_A^* = \mathcal{H}_A)\). A Hamiltonian of this form was introduced and analysed in Harris et al. [9].

3. Doped magnets

A typical cuprate such as \(\text{La}_{2-x}\text{Sr}_x\text{CuO}_4\) consists of stacked planes of Cu atoms. Within a plane, the Cu atoms are arranged in a square lattice. In the pure compound \(\text{La}_{2}\text{CuO}_4\), there is one electron available per Cu atom. If electron–electron interactions were weak, the electrons could hop from atom to atom via tunnelling. However, in the cuprates, the electron–electron repulsion is strong, forbidding double occupancy of the Cu sites. Each site is, therefore, occupied by a single electron. The electrons are locked in place and immobile. A material like this is called a ‘Mott insulator’. The only degree of freedom is the electron spin that can point up or down at each site. The decidedly unequal competition between hopping and electron–electron repulsion tends to make the spins align anti-ferromagnetically. The undoped cuprates may, therefore, be described by the anti-ferromagnetic Heisenberg Hamiltonian (e.g. [10])

In the doped compound \(\text{La}_{2-x}\text{Sr}_x\text{CuO}_4\), there are only \(1 - x\) electrons per site and therefore a fraction \(x\) of the sites are unoccupied. The absence of electrons (‘holons’) can hop and when the density of holons is sufficiently high, the materials are observed to exhibit strange metallic and then superconducting behaviour. The competition between hopping and electron–electron repulsion for the doped compounds is described by the \(t–J\) Hamiltonian. In the next section, the \(t–J\) Hamiltonian is formulated in a way that is particularly well suited to our present purpose.

(a) Supersymmetric formulation of the \(t–J\) model

In the parent compound, there are two possible states for each site: spin up or spin down. In the doped material, each site has three possible states: spin up, spin down or missing electron. The missing electron state corresponds to zero spin and a positive charge \(+e\) on the site. In the following, it will be useful to consider a spin \(s\) generalization, wherein there are \(4s + 1\) states per site. The site may either be in one of the \(2s + 1\) states \(|s, m\rangle\) with \(m = -s, \ldots, s\), or in one of the \(2s\) states \(|s - \frac{1}{2}, m\rangle\) with \(m = -s, \ldots, s - \frac{1}{2}\). If the site is in a spin \(s\) state, \(|s, m\rangle\), the total spin is \(s\), the \(z\)-component of the spin is \(m\) and the site is assumed to have no charge. On the other hand, if it is in a spin \(s - \frac{1}{2}\) state, \(|s - \frac{1}{2}, m\rangle\), the total spin is \(s - \frac{1}{2}\), its \(z\)-component is \(m\) and the site has a positive charge \(+e\) owing to the lack of one electron. In summary, whereas the basic states per site of a spin \(s\) magnet are a single spin \(s\) multiplet \(|s, m\rangle\), the basic states per site for our \(t–J\) model are a ‘super-multiplet’: a pair of multiplets with spin \(s\) and spin \(s - \frac{1}{2}\). The physically relevant case is \(s = \frac{1}{2}\).

Having specified the basic states at each site, we must now describe the basic operators out of which the \(t–J\) Hamiltonian will be built. For a magnet, these operators are \(\hat{S}_+, \hat{S}_-\) and \(\hat{S}_z\). They satisfy the \(\text{su}(2)\) angular momentum algebra (equation (2.2)), and their action on the states \(|s, m\rangle\) of a spin \(s\) multiplet is well known (equation (2.3)). Now it turns out there is a super-algebra that is a natural generalization of the \(\text{su}(2)\) algebra and the \(t–J\) model can be written (super)naturally in terms of the elements of this algebra; this appears to have been first noted in Weigmann [7], and subsequently solved exactly in one dimension in Bares & Blatter [6].

The super-algebra has eight elements. Six of them are raising and lowering operators (also known as Weyl elements): \(\hat{S}_+, \hat{S}_-, \hat{R}_+, \hat{R}_-, \hat{T}_+\) and \(\hat{T}_-\). The remaining two are the Cartan elements \(\hat{A}\) and \(\hat{S}_z\). Because this is a super-algebra, the elements may also be grouped differently into commuting elements \((\hat{S}_+, \hat{R}_-, \hat{S}_z, \hat{A})\) and anti-commuting elements \((\hat{R}_+, \hat{R}_-, \hat{T}_+, \hat{T}_-)\). Just as the \(\text{su}(2)\) algebra is defined by the commutation relations of its elements equation (2.2), so the super-algebra is defined by the commutation or anti-commutation relations amongst all pairs of its elements. First, there are the diagonal Weyl element relations

\[
[S_+, S_-] = 2S_z, \quad \{R_+, R_-\} = A + S_z \quad \text{and} \quad \{T_+, T_-\} = A - S_z.
\]  

(3.1)
As usual, square brackets denote commutators and curly brackets denote anti-commutators. Next, there are the off-diagonal Weyl commutation relations,

\[
\begin{align*}
[S_+, R_+] &= -T_+, \quad [S_-, R_+] = 0, \\
[S_+, R_-] &= 0, \quad [S_-, R_-] = T_-
\end{align*}
\] (3.2)

and the off-diagonal Weyl anti-commutation relations,

\[
\begin{align*}
\{R_+, T_+\} &= 0, \quad \{R_-, T_+\} = S_+, \\
\{R_+, T_-\} &= S_- \quad \text{and} \quad \{R_-, T_-\} = 0.
\end{align*}
\] (3.3)

The Cartan elements \( A \) and \( S_z \) commute with each other; \([A, S_z] = 0\). The final set of defining relations are the commutators of the Weyl and Cartan elements,

\[
\begin{align*}
[S_+, S_z] &= -S_+, \quad 2[R_+, S_z] = R_+, \quad 2[T_+, S_z] = -T_+, \\
[S_-, S_z] &= S_-, \quad 2[R_-, S_z] = -R_-, \quad 2[T_-, S_z] = T_-, \\
[S_+, A] &= 0, \quad 2[R_+, A] = -R_+, \quad 2[T_+, A] = -T_+, \\
[S_-, A] &= 0, \quad 2[R_-, A] = R_- \quad \text{and} \quad 2[T_-, A] = T_-.
\end{align*}
\] (3.4)

These relations serve to define the algebra.

Now let us describe the action of the algebra elements on the states of a super-multiplet. \( S_+ \) and \( S_- \) simply raise and lower the \( z \)-component of the spin in either multiplet,

\[
\begin{align*}
S_+|s,m\rangle &= (s - m)^{1/2}(s + m + 1)^{1/2}|s,m + 1\rangle, \\
S_+|s - \frac{1}{2},m\rangle &= (s - \frac{1}{2} - m)^{1/2}(s + \frac{1}{2} + m)^{1/2}|s - \frac{1}{2},m + 1\rangle, \\
S_-|s,m\rangle &= (s - m + 1)^{1/2}(s + m)^{1/2}|s,m - 1\rangle \\
\text{and} \\
S_-|s - \frac{1}{2},m\rangle &= (s + \frac{1}{2} - m)^{1/2}(s - \frac{1}{2} + m)^{1/2}|s - \frac{1}{2},m - 1\rangle.
\end{align*}
\] (3.5)

\( R_+ \) and \( R_- \) switch states between multiplets,

\[
\begin{align*}
R_+|s,m\rangle &= (s + m)^{1/2}|s - \frac{1}{2},m - \frac{1}{2}\rangle, \\
R_+|s - \frac{1}{2},m\rangle &= 0, \\
R_-|s,m\rangle &= 0
\end{align*}
\] (3.6)

and

\[
\begin{align*}
R_-|s - \frac{1}{2},m\rangle &= (s + \frac{1}{2} + m)^{1/2}|s,m + \frac{1}{2}\rangle.
\end{align*}
\]

Note that \( R_+ \) lowers the \( z \)-component of spin by half when it changes from spin \( s \) to spin \( s - \frac{1}{2} \). \( T_+ \) and \( T_- \) also switch states between multiplets,

\[
\begin{align*}
T_+|s,m\rangle &= (s - m)^{1/2}|s - \frac{1}{2},m + \frac{1}{2}\rangle, \\
T_+|s - \frac{1}{2},m\rangle &= 0, \\
T_-|s,m\rangle &= 0
\end{align*}
\] (3.7)

and

\[
\begin{align*}
T_-|s - \frac{1}{2},m\rangle &= (s + \frac{1}{2} - m)^{1/2}|s,m - \frac{1}{2}\rangle.
\end{align*}
\]
but whereas $R_+$ lowers the $z$-component by half, $T_+$ raises it. Finally, the states of the super-multiplet are eigenstates of $A$ and $S_z$,

$$
A[s, m] = s[s, m],
$$
$$
A[s - \frac{1}{2}, m] = (s + \frac{1}{2})[s - \frac{1}{2}, m],
$$
$$
S_z[s, m] = m[s, m],
$$
$$
S_z[s - \frac{1}{2}, m] = m[s - \frac{1}{2}, m].
$$

(3.8)

Thus, the $A$ value distinguishes the multiplets; the $S_z$ value specifies the state within the multiplet. Equations (3.5)–(3.8) fully describe the action of the super-algebra elements on the states of the super-multiplet. The normalization factors in these equations follow inexorably from the commutation and anti-commutation relations that define the super-algebra. Note that the action of $S_+, S_- \text{ and } S_z$ is exactly as one would expect from the textbook theory of angular momentum; this is because these operators constitute an $su(2)$ subalgebra of our super-algebra.

We can now write the $t-J$ Hamiltonian in supersymmetric form,

$$
H_{t-J} = -\tau \sum_{r, \delta} \{ R_+(r + \delta)R_-(r) + R_+(r)R_-(r + \delta)
$$
$$
+ T_+(r + \delta)T_-(r) + T_+(r)T_-(r + \delta)\}
$$
$$
+ J \sum_{r, \delta} \left[ S_z(r)S_z(r + \delta) - \{A(r) - 2s\}\{A(r + \delta) - 2s\}
$$
$$
+ \frac{1}{2}S_+(r + \delta)S_-(r) + \frac{1}{2}S_+(r)S_-(r + \delta) \right].
$$

(3.9)

(For the traditional/non-supersymmetric expression, see for example §3.2 of [11].) We assume the super-spins occupy the sites of a square lattice in a plane. The lattice position vectors are $r = m\hat{e}_x + n\hat{e}_y$, where $m$ and $n$ are integers and the sum over $r$ in equation (3.9) is over $m$ and $n$. $\delta$ denotes the four nearest neighbour displacements $\pm \hat{e}_x \text{ and } \pm \hat{e}_y$; the sum over $\delta$ in equation (3.9) is over these four values. The super-spin operators at different sites are assumed to commute, and at a given site, they are assumed to obey the super-algebra defined by equations (3.1)–(3.4). Thus, the $t-J$ Hamiltonian couples super-spins at neighbouring sites.

Finally, a word about the symmetry of the Hamiltonian, $H_{t-J}$. The Heisenberg Hamiltonian $H_F$ equation (2.18) has rotational symmetry. Formally, this is demonstrated by defining the total spin operators

$$
S^\text{tot}_+ = \sum_r S_+(r)
$$

(3.10)

(and $S^{\text{tot}}_-$ similarly) and verifying that $[H_F, S^{\text{tot}}_+] = 0$ (as well as $[H_F, S^{\text{tot}}_-] = 0$ and $[H_F, S^{\text{tot}}_z] = 0$). In the same way, we can define the total super-spin operator

$$
R^\text{tot}_+ = \sum_r R_+(r),
$$

(3.11)

and similarly for all other elements of the super-algebra. For the $t-J$ Hamiltonian to be supersymmetric, it would have to satisfy $[H_{t-J}, R^{\text{tot}}_+] = 0, [H_{t-J}, S^{\text{tot}}_+] = 0$ and so on for all eight elements of the super-algebra. This condition is not met, except for special values of the parameters $t$ and $J$, namely $|2\tau| = |J|$. The $t-J$ Hamiltonian is certainly not supersymmetric for the experimentally relevant values. Thus, although the Hamiltonian is built out of supersymmetric algebra elements, it is not generally supersymmetric. In this respect it is similar to supersymmetric extensions of the standard model for which also supersymmetry is broken.
(b) Dysonization of the $t$–$J$ Hamiltonian

Dyson’s key insight was to define magnons as bosonic with respect to a non-standard inner product. For the $t$–$J$ model, we wish to take that scheme one step further and define a ‘Dyson fermion’ in addition to the Dyson bosons we have already alluded to.

In order to represent the $t$–$J$ Hamiltonian in terms of Dyson bosons and fermions, first let us consider a single super-multiplet corresponding to the states at a single site. The basis states for a super-multiplet that we have so far adopted are the $4s + 1$ states $|s,m\rangle$ and $|s - \frac{1}{2}, \mu\rangle$, where $m = -s, \ldots, +s$ and $\mu = -(s - \frac{1}{2}), \ldots, s - \frac{1}{2}$.

Following Dyson, we now introduce the alternative basis states

$$|u, 0\rangle = F_{u,0}|s, -s + u\rangle \quad \text{and} \quad |u, 1\rangle = F_{u,1}|s - \frac{1}{2}, -(s - \frac{1}{2}) + u\rangle,$$  \hspace{1cm} (3.12)

where $u = 0, \ldots, 2s$ for the $|u, 0\rangle$ states and $u = 0, \ldots, 2s - 1$ for the $|u, 1\rangle$ states. Thus, $|0, 0\rangle$ corresponds to having a spin $s$ at the site that is maximally down; $|u, 0\rangle$ corresponds to raising the spin $u$ times. Similarly, $|0, 1\rangle$ corresponds to having a spin $s - \frac{1}{2}$ at the site that is maximally down; $|u, 1\rangle$ corresponds to raising that spin $u$ times. The states $|u, 0\rangle$ are neutral; the states $|u, 1\rangle$ correspond to having a net charge $+e$ on the site. Usually, these sites are described as holons; in light of the supersymmetry discussion above, it seems natural to associate the charge with the presence of a non-Hermitian ‘Dyson fermion’. Thus, the filling fraction of Dyson fermions (i.e. the number of Dyson fermions per lattice site) is equal to the doping parameter $x$.

The states in this basis are orthogonal to each other but not normalized,

$$\langle u, a | v, b \rangle = F_{u,a}^2 \delta_{ab} \delta_{uv}.$$  \hspace{1cm} (3.13)

The normalization factors $F_{u,a}$ are chosen judiciously,

$$S^+|u, a\rangle = \sqrt{2s} \sqrt{u + 1}|u + 1, a\rangle,$$  \hspace{1cm} (3.14)

so as to maintain the action of $S^+$ as a bosonic raising operator. This is accomplished by defining

$$|u, a\rangle = \frac{1}{\sqrt{2s}} \frac{1}{\sqrt{|u|}} (S^+)^u|0, a\rangle,$$  \hspace{1cm} (3.15)

which corresponds to the choice

$$F_{u,a} = \left(1 - \frac{1}{2s}\right)^{1/2} \left(1 - \frac{2}{2s}\right)^{1/2} \ldots \left(1 - \frac{u - 1 + a}{2s}\right)^{1/2}. \hspace{1cm} (3.16)$$

The $|u, a\rangle$ basis is fully specified by equations (3.12) and (3.16) or equivalently by equation (3.15).

We may now determine the action of all the super-spin operators in this basis. The results are

$$S^+|u, a\rangle = \sqrt{2s} \sqrt{u + 1}|u + 1, a\rangle,$$

$$S^-|u, a\rangle = \sqrt{2s} \left[1 - \frac{u - 1 + a}{2s}\right]u^{1/2}|u - 1, a\rangle,$$

$$S_\pm|u, a\rangle = \left(-s + u + \frac{a}{2}\right)|u, a\rangle$$

and

$$A|u, a\rangle = a|u, a\rangle.$$  \hspace{1cm} (3.17)
for the commuting elements of the super-algebra, and
\[
\begin{align*}
T^+ |u, 0\rangle &= \sqrt{2s} |u, 1\rangle, \\
T^+ |u, 1\rangle &= 0, \\
T^- |u, 0\rangle &= 0, \\
T^- |u, 1\rangle &= \left[ 1 - \frac{u}{2s} \right] \sqrt{2s} |u, 0\rangle,
\end{align*}
\]
(3.18)
and
\[
\begin{align*}
R^+ |u, 0\rangle &= u^{1/2} |u - 1, 1\rangle, \\
R^+ |u, 1\rangle &= 0, \\
R^- |u, 0\rangle &= 0, \\
R^- |u, 1\rangle &= (u + 1)^{1/2} |u + 1, 0\rangle,
\end{align*}
\]
for the anti-commuting elements. Now consider a different Hilbert space inhabited by a single Bose creation and annihilation operator pair \((b, b^\dagger)\) and a Fermi pair \((a, a^\dagger)\) that satisfy the canonical commutation relations
\[
\begin{align*}
[b, b^\dagger] &= 1, \\
\{a, a^\dagger\} &= 1 \text{ and } a^2 = a^\dagger 2 = 0.
\end{align*}
\]
(3.19)
We also suppose \([a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0\). The creation and annihilation operators are adjoints of each other under the kinematical inner product in this Hilbert space. One can show inexorably from these assumptions that the basic states of this Hilbert space are \(|u, 0\rangle\) and \(|u, 1\rangle\), where \(u = 0, 1, 2, \ldots\). The state \(|0, 0\rangle\) has the defining characteristic
\[
b|0, 0\rangle = a|0, 0\rangle = 0;
\]
(3.20)
it contains neither a \(b\) boson not an \(a\) fermion. The state
\[
|u, 0\rangle = \frac{1}{\sqrt{u!}} (b^\dagger)^u |0, 0\rangle
\]
(3.21)
contains \(u\) bosons and no fermions. The state
\[
|u, 1\rangle = \frac{1}{\sqrt{u!}} (b^\dagger)^u a^\dagger |0, 0\rangle
\]
(3.22)
contains \(u\) bosons and one fermion. These states are orthonormal under the kinematic inner product
\[
(u, a | v, b)_{\text{kin}} = \delta_{u,v} \delta_{a,b}.
\]
(3.23)
We now establish the following mapping between the states of a super-multiplet and the Bose–Fermi Hilbert space discussed above. The mapping is
\[
|u, a\rangle \rightarrow |u, a\rangle.
\]
(3.24)
Here, \(u = 0, \ldots, 2s\) for \(a = 0\) and \(u = 0, \ldots, 2s - 1\) for \(a = 1\). States with more bosons have no counterpart in the super-spin space.

As before, this correspondence exports a dynamical inner product to the Bose–Fermi Hilbert space
\[
(u, a | v, b)_{\text{dyn}} = F^2_{uv} \delta_{uv} \delta_{ab}.
\]
(3.25)
We assume \(F_{u,0} = 0\) for \(u > 2s\) and \(F_{u,1} = 0\) for \(u > 2s - 1\). Thus, states with a greater number of bosons are weightless.
The mapping equation (3.24) also allows us to establish a correspondence between super-spin and Bose and Fermi creation and annihilation operators. The correspondence follows from equations (3.17) and (3.18):

\[
\begin{align*}
S^+ & \to \sqrt{2s} b^\dagger, \\
T^+ & \to \sqrt{2sa^\dagger}, \\
S^- & \to \left[1 - \frac{b^\dagger b + a^\dagger a}{2s}\right] \sqrt{2sb}, \\
T^- & \to \left[1 - \frac{b^\dagger b + a^\dagger a}{2s}\right] \sqrt{2sa}, \\
S_z & \to (-s + b^\dagger b + \frac{1}{2} a^\dagger a), \\
R^+ & \to ba^\dagger, \\
A & \to a^\dagger a \text{ and } R^- \to ab^\dagger.
\end{align*}
\] (3.26)

(i) Ferromagnetic $t$--$J$ model

Now let us consider the $t$--$J$ model equation (3.9). For the cuprates we are interested in anti-ferromagnetic coupling ($J > 0$) but it is instructive to first consider the case of ferromagnetic coupling, $J < 0$.

We introduce a single boson $b(r), b^\dagger(r)$ and a single fermion $a(r), a^\dagger(r)$ at each site of the lattice. Bose and Fermi creation and annihilation operators at the same site are taken to be adjoints of each other under the kinematic inner product. Using the correspondence between super-spin operators and Bose and Fermi operators, equation (3.26), we may write the $t$--$J$ Hamiltonian as

\[
H_{t-J} = -2\tau s \sum_{r,\delta} \left[ a^\dagger (r + \delta) a(r) + a^\dagger (r) a(r + \delta) \right] + \frac{1}{2}Js \sum_{r,\delta} a^\dagger (r) a(r) \\
+ \frac{1}{2}Js \sum_{r,\delta} \left[ b^\dagger (r) b(r) - b^\dagger (r + \delta) b(r) \right] + \ldots
\] (3.27)

In equation (3.27), we have written out the leading quadratic term in the Dyson representation of the ferromagnetic $t$--$J$ Hamiltonian. At this level, it is a theory of non-interacting bosonic spin waves (‘magnons’) and fermions with charge $+e$ (‘magninos’).

The interaction terms that were omitted in equation (3.27) and are presumably small in this representation, are given by

\[
H_{\text{int}} = -\tau \sum_{r,\delta} b^\dagger (r) b(r) a(r) a^\dagger (r + \delta) - \tau \sum_{r,\delta} \left[ b(r + \delta) b^\dagger (r + \delta) a(r) + b^\dagger (r) b(r + \delta) a^\dagger (r + \delta) \right] \\
- \frac{J}{2} \sum_{r,\delta} \left[ b^\dagger (r) b(r) + \frac{1}{2} a^\dagger (r) a(r) \right] \left[ b^\dagger (r + \delta) b(r + \delta) + \frac{1}{2} a^\dagger (r + \delta) a(r + \delta) \right] \\
+ \frac{J}{4} \sum_{r,\delta} \left[ a^\dagger (r) a(r) + b^\dagger (r) b(r) \right] \left[ b^\dagger (r + \delta) b(r) \right].
\] (3.28)

The full $t$--$J$ Hamiltonian, $H_{t-J}$, is not self-adjoint under the kinematic inner product ($H_{t-J}^* \neq H_{t-J}$); however, it is self-adjoint under the dynamical inner product, $H_{t-J}^* = H_{t-J}$.

(ii) Anti-ferromagnetic $t$--$J$ model

For the anti-ferromagnetic $t$--$J$ model, as for the Heisenberg anti-ferromagnet, it is convenient to imagine a pair of interpenetrating square lattices. The $t$--$J$ Hamiltonian may then be re-written as

\[
H_{t-J} = -\tau \sum_{r,\delta} \left[ R_{+}^{(2)} (r + \delta) R_{-}^{(1)} (r) + R_{+}^{(1)} (r) R_{-}^{(2)} (r + \delta) \right] - \tau \sum_{r,\delta} \left[ T_{+}^{(2)} (r + \delta) T_{-}^{(1)} (r) + T_{+}^{(1)} (r) T_{-}^{(2)} (r + \delta) \right] \\
+ \frac{J}{2} \sum_{r,\delta} \left[ S_{+}^{(2)} (r + \delta) S_{-}^{(1)} (r) + S_{+}^{(1)} (r) S_{-}^{(2)} (r + \delta) \right] \\
+ \frac{J}{2} \sum_{r,\delta} \left[ S_{-}^{(2)} (r + \delta) S_{+}^{(1)} (r) + S_{-}^{(1)} (r) S_{+}^{(2)} (r + \delta) \right].
\] (3.29)
The sum over $r$ in equation (3.29) extends over the sites of the first lattice; the sum over $\delta$ extends over the four nearest neighbours of each site. The superscripts ‘(1)’ and ‘(2)’ over the super-spin operators serve to remind us that the spin is on lattice one or lattice two, respectively.

At least for light doping, it makes sense to assume that the Néel state is a good starting point for the ground state of the $t$–$J$ model. In the Néel state, the spin is maximally down at each site of the first lattice; it is maximally up at each site of the second lattice. The magnitude of the spin is $s - 1/2$ at sites occupied by a hole. It is $s$ at all other sites. In representing the Néel state in terms of Dyson bosons and fermions, we shall take the Néel state to have zero bosons and to have Dyson fermions at all the sites with holes.

To this end, we establish a second mapping between the states of a single super-spin and the Hilbert space of a single boson and fermion. In this mapping, we identify the states with spin fermions at all the sites with holes.

Dyson bosons and fermions, we shall take the Néel state to have zero bosons and to have Dyson fermions at all the sites with holes.

To this end, we establish a second mapping between the states of a single super-spin and the Hilbert space of a single boson and fermion. In this mapping, we identify the states with spin fermions at all the sites with holes.

As before, we then establish a mapping $|u, 0\rangle = G_{u|0}|s, s - u\rangle$ and $|u, 1\rangle = G_{u|1}|s - 1/2, s - 1/2 - u\rangle$ (3.30) in place of equation (3.12). This time we choose

$$
G_{u,a} = \left( 1 - \frac{1}{2s} \right)^{-1/2} \left( 1 - \frac{2}{2s} \right)^{-1/2} \ldots \left( 1 - \frac{u - 1 + a}{2s} \right)^{-1/2}.
$$

(3.31)

As before, we then establish a mapping $|u, a\rangle$ between the states of the super-spin and the states $|u, a\rangle$ of a Bose–Fermi system. By virtue of this correspondence, we obtain a second mapping between super-spin and Bose and Fermi operators,

$$
\begin{align*}
S^+ & \rightarrow \sqrt{2s} b, & S^- & \rightarrow \sqrt{2s} b^+ \left( 1 - \frac{b^+ b + a^+ a}{2s} \right), \\
S_z & \rightarrow \left( s - b^+ b - \frac{1}{2} a^+ a \right), & A & \rightarrow a^+ a, \\
R^+ & \rightarrow \sqrt{2s} \left( 1 - \frac{b^+ b}{2s} \right), & R^- & \rightarrow \sqrt{2s} a, \\
T^+ & \rightarrow ba^+ & T^- & \rightarrow b^+ a.
\end{align*}
$$

(3.32)

In order to write the $t$–$J$ Hamiltonian in terms of bosons and fermions, we use the first correspondence equation (3.26) on the first lattice and the second correspondence equation (3.32) on the second lattice. Keeping the leading terms up to cubic order, we obtain a novel representation of the $t$–$J$ Hamiltonian in terms of bosons and fermions,

$$
\begin{align*}
h_{t-J} & = \frac{J_S}{s} \sum_{r, \delta} \left[ b_1^r(r)b_1^r(r) + b_2^r(r + \delta)b_2^r(r + \delta) \right] + \frac{J_S}{s} \sum_{r, \delta} \left[ a_1^r(r)a_1^r(r) + a_2^r(r + \delta)a_2^r(r + \delta) \right] \\
& \quad + \frac{J_S}{s} \sum_{r, \delta} \left[ b_1^r(r)b_2^r(r + \delta) + b_2^r(r + \delta)b_1^r(r) \right] - \frac{\sqrt{2s}}{s} \sum_{r, \delta} \left[ a_1^r(r)a_2^r(r + \delta)b_1^r(r) + a_2^r(r + \delta)a_1^r(r)b_1^r(r) \right] \\
& \quad - \frac{\sqrt{2s}}{s} \sum_{r, \delta} \left[ a_1^r(r)a_2^r(r + \delta)b_2^r(r + \delta) + a_2^r(r + \delta)a_1^r(r)b_2^r(r + \delta) \right] + \ldots
\end{align*}
$$

The remaining interaction terms, which are quartic and quintic, are presumably small in this representation, but we leave these calculations for future work, as our purpose here is simply to construct the relevant formalism.

The essential physics of the $t$–$J$ model in this regime is thus revealed to be that of charged non-Hermitian fermions hopping in a background of spin waves. This represents a novel formulation of the problem of weakly doped anti-ferromagnets that has been extensively studied beginning with the seminal work of Kane et al. [12]. It is a tantalizing possibility that the non-Hermitian quasi-particles defined here may shed new light on the underlying physics of high $T_c$ materials.
4. Conclusion

The applications of non-Hermitian quantum mechanics may extend beyond the realm of fundamental physics into the emergent world of condensed matter. Dyson unwittingly discovered non-Hermitian quantum mechanics in 1956 when he found that high-precision calculations of interacting spin waves in a ferromagnet were facilitated by use of a non-Hermitian Hamiltonian [2]. Dyson’s technique of defining quasi-particles with respect to a non-standard inner product allows for a novel way of writing the $t$–$J$ Hamiltonian; this new form of the $t$–$J$ Hamiltonian may prove more wieldy to calculations and even shed some light on the physics that underlies high-temperature superconductivity, arguably the most outstanding problem in theoretical condensed matter physics [5].

References