Feedback control of unstable periodic orbits in equivariant Hopf bifurcation problems

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Symmetry-breaking Hopf bifurcation problems arise naturally in studies of pattern formation. These equivariant Hopf bifurcations may generically result in multiple solution branches bifurcating simultaneously from a fully symmetric equilibrium state. The equivariant Hopf bifurcation theorem classifies these solution branches in terms of their symmetries, which may involve a combination of spatial transformations and temporal shifts. In this paper, we exploit these spatio-temporal symmetries to design non-invasive feedback controls to select and stabilize a targeted solution branch, in the event that it bifurcates unstably. The approach is an extension of the Pyragas delayed feedback method, as it was developed for the generic subcritical Hopf bifurcation problem. Restrictions on the types of groups where the proposed method works are given. After addition of the appropriately optimized feedback term, we are able to compute the stability of the targeted solution using standard bifurcation theory, and give an account of the parameter regimes in which stabilization is possible. We conclude by demonstrating our results with a numerical example involving symmetrically coupled identical nonlinear oscillators.

1. Introduction

The feedback control scheme of Pyragas [1], also known as ‘time-delayed autosynchronization’, has successfully been applied in a wide variety of experimental and theoretical contexts (see [2] for a review). In particular, Pyragas feedback has been used to stabilize unstable periodic orbits (UPOs) that arise via a generic subcritical
Hopf bifurcation [3–5]. This problem is particularly amenable to a detailed analysis, as solutions and their stability can be found analytically and the bifurcation problem described in detail. Much of the analysis revolves around computation of the stability of the origin and an invocation of generic bifurcation theory to determine the stability of bifurcating periodic orbits.

Pyragas-type feedback control is particularly attractive because it exploits the (temporal) symmetry of the targeted UPO to ensure that control is achieved non-invasively. Specifically, if the period of the UPO is $\tau$, then the feedback is proportional to $y(t) - y(t - \tau)$, where $y(t)$ is the system state vector; by construction this feedback vanishes when the system is in the desired periodic state. The role of this feedback is to alter the linear stability properties of the UPO, without changing the periodic orbit itself.

Motivated by the application of the method to the case of generic subcritical Hopf bifurcation, we now pursue a related problem: is there a systematic way that Pyragas feedback control can be extended to target UPOs that arise in systems containing special symmetry properties? Systems of ordinary differential equations that are equivariant with respect to a group of symmetries arise naturally in many physical situations (see [6], and references therein). The equivariance condition implies that, generically, multiple pairs of complex conjugate eigenvalues may cross the imaginary axis as a parameter is varied through a symmetry-breaking bifurcation point. The equivariant Hopf bifurcation theorem [7] gives existence conditions for particular periodic solution branches to bifurcate simultaneously from the bifurcation point; these conditions are stated in terms of the subgroup of spatio-temporal symmetries retained by the periodic solutions. (The theorem does not address the stability of the bifurcating solutions, which can no longer be inferred from the direction of bifurcation from a stable equilibrium.) The goal of this paper is to develop and investigate the effectiveness of a Pyragas-type feedback that exploits the spatio-temporal symmetry properties of the UPOs associated with equivariant Hopf bifurcation problems.

In this paper, we determine properties of symmetry groups for which the design of a single Pyragas-type feedback control term, appropriately optimized, may be effective in stabilizing a branch of periodic orbits that bifurcates unstably in a generic symmetry-breaking Hopf bifurcation of a stable symmetric equilibrium. Previous work in this area has focused either on travelling wave solutions of the complex Ginzburg–Landau equations [8,9] or on symmetric solutions in coupled oscillator problems [10]. Fiedler et al. [10] consider the use of Pyragas-type feedback in stabilizing solutions arising in a model of two coupled oscillators. Owing to the symmetries in the problem, the delay time can be set to half the period. A similar type of feedback is also used by Nakajima & Ueda [11], who consider the stabilization of particular symmetric periodic solutions in the Lorenz and Duffing equations.

We take a more general approach, and consider equations equivariant under some symmetry group $\Gamma$. In the examples we give, $\Gamma$ is a discrete group, but our approach is also applicable to continuous groups. The methods we use to determine whether or not stability can be achieved follow the same path as developed for the generic Hopf bifurcation problem [3,4]; that is, we analyse the stability of the zero solution, and deduce results about the stability of bifurcating branches using generic bifurcation theory. This is possible because the Pyragas-type feedback term destroys enough of the symmetry of the problem to remove the inherent degeneracy of the Hopf bifurcation problem; it retains only the symmetries associated with the targeted bifurcating UPO.

We explore under what conditions stabilization is possible, and show that there are two mechanisms by which it can fail. The first of these is related to a failure of the method to stabilize periodic orbits near the generic subcritical Hopf bifurcation problem, as described by Fiedler et al. [3]. Their analysis shows that if the imaginary part of the cubic coefficient in the Hopf normal form is zero, then stabilization of the target orbit can never occur, for any choice of the gain parameter. In our symmetric case, if the target orbit bifurcates subcritically, this condition is worsened so that this same coefficient cannot lie in an open interval which includes zero, and the size of this interval may be determined explicitly by the representation of the group of symmetries of the targeted orbit. The second failure mechanism gives a restriction on the types of groups for which this stabilization mechanism can work. Namely, in order for stabilization to be
possible, we must be able to find a group element which fixes only the two-dimensional subspace containing the target orbit. This condition is needed to turn the relevant bifurcation problem into a generic one by lifting the degeneracy associated with the full symmetry group $\Gamma$. This discussion is continued in depth in §2c.

The remainder of this paper is arranged as follows. As background, §2 reviews the results of Fiedler et al. [3] concerning Pyragas feedback control near a subcritical Hopf bifurcation in normal form. It also provides a brief review of the equivariant Hopf bifurcation theorem, and shows how we design the feedback terms to target the appropriate periodic solution. In §§3 and 4, we present the calculations required to compute the stability of the targeted solution branch. We intersperse our calculations for the general case with those for a specific example with $\Gamma = \mathbb{D}_N$. We finish in §5 by giving a numerical example. Section 6 concludes.

2. Problem set-up and design of feedback

(a) Background

Before considering the symmetric problem, we first review the results of Fiedler et al. [3] on the use of Pyragas feedback control to stabilize UPOs arising from a subcritical Hopf bifurcation. They use a slightly different (scaled) form of the equations, but here we consider the following, as in [4]:

$$\dot{z} = (p(\lambda) + i\omega(\lambda))z + (a + ib)|z|^2z + \rho_0 e^{i\beta}[z(t - \tau(\lambda)) - z(t)], \quad z \in \mathbb{C},$$

(2.1)

where $\rho_0 e^{i\beta} \equiv \rho \in \mathbb{C}$ is the gain parameter, $\lambda, a, b, p(\lambda), \omega(\lambda) \in \mathbb{R}$, and $\tau(\lambda)$ is the period of the bifurcating periodic orbit in the uncontrolled system (that is, when $\rho_0 = 0$).

Periodic solutions of the uncontrolled system ($\rho_0 = 0$) have the form $z = r e^{i\omega t}$, which gives (splitting into real and imaginary parts)

$$0 = p(\lambda) + ar^2$$

and

$$\alpha = \omega(\lambda) + br^2.$$

We write $p(\lambda) = p_1 \lambda + O(\lambda^2)$, and, provided $p_1 \neq 0$, we are free to choose our control parameter $\lambda$ so that $p_1 = 1$. We also expand $\omega(\lambda)$ as

$$\omega(\lambda) = \omega_0 + \omega_1 \lambda + O(\lambda^2), \quad \omega_0 > 0.$$

(2.2)

We then find to lowest order that $r = \sqrt{-\lambda/\bar{a}}$, and $\alpha = \omega_0 + (\omega_1 - b/\bar{a})\lambda$. Thus, the period of the orbit is

$$\tau(\lambda) = \frac{2\pi}{\omega_0 + (\omega_1 - c)\lambda} = \tau_0 + \tau_1 \lambda + O(\lambda^2),$$

where $\tau_0 = \frac{2\pi}{\omega_0}, \quad \tau_1 = \frac{-2\pi(\omega_1 - c)}{\omega_0^2}$

(2.3)

and

$$c \equiv \frac{b}{\bar{a}}.$$

(4.4)

Here, $a$ and $b$ are the real and imaginary parts of the cubic nonlinearity in (2.1). The assumption that the orbit is subcritical (i.e. bifurcates into $\lambda < 0$) means that $a > 0$. The parameter $c$ determines how the period of the UPO increases or decreases with an increase in the amplitude of the oscillations. (Note that the expression for $\tau$ in (2.3) still applies even in the case where the orbit bifurcates supercritically.)

Computations of the stability of the origin in (2.1) (details given in [3,4,12]) show that there exists a $\rho_0^c$ such that close to $\lambda = 0$ (and subject to other conditions, discussed below), if $\rho_0 < \rho_0^c$, the origin is stable in $\lambda < 0$ and unstable in $\lambda > 0$. If $\rho_0 > \rho_0^c$, the origin is unstable in $\lambda < 0$ and
stable in $\lambda > 0$. Since the location of the target orbit is unaffected by the feedback, we know that it always bifurcates into $\lambda < 0$, and thus if $\rho_0 > \rho_0^c$, generic bifurcation theory tells us that the orbit must bifurcate stably. The computations give

$$\rho_0^c = \frac{-1}{\tau_0 (\cos \beta + c \sin \beta)}.$$  

The other conditions mentioned above restrict the allowed range of $\beta$ and the amount by which $\rho_0$ can exceed $\rho_0^c$ to ensure that no further Hopf bifurcations, induced by the delay term in (2.1), have destabilized the origin. Additionally, for some values of $\beta$, $\rho_0^c$ may be negative or undefined. Brown et al. [4] further showed that the Hopf bifurcation at $\lambda = 0$, $\rho_0 = \rho_0^c$ is degenerate because the eigenvalues do not in fact cross the imaginary axis as the parameter $\lambda$ is varied. Analysis of this degeneracy allows us to determine the other bifurcation curves in a neighbourhood of this co-dimension-2 point, and hence we can describe this problem in full detail via an unfolding of the degenerate Hopf bifurcation.

The results of Fiedler et al. [3] described above were all obtained by adding the feedback terms directly to the equations in normal form on a centre manifold. Brown et al. [4] were able to show that if feedback terms were instead added to the original equations (not in normal form, and possibly on a space of higher dimension than the centre manifold), then even though a centre manifold reduction of these equations (with feedback) does not result in the equation considered by Fiedler et al. [3], the bifurcation structure remains the same. Explicitly, we considered a system of the form

$$\dot{x} = g(x, \lambda), \quad x \in \mathbb{R}^n,$$  

which contains a subcritical Hopf bifurcation at $\lambda = 0$, and considered adding feedback to obtain the equation

$$\dot{x} = g(x, \lambda) + K [x(t - \tau(\lambda)) - x(t)],$$  

where again $\tau(\lambda)$ is chosen to match the period of the bifurcating orbit in the uncontrolled system. $K$ is a gain matrix, and is chosen so that the feedback acts only in the two centre directions. That is, $K$ consists of a single Jordan block of the form

$$\begin{pmatrix} \rho_0 \cos \beta & -\rho_0 \sin \beta \\ \rho_0 \sin \beta & \rho_0 \cos \beta \end{pmatrix}$$  

and zeros elsewhere. We assumed that, in the uncontrolled system (2.5), all other directions are stable, and they are then unaffected by the addition of feedback and remain stable in the controlled system (2.6). We performed a centre manifold reduction of (2.6) at the Hopf bifurcation at $\lambda = 0$, and showed that the bifurcation structure of this system is identical to that of (2.1). That is, the Hopf bifurcation at $\lambda = 0$ changes from subcritical to supercritical at the same value of $\rho_0 = \rho_0^c$, and the other conditions on parameters required for stabilization remain the same.

In the following, we consider adding feedback to equivariant Hopf bifurcations in normal form, and are able to use the analysis in [4] to determine that the results will be the same if we were to add suitably chosen feedback to the original equations instead. We demonstrate this explicitly with a numerical example involving a ring of symmetrically coupled, identical nonlinear oscillators in §5.

(b) Problem set-up

We now consider a generic $\Gamma$-equivariant Hopf bifurcation problem. Suppose the original equations are written as

$$\dot{y} = \tilde{f}(y; \lambda) \quad \text{and} \quad y \in Y$$  

for some vector space $Y$ and parameter $\lambda \in \mathbb{R}$. We consider a symmetric equilibrium solution of (2.8), one which is invariant under the group $\Gamma$, and assume that it undergoes a Hopf bifurcation as $\lambda$ is varied. The representation of $\Gamma$ on $Y$ determines the number, $n$, of pairs of eigenvalues
which generically cross the imaginary axis simultaneously as $\tilde{\lambda}$ passes through a Hopf bifurcation point. The centre manifold is then $\mathbb{C}^n$, and the action of $\Gamma$ on $\mathbb{C}^n$ in this generic case is called ‘$\Gamma$-simple’ [7]. We can write the equations (on the centre manifold) in normal form as

$$\dot{z} = f(z; \lambda) \quad \text{and} \quad z \in \mathbb{C}^n,$$  

(2.9)

where $\lambda \in \mathbb{R}$ is a parameter. The equivariance condition implies that $f(\gamma(z)) = y(f(z))$ for all $y \in \Gamma$. (Here, $\gamma(z)$ represents a linear transformation of the coordinates on the centre manifold determined by the representation of $\Gamma$ restricted to the centre subspace.) The symmetric equilibrium solution is $z = 0$ for all $\lambda$; that is, $f(0; \lambda) = 0$. We assume, without loss of generality, that the bifurcation point is at $\lambda = 0$, and that, as $\lambda$ increases through zero, $n$ pairs of complex conjugate eigenvalues pass through the imaginary axis with non-zero speed. In particular, we assume that the fully symmetric state ($z = 0$) is stable for $\lambda < 0$ and unstable for $\lambda > 0$.

Just as in the generic Hopf bifurcation problem, a normal form transformation induces a normal form $S^1$ symmetry, where $S^1$ is the circle group. The equivariant Hopf bifurcation theorem [7] states that if a subgroup $\Sigma \subset \Gamma \times S^1$ leaves a two-dimensional subspace invariant (i.e. $\dim \text{Fix} \Sigma = 2$), then there exists a unique branch of periodic orbits bifurcating at $\lambda = 0$ from $z = 0$, such that the isotropy subgroup of the periodic orbits is $\Sigma$. If this solution branch bifurcates supercritically, then it is stable to perturbations in the two-dimensional fixed-point space. However, it may be unstable to other perturbations in the centre manifold that are transverse to this subspace.

The normal form $S^1$ symmetry acts linearly as a rotation of the phase space coordinates about the equilibrium at the origin; in normal form, the periodic orbits are circles, and this rotation is equivalent to a phase shift or a translation of time. In the original equations (2.8), the orbits will not in general be circular. Yet the equivariant Hopf theorem [7] states that any symmetries of the periodic solutions of the normal form equations that involved elements of the group $S^1$ correspond to exact time translation symmetries of the periodic solutions of the original equations (2.8). Specifically, we write a group element $s \in \Sigma \subset \Gamma \times S^1$ as $s = (\sigma, \theta)$, where $\sigma \in \Gamma$ are the spatial symmetries and $\theta \in S^1$ is called the twist; the action of $s$ on a periodic solution of the original system of equations (2.8) is $s(y(t)) = \sigma(y(t - \theta t))$, where $\tau$ is the period of the solution.

Now suppose a particular branch of periodic solutions of (2.8), $y^*(t)$, bifurcates unstably, and suppose $y^*(t)$ has symmetry $\Sigma \subset \Gamma \times S^1$. Let the period of the bifurcating orbit be $\tau$. Then $s(y^*(t)) = \sigma(y^*(t - \theta \tau)) = y^*(t)$ for all $s \in \Sigma$. We add feedback to the original system (2.8), to obtain the system

$$\dot{y} = \tilde{f}(y; \tilde{\lambda}) + K[\sigma y(t - \theta \tau(\tilde{\lambda})) - y(t)],$$  

(2.10)

where, as in [4], we choose $K$ so that it acts only in the $(2n)$ centre directions; that is, it is made up of $n$ Jordan blocks of the form (2.7) and zeros elsewhere, and $(\sigma, \theta) = s \in \Sigma$ is a suitably chosen symmetry group element. Note that the action of $\sigma$ on the stable directions is unimportant because there is no feedback in these directions. The results of [4] can then be applied; that is, the bifurcation structure of the (symmetric) Hopf bifurcation at $\tilde{\lambda} = 0$ in (2.10) will be the same as if we had added feedback to the normal form (2.9). This would result in the equation

$$\dot{z} = \tau(\lambda)[f(z; \lambda) + \rho_0 e^{i\beta}(\sigma z(t - \theta) - z(t))],$$  

(2.11)

(where we have rescaled time so that the period of the orbit is equal to 1), and hence it is sufficient to consider this equation instead.

We discuss further how the particular symmetry element $s = (\sigma, \theta) \in \Sigma$ is optimally chosen in the following sections. Note that the feedback term vanishes on the targeted solution, and hence the feedback is non-invasive and does not alter the location of the targeted orbit in phase space or in parameter space. More complicated feedback terms, for example including multiple symmetry elements, possibly with multiple distinct time delays, are of course possible. Here, we examine the simplest form of a single feedback term only, in order to determine in which situations this feedback can be used to stabilize targeted unstable periodic solutions that bifurcate from a stable equilibrium solution.
The guiding principle for designing the feedback term to work as a stabilization mechanism is to make sure that it will ‘split’ the multiple Hopf bifurcations at \( \lambda = 0 \). That is, we know that, after the feedback is added, there will still be a Hopf bifurcation at \( \lambda = 0 \), because the location of the original target periodic orbit in parameter space, as well as phase space, is unchanged. In order for our stabilization mechanism to work, we require that this Hopf bifurcation is generic; that is, only a single pair of eigenvalues cross the real axis at \( \lambda = 0 \). If the original orbit bifurcates supercritically, this splitting will be sufficient for stabilization, so long as the bifurcation at \( \lambda = 0 \) is from a stable equilibrium. If the original orbit bifurcates subcritically, we also need to change the criticality of the bifurcation using the time-delayed term, and in this regard we incorporate the approach developed in [3,4]. Further details of these ideas are given in later sections.

(c) Groups

Before we continue with the computations, we present a brief discussion about the types of groups for which our stabilization method will fail to work. As discussed above, the stabilization mechanism requires that the symmetric Hopf bifurcation at \( \lambda = 0 \) be split so that the bifurcation to the target orbit is a generic one associated with a single complex conjugate pair. In order to do this with a single feedback term for a UPO with symmetry \( \Sigma \subset \Gamma \times S^1 \), we must be able to choose an element \( s \in \Sigma \) which satisfies

\[
\text{Fix } s = \text{Fix } \Sigma, \tag{2.12}
\]

where \( \text{Fix } \Sigma \) is the subspace of \( \mathbb{C}^n \) that is invariant under the group of transformations \( \Sigma \) (acting on the centre coordinates). More precisely, recall that, for a group acting on \( z \in \mathbb{C}^n \), the isotropy subgroup \( H_z \) is

\[
H_z = \{ h \in H : hz = z \},
\]

and for \( H_z \) an isotropy subgroup, the fixed-point subspace \( \text{Fix } H_z \) is

\[
\text{Fix } H_z = \{ \zeta \in \mathbb{C}^n : h\zeta = \zeta \forall h \in H_z \}.
\]

As we are considering periodic solutions that are guaranteed by the equivariant Hopf bifurcation theorem, we know that \( \dim(\text{Fix } \Sigma) = 2 \).

Condition (2.12) can clearly be satisfied for cyclic subgroups \( \Sigma \), when one can choose \( s \) to be a generator of \( \Sigma \). In §3, we use the group \( \mathbb{D}_N \) as an example. This group falls into the category that all subgroups \( \Sigma \) associated with bifurcating branches are cyclic. It can also be shown that in the example given by Dias & Rodrigues [13] of Hopf bifurcation with \( S_N \) symmetry, for all subgroups \( \Sigma \) associated with a bifurcating branch guaranteed by the equivariant Hopf bifurcation theorem (that is, those listed in table 1 in [13]), it is always possible to choose an element \( s \) with this property, even though not all of these subgroups are cyclic.

However, there are a large class of groups for which it is not possible to choose \( s \) so that condition (2.12) is satisfied, and we now give an example of such a group representation. The group we consider is the alternating group \( A_5 \), and we consider a representation arising in the analysis, by Dias & Rodrigues [13], of the Hopf bifurcation problem with \( S_N \) symmetry. The action of \( A_5 \) is by even permutations of coordinates on the space

\[
\mathbb{C}^5,0 = \{(z_1, \ldots, z_5) \in \mathbb{C}^5 | z_1 + \cdots + z_5 = 0 \}.
\]

We can see that this action is \( \Gamma \)-simple as we can write

\[
\mathbb{C}^5,0 \cong \mathbb{R}^5,0 \oplus \mathbb{R}^5,0
\]

and \( A_5 \) acts absolutely irreducibly on

\[
\mathbb{R}^5,0 = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 | x_1 + \cdots + x_5 = 0 \}.
\]
Now, consider the purely spatial subgroup $A_4 \subset A_5 \times S^1$, which fixes the two-dimensional subspace
\[
\text{Fix } A_4 = \{(z_1, z_1, z_1, -4z_1) | z_1 \in \mathbb{C})
\]
All non-trivial elements of $A_4$ are either 3-cycles or a pair of 2-cycles, and it is simple to show that both of these types of elements have representations that have repeated eigenvalues equal to 1. Thus, we conclude that there is no single element with $\text{Fix } s = \text{Fix } \Sigma$.

(d) Framework for analysis

As discussed above, the stability of the targeted solution can be determined entirely by computation of the stability of the origin, provided that we have split the multiple bifurcations at $\lambda = 0$. To compute the stability of the origin, we linearize equation (2.11) about $z = 0$ and diagonalize with respect to the action of the spatial part of the symmetry $s$, $\sigma$. This results in $n$ complex equations
\[
\dot{z}_j = \tau(\lambda)(p(\lambda) + i o(\lambda))z_j + \tau(\lambda)\rho(e^{i\phi_j}z_j(t - \theta) - z_j(t)), \quad z_j \in \mathbb{C}, \quad j = 1, \ldots, n
\]  
(2.13)
where $e^{i\phi_j}$ are the eigenvalues of the matrix representing the spatial symmetry $\sigma$. By definition, the target orbit lies in the subspace $\text{Fix } \Sigma$, and, without loss of generality, we can write $\text{Fix } \Sigma = \{(z_1, 0, \ldots, 0)\}$ (recall that $\dim \text{Fix } \Sigma = 2$ by the equivariant Hopf theorem).

There will be four cases, which we will consider separately.

Case 1: The targeted solution branch bifurcates supercritically (that is, it exists in $\lambda > 0$) and the element $s$ from the subgroup $\Sigma$ contains no twist (i.e. this symmetry is purely spatial).

Case 2: The targeted solution branch bifurcates supercritically, and $s$ contains a non-trivial twist.

Case 3: The targeted solution branch bifurcates subcritically (that is, it exists in $\lambda < 0$) and $s$ contains no twist.

Case 4: The targeted solution branch bifurcates subcritically and $s$ contains a non-trivial twist.

In all cases, the twist is given by
\[
\theta = \frac{\phi_1}{2\pi},
\]  
(2.14)
where $e^{i\phi_1}$ is the eigenvalue of $\sigma$ (which is the spatial part of the symmetry $s \in \Sigma$) corresponding to the eigenspace $\text{Fix } \sigma$.

In cases 2 and 4, where the twist is non-trivial, we choose to have the time-delay $\theta = \phi_1/2\pi$.

In cases 1 and 3, where there is no twist, then $\theta$ is the identity element in $S^1$. In these cases, the eigenvalue associated with $\sigma$, corresponding to the eigenspace $\text{Fix } \sigma$, is $e^{i\phi_1} = 1$ in (2.13). If we choose $\theta = 0$ as the identity element in $S^1$, then there is no time delay, while a choice of $\theta = 1$ corresponds to a delay by one period of the targeted orbit. We treat the sub- and supercritical cases differently in this regard. In the supercritical case (case 1), it turns out that stabilization is possible without a delay term, and so we choose $\theta = 0$, which greatly simplifies the problem. However, in the subcritical case (case 3), spatial feedback alone is insufficient for stabilization, and so we choose $\theta = 1$, resulting in a time-delayed term in the feedback.

The desired effect of the feedback control also differs between these four cases. In cases 1 and 2, the targeted orbit bifurcates supercritically, and we can create a generic non-symmetric Hopf bifurcation at $\lambda = 0$ by using the feedback to ‘split’ the symmetric bifurcation so that a single pair of eigenvalues crosses the imaginary axis. The bifurcating periodic solution will be stable if the bifurcation is supercritical from a stable equilibrium (i.e. the other Hopf bifurcations are moved into $\lambda > 0$).

In the subcritical cases 3 and 4, we again break the symmetry of the bifurcation, but this is not sufficient for stabilization. In addition, we need to change the stability of the origin, so that (in a neighbourhood of $\lambda = 0$) the origin is unstable for $\lambda < 0$ and stable for $\lambda > 0$, as in the Pyragas control for a generic subcritical Hopf example [3]. Since the feedback vanishes on the
targeted solution, the targeted solution will still exist in \( \lambda < 0 \), and the change in stability of the origin corresponds to a change in criticality of the bifurcation, and hence the solution will now bifurcate stably.

We continue the computations required to determine the stability of the targeted orbit in §4.

### 3. Set-up for \( \mathbb{D}_N \) example

As an example system for demonstrating our set-up and results, we consider the group \( \mathbb{D}_N \) (the symmetry group of a regular \( N \)-gon) and an action on a four-dimensional centre subspace \( \mathbb{C}^2 \) (i.e. \( n = 2 \)). The symmetry group \( \mathbb{D}_N \) arises, for example, when a ring of identical oscillators has symmetric, nearest-neighbour coupling. (An example of such a coupled nonlinear oscillator problem, with \( \mathbb{D}_3 \) symmetry, is presented in §5b.)

The details of the group action are taken from an example of Hopf bifurcation with \( \mathbb{D}_N \) symmetry given in [7]. The group is generated by a rotation by \( 2\pi/N \) that we call \( \zeta \), and a reflection \( \kappa \), which act on \( \mathbb{C}^2 \) as

\[
\zeta(z_1, z_2) = (e^{2\pi i/N}z_1, e^{-2\pi i/N}z_2)
\]

and

\[
\kappa(z_1, z_2) = (z_2, z_1).
\]

The \( S^1 \) symmetry acts in normal form as a phase shift

\[
\Theta(z_1, z_2) = (e^{2\pi i/\eta}z_1, e^{2\pi i/\eta}z_2).
\]

In the original problem, this manifests itself as a time-shift by an appropriate fraction of a period. In §5b, we give an example showing this explicitly.

As shown in [7], the group \( \mathbb{D}_N \times S^1 \) has three isotropy subgroups, not related by symmetry, for which the fixed-point subspace is two dimensions. One is an order-\( N \) cyclic group \( \mathbb{Z}_N \), and the other two are order-2 groups. The equivariant Hopf bifurcation theorem guarantees that these three types of periodic solutions, possessing the associated symmetries, will bifurcate simultaneously from the trivial, symmetric equilibrium in a Hopf bifurcation.

For all values of \( N \), the \( \mathbb{Z}_N \) subgroup is generated by the element \( (\zeta, -1/N) \); that is, a spatial rotation of \( 2\pi/N \) followed by a phase shift of \(-1/N\). The actions of the order-2 groups differ slightly in the three cases \( N \) odd, \( N = 2 \mod 4 \) and \( N = 0 \mod 4 \). In the case \( N \) odd, the first order-2 group is generated by the element \( \kappa \), and hence is purely spatial. The second order-2 group is generated by the element \( (\kappa, 1/2) \); that is, a spatial reflection followed by a phase shift of \( 1/2 \). The actions of the order-2 groups in the case where \( N \) is even are similar, and details can be found in [7]. The three isotropy subgroups and fixed-point subspaces for \( N \) odd are given in table 1.

### 3.1. Set-up for \( \mathbb{D}_N \) example

Each of the isotropy subgroups in table 1 is cyclic, and thus satisfies the conditions on the group given in the previous section; that is, that there exists a group element \( s \in \Sigma \) for which \( \text{Fix} \ s = \text{Fix} \ \Sigma \). The two \( \mathbb{Z}_2 \) subgroups are of order 2, and so there is only one non-trivial group element and hence no choice about the element \( s \) which is to be used in the feedback. The \( \mathbb{Z}_N \) subgroup has \( N \) elements, and so we have a choice of \( s \). To demonstrate the construction of feedback described earlier, we write down the equations with appropriate feedback for the \( \mathbb{Z}_N \) and \( \mathbb{Z}_2(\kappa) \) groups, in both supercritical and subcritical cases.

### Table 1. The isotropy subgroups of \( \mathbb{D}_N \) (\( N \) odd) with fixed-point subspace of dimension 2. The action of the elements \( \zeta, \kappa \) and the normal form symmetry \( \Theta \) are given in equations (3.1)–(3.3).

<table>
<thead>
<tr>
<th>subgroup</th>
<th>twist</th>
<th>fixed-point subspace</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_N(\zeta, -1/N) )</td>
<td>( 1/N )</td>
<td>( \langle z_1, 0 \rangle )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2(\kappa) )</td>
<td>0</td>
<td>( \langle z_1, z_1 \rangle )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2(\kappa, \frac{1}{2}) )</td>
<td>( \frac{1}{2} )</td>
<td>( \langle z_1, -z_1 \rangle )</td>
</tr>
</tbody>
</table>
(a) $\Sigma = \mathbb{Z}_N$

This subgroup has symmetry elements with spatial parts $e^{im}$, where $m = 0, \ldots, N - 1$, and the representation of each is a diagonal matrix with eigenvalues $e^{\pm 2\pi im/N}$. Each element has an associated twist of $m/N$. Thus, choosing $s = e^{im}$, for some $m$, the four-dimensional equations on the centre manifold, linearized and with added feedback, are

$$
\dot{z}_1 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_1 + \tau(\lambda)\rho \left( e^{2\pi im/N}z_1 \left( t - \frac{m}{N} \right) - z_1(t) \right) \quad (3.4a)
$$

and

$$
\dot{z}_2 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_2 + \tau(\lambda)\rho \left( e^{-2\pi im/N}z_2 \left( t - \frac{m}{N} \right) - z_2(t) \right) \quad (3.4b)
$$

We use these equations in both cases when the $\mathbb{Z}_N$ solution branch bifurcates supercritically or subcritically. We further discuss how to optimize the choice of group element (that is, the choice of $m$) in §5.

(b) $\Sigma = \mathbb{Z}_2(\kappa)$

This subgroup consists of purely spatial symmetries, and thus has no twist. The element $\kappa$ has a representation with eigenvalues $\pm 1$. In the case when the branch bifurcates supercritically, we use only spatial feedback with no time delay, giving equations of the form

$$
\dot{z}_1 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_1 + \tau(\lambda)\rho(z_1(t) - z_1(t))
$$

and

$$
\dot{z}_2 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_2 + \tau(\lambda)\rho(-z_2(t) - z_2(t)).
$$

In this case, it is trivial to see that the first equation is unchanged, and so it still contains a supercritical Hopf bifurcation at $\lambda = 0$. The Hopf bifurcation in the second equation is shifted, and so for stabilization of the targeted orbit to occur, all that remains to check is that this Hopf bifurcation has moved into $\lambda > 0$, so that the Hopf bifurcation at $\lambda = 0$ in the first equation is from a stable equilibrium.

In the subcritical case, we also include a time delay of one full period in the feedback, giving equations

$$
\dot{z}_1 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_1 + \tau(\lambda)\rho(z_1(t - 1) - z_1(t))
$$

and

$$
\dot{z}_2 = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_2 + \tau(\lambda)\rho(-z_2(t - 1) - z_2(t)).
$$

4. Computations in general case

In this section, we continue the computations from §2d for a generic equivariant Hopf bifurcation. Recall that we are interested in determining the stability of the origin for the system of $n$ equations

$$
\dot{z}_j = \tau(\lambda)(p(\lambda) + i\omega(\lambda))z_j + \tau(\lambda)\rho_0 e^{i\phi_j} \left( e^{i\theta}z_j(t - \theta) - z_j(t) \right), \quad z_j \in \mathbb{C}, \quad j = 1, \ldots, n,
$$

where $p(\lambda) = \lambda + O(\lambda^2)$ and $\omega(\lambda)$ and $\tau(\lambda)$ are given by equations (2.2) and (2.3).

The number of unstable directions associated with the equilibrium at the origin changes at bifurcations. We determine the location of these bifurcations in the $(\lambda, \rho_0)$ parameter plane by writing $z_j = e^{i\eta_j}$, with $\eta_j = \mu_j + i\nu_j$; bifurcation sets are associated with $\mu_j = 0$. (Note that this
includes steady-state bifurcations if \( v_j = 0 \) for some \( j \).) Separating into real and imaginary parts gives

\[
\mu_j = \tau(\lambda)p(\lambda) + \tau(\lambda)\rho_0[e^{-\mu_j \theta} \cos(\beta + \phi_j - v_j \theta) - \cos \beta]
\]

(4.1a)

and

\[
v_j = \tau(\lambda)\omega(\lambda) + \tau(\lambda)\rho_0[e^{-\mu_j \theta} \sin(\beta + \phi_j - v_j \theta) - \sin \beta].
\]

(4.1b)

Setting \( \mu_j = 0 \), we obtain the following parametric equations for the bifurcation curves in the \((\lambda, \rho_0)\)-plane:

\[
0 = p(\lambda) + \rho_0[\cos(\beta + \phi_j - v_j \theta) - \cos \beta]
\]

(4.2a)

and

\[
v_j = \tau(\lambda)\omega(\lambda) + \tau(\lambda)\rho_0[\sin(\beta + \phi_j - v_j \theta) - \sin \beta].
\]

(4.2b)

These curves are parametrized by the Hopf frequency \( v_j \).

In the following sections, we use these equations to determine the location of Hopf bifurcations and hence deduce conditions that ensure the targeted orbit is stable for \(|\lambda|\) sufficiently small.

(a) Location of Hopf curves in \((\lambda, \rho_0)\) plane

Each of the \( n \) equations in (4.2) has multiple solution branches, swept out in the \((\lambda, \rho_0)\)-plane as the frequency \( v_j \) increases. In this section, we discuss the arrangement of these curves in this parameter plane. While the delay \( \tau(\lambda) \) is determined by the original Hopf bifurcation problem (e.g. in terms of the nonlinear coefficient \( c \) defined by (2.4) and (2.1)), we are free to choose \( \beta \) and \( \rho_0 \). In particular, we wish to determine the possible choices of \( \beta \) and \( \rho_0 \) for which the target orbit bifurcates stably at \( \lambda = 0 \).

First, note that, as a consequence of the way in which the problem is set up, each of the \( n \) equations in (4.2) has a solution at \( \lambda = \rho_0 = 0 \), and each of these has \( v_j = 2\pi \). The solution corresponding to the equation with \( j = 1 \) continues along the line \( \lambda = 0 \), for all \( \rho_0 \geq 0 \), with \( v_1 = 2\pi \).

Simple substitution of these values into the equation can confirm this.

Before we give general locations of the Hopf curves, we discuss a couple of degenerate cases, which we avoid by choosing \( \beta \) appropriately. These cases arise when \( \cos(\beta + \phi_j - 2\pi \theta) - \cos \beta = 0 \) and hence at \( \lambda = \rho_0 = 0 \) the second term in equation (4.2a) is zero to quadratic order. This can occur in two ways. In the first case, \( \phi_j - 2\pi \theta = 2\pi k \) for some \( k \in \mathbb{Z} \). By our assumptions on the group in §2c this only occurs when \( j = 1 \), and this curve through \( \lambda = \rho_0 = 0 \) is at \( \lambda = 0 \) for all \( \rho_0 > 0 \). In the second case, \( \beta = -(\beta + \phi_j - 2\pi \theta) + 2\pi k \), or equivalently, in order to avoid this degeneracy, we choose

\[
\beta \neq \pi(k + \theta) - \frac{\phi_j}{2}, \quad j = 1, \ldots, n
\]

(4.3)

for any integer \( k \). Since there are a finite number of \( \phi_j \), we can choose \( \beta \) to satisfy this condition.

We now give two lemmas regarding the arrangement of the Hopf bifurcation curves in \((\lambda, \rho_0)\) space. The first describes the slope of the Hopf curves through \( \lambda = \rho_0 = 0 \) with \( j = 2, \ldots, n \), and the second gives the points at which the Hopf curves cross the \( \lambda = 0 \) axis.

**Lemma 4.1.** Define the slope of each of the Hopf curves given by equations (4.2) through the point \( \lambda = \rho_0 = 0 \) as

\[
\rho_j'(0) = \left. \frac{d\rho_j}{d\lambda} \right|_{\lambda=\rho_0=0}.
\]

Then

\[
\rho_j'(0) = \frac{1}{\cos(\beta + \phi_j - 2\pi \theta) - \cos \beta}, \quad j = 2, \ldots, n.
\]
Proof. Differentiate equation (4.2a) with respect to $\lambda$, and set $\lambda = \rho_0 = 0$ (which implies that $\nu_j = 2\pi$, from above), to find

$$0 = 1 - \frac{d\rho_0}{d\lambda} [\cos(\beta + \phi_j - 2\pi \theta) - \cos \beta].$$

Recall that we choose $\beta$ so that $\cos(\beta + \phi_j - 2\pi \theta) - \cos \beta \neq 0$, so rearranging gives the desired expression.

\[\text{Lemma 4.2.}\]

1. If $\theta \sin \beta \neq 0$, then, in the two-parameter space $(\lambda, \rho_0)$, Hopf bifurcation curves cross the $\rho_0$ axis at a sequence of points

$$\rho_0 = \rho_j^k = \frac{2\pi \theta + 2\pi k - 2\phi_j}{2\theta \tau_0 \sin \beta}, \quad k \in \mathbb{Z}, \quad j = 1, \ldots, n.$$

2. If $\theta \sin \beta = 0$, then there are no Hopf curves which cross the line $\lambda = 0$ in $(\lambda, \rho_0)$ space, except the original one at $\lambda = \rho_0 = 0$.

Proof. Substitute $\lambda = 0$ into (4.2), giving

$$0 = \rho_0 (\cos(\beta + \phi_j - \nu_j \theta) - \cos \beta) \quad (4.4a)$$

and

$$\nu_j = 2\pi + \tau_0 \rho_0 (\sin(\beta + \phi_j - \nu_j \theta) - \sin \beta). \quad (4.4b)$$

For $\rho_0 \neq 0$, we require $\cos \beta = \cos(\beta + \phi_j - \nu_j \theta)$, which means that either (a) $\beta + \phi_j - \nu_j \theta = 2\pi k + \beta$ or (b) $\beta + \phi_j - \nu_j \theta = 2\pi k - \beta$, for some $k \in \mathbb{Z}$.

Substituting case (a), for which $\beta + \phi_j - \nu_j \theta = 2\pi k + \beta$, into (4.4b) implies that $\nu_j = 2\pi$. Substituting this back into (a) gives $\phi_j - 2\pi \theta = 2\pi k$. This is only solved when $j = 1$ (by the assumption made in §2c), and again gives the solution with $\lambda = 0$ for all $\rho_0 > 0$.

In case (b), we substitute $\beta + \phi_j - \nu_j \theta = 2\pi k - \beta$ into (4.4b), and find

$$2\beta + \phi_j - 2\pi k = 2\pi \theta - 2\rho_0 \tau_0 \theta \sin \beta, \quad (4.5)$$

which, if $\theta \sin \beta \neq 0$, can be rearranged to give the desired expression.

If $\sin \beta = 0$, then $\beta = \pi l$, for some $l \in \mathbb{Z}$. Substituting this into (4.5) gives $\phi_j = 2\pi (\theta + k - l)$, which, as discussed above, is only satisfied at $j = 1$ and gives the solution at $\lambda = 0$ for all $\rho_0 > 0$.

If $\theta = 0$, then (4.5) gives $2\beta + \phi_j = 2\pi k$, which by (4.3) does not occur; thus there will be no further solutions.

(b) Stability of zero solution in a neighbourhood of $\lambda = 0$

We know that, for any choice of $\rho_0$, the number of unstable eigenvalues of the solution $z = 0$ changes at $\lambda = 0$, as there is always a Hopf bifurcation at $\lambda = 0$ for all $\rho_0$ in the equations (4.2) for $j = 1$. The following lemma describes how the stability of the origin changes at this point.

\[\text{Lemma 4.3.}\]

Define

$$\mu_1'(0) = \frac{d\mu_1}{d\lambda} \bigg|_{\lambda=0}$$

1. If $\theta (\cos \beta + c \sin \beta) = 0$, then $\mu_1'(0) > 0$ for all $\rho_0 \geq 0$.
2. If $\theta (\cos \beta + c \sin \beta) \neq 0$, let

$$\rho_0^c = \frac{-1}{\tau_0 \theta (\cos \beta + c \sin \beta)}.$$

(a) If $\rho_0^c > 0$, then

- If $0 \leq \rho_0 < \rho_0^c$, then $\mu_1'(0) > 0$.
- If $\rho_0 > \rho_0^c$, then $\mu_1'(0) < 0$. 

(b) If $\rho_0^c < 0$, then

- If $\rho_0 < \rho_0^c$, then $\mu_1'(0) < 0$.
- If $\rho_0 > \rho_0^c$, then $\mu_1'(0) > 0$. 

(c) If $\rho_0^c = 0$, then $\mu_1'(0)$ is undefined.
— If $\rho_0 = \rho_0^k$, then $\mu'_1(0) = 0$.

(b) If $\rho_0^k < 0$ then $\mu'_1(0) > 0$ for all $\rho_0 \geq 0$.

Proof. Differentiate equations (4.1) for $j = 1$ with respect to $\lambda$, for fixed $\rho_0$. Recall that $\phi_1 = 2\pi \theta$, and set $\lambda = 0$, $\mu_1 = 0$ and $\nu_1 = 2\pi$. We find

$$\frac{d\mu_1}{d\lambda} = \tau_0 + \tau_0 \rho_0 \left( -\theta \frac{d\mu_1}{d\lambda} \right) \cos \beta + \tau_0 \rho_0 \left( -\theta \frac{d\nu_1}{d\lambda} \right) (-\sin \beta)$$

and

$$\frac{d\nu_1}{d\lambda} = \tau_1 \omega_0 + \tau_0 \omega_1 + \tau_0 \rho_0 \left( -\theta \frac{d\mu_1}{d\lambda} \right) \sin \beta + \tau_0 \rho_0 \left( -\theta \frac{d\nu_1}{d\lambda} \right) (\cos \beta).$$

Note that $\tau_1 \omega_0 + \tau_0 \omega_1 = \tau_0 p_1 c$. Eliminating $d\nu_1/d\lambda$ and rearranging gives

$$\frac{d\mu_1}{d\lambda} \bigg|_{\lambda = 0} = \frac{\tau_0 \left[ \tau_0 \rho_0 \theta (\cos \beta + c \sin \beta) + 1 \right]}{(1 + \tau_0 \rho_0 \theta \cos \beta)^2 + (\tau_0 \rho_0 \theta \sin \beta)^2}.$$ 

Since the denominator is positive, and $\tau_0 > 0$, the statements of the lemma clearly follow. ■

(c) Criteria for stabilization

We are now in a position to give criteria under which the targeted orbit is stable. Recall that the parameter $c$ is fixed by the original problem, and it is the ratio of the imaginary and real parts of the cubic coefficient in the normal form (2.1). Note that the parameter $c$ only enters the computations through the delay time $\tau$.

We are free to choose the feedback gain parameters $\beta$ and $\rho_0$, and, in any given problem, we may also have some freedom in choosing the symmetry element $s$ used in the feedback, which specifies the $\phi_j$ associated with this representation and the time lag delay $\theta$. In the following section, we continue our example of Hopf bifurcation with $D_N$ symmetry, where we also give a discussion on choosing an optimal $s$.

One hurdle to overcome in writing down general criteria is that we wish to compute the smallest positive value of the $\rho_j^k$; that is, the smallest value of $\rho_0 > 0$ for which there is an additional delay-induced Hopf bifurcation at $\lambda = 0$. This is complicated because the $\rho_j^k$ depend on $\phi_j$, $\theta$ and $\beta$, and the smallest positive one depends on these in a discontinuous fashion. For convenience, we write

$$\hat{\rho}_0 = \min_{j,k,\rho_j^k > 0} \rho_j^k.$$ 

We now consider the two cases where the targeted orbit bifurcates supercritically or subcritically separately.

(i) Supercritical case

As discussed in §2b, if the targeted orbit bifurcates supercritically, then the feedback is used to split apart the multiple bifurcations at $\lambda = 0$. In addition, we need to ensure that, at the choice of parameters used, no additional instabilities have been introduced by further Hopf bifurcations. For this, we require that $\mu'_1(0) > 0$, and that $\rho_0 < \hat{\rho}_0$. We further require that, of the $n$ Hopf curves which pass through $\lambda = \rho_0 = 0$, the only one which is vertical (i.e. exists for $\lambda = 0$ and all $\rho_0 > 0$) is that for $j = 1$, and that the other $n - 1$ curves have positive slope. This desired arrangement of Hopf bifurcation curves is shown schematically in figure 1.
Figure 1. The schematic arrangement of Hopf bifurcation curves required for stabilization. Numbers indicate the number of unstable eigenvalues of the origin. The red (solid) curves are those arising in the $j = 1$ equations in (4.2), and the black (dashed) curves arise in equations with $j = 2, \ldots, n$. We require that these $n - 1$ Hopf curves passing through the origin have positive slope so that for $\rho_0 > 0$ there is only a single Hopf bifurcation at $\lambda = 0$ and it is from a stable equilibrium. In the case that the targeted orbit bifurcates supercritically, we need to choose $\rho_0 < \rho_0^c$. In the case that the targeted orbit bifurcates subcritically, we need to choose $\rho_0^c < \rho_0 < \rho_0$. (Online version in colour.)

In order to obtain these conditions, we require that
\[ \rho_j'(0) > 0, \quad j = 2, \ldots, n \] (4.6)
and
\[ \phi_j \neq 2(\pi(k + \theta) - \beta), \quad k \in \mathbb{Z}, \ j = 2, \ldots, n, \] (4.7)
and then we must choose a $\rho_0$ so that
\[ \mu_1'(0) > 0, \] (4.8)
and, if $\theta \sin \beta \neq 0$, we also require that
\[ 0 < \rho_0 < \hat{\rho}_0. \] (4.9)

In order to satisfy (4.8), lemma 4.3 tells us we must have at least one of the following:
\[ \theta(\cos \beta + c \sin \beta) = 0, \] (4.10)
\[ 0 < \rho_0 < \rho_0^c \] (4.11)
or
\[ \rho_0^c < 0. \] (4.12)

We note that these conditions can be satisfied by choosing $\beta = 0$, and any value of $\rho_0 > 0$. This can be seen by noting that, for (4.6), we must have $\cos \beta > \cos(\beta + \phi_j)$ for $j = 2, \ldots, n$, and for (4.7) we must have $\beta \neq k\pi - \phi_j/2$ for any $k$ and for $j = 2, \ldots, n$. We know (by the conditions on the group) that, for $j \neq 1$, $\phi_j \neq 0 \mod 2\pi$, and so $k\pi - \phi_j/2 \neq 0 \mod \pi$, and hence these are clearly satisfied if $\beta = 0$. Finally, for (4.8), either $\theta = 0$ and (4.10) is satisfied, or $\theta \neq 0$ and then $\rho_0^c < 0$, and (4.12) is satisfied.

(ii) Subcritical case

If the targeted orbit bifurcates subcritically, then, as well as splitting apart the multiple bifurcations at $\lambda = 0$, the feedback must also change the criticality of the Hopf bifurcation at $\lambda = 0$ (in the $j = 1$ equation) so that the targeted orbit now bifurcates supercritically. This change of criticality is the same as is required in the generic subcritical Hopf bifurcation example considered...
by Fiedler et al. [3]. This is achieved by changing the stability of the origin so that $\mu'_1(0) < 0$, which means, from lemma 4.3, that we must have

$$\rho_0 > \rho_0^c > 0.$$  \hfill (4.13)

In addition, we again require that $\rho_j'(0) > 0$, which means that we must choose $\beta$ so that

$$\cos(\beta + \phi - 2\pi \theta) - \cos \beta > 0 \quad \text{and} \quad j = 2, \ldots, n$$  \hfill (4.14)

and $\beta$ is also again restricted by

$$\phi_j \neq 2(\pi(k + \theta) - \beta) \quad \text{and} \quad k \in \mathbb{Z}, \quad j = 2, \ldots, n.$$  \hfill (4.15)

Finally, we must ensure that no additional instabilities are introduced by the feedback, so we must choose $\rho_0 < \hat{\rho}_0$. Again, figure 1 shows the desired arrangement of the Hopf bifurcation curves.

We begin with condition (4.14). Write $\psi_j = \pi \theta - \phi_j / 2 \mod \pi$ (note that $\psi_1 = 0$), then rearranging we can write (4.14) as

$$\cos \beta > \sin \beta \cot \psi_j, \quad j = 2, \ldots, n.$$  \hfill (4.16)

We claim that this is equivalent to

$$\beta \in \bigcap_{j=2}^{n} [(0, \psi_j) \cup (\psi_j + \pi, 2\pi)].$$  \hfill (4.17)

This can be seen by first considering $\beta \in (0, \pi)$, then (4.16) becomes

$$\cot \beta > \cot \psi_j \Rightarrow \beta \in (0, \psi_j), \quad j = 2, \ldots, n$$

or, if we choose $\beta \in (\pi, 2\pi)$, (4.16) becomes

$$\cot \beta < \cot \psi_j \Rightarrow \beta \in (\psi_j + \pi, 2\pi), \quad j = 2, \ldots, n.$$  

Finally, note that (4.16) is trivially satisfied if $\beta = 0$, and trivially fails if $\beta = \pi$.

Condition (4.15) can be rewritten as

$$\beta \neq \pi k + \psi_j, \quad j = 2, \ldots, n$$  \hfill (4.18)

and gives no further restrictions on $\beta$ other than those already given by (4.17).

The remaining conditions require us to choose a $\rho_0$ satisfying

$$\rho_0^c < \rho_0 < \hat{\rho}_0.$$  \hfill (4.19)

A necessary condition for us to be able to do this is that $\rho_0^c < \hat{\rho}_0$. We can write

$$\rho_j^k \equiv \frac{\psi_j + \pi k - \beta}{\theta \tau_0 \sin \beta}, \quad k \in \mathbb{Z}, \quad j = 1, \ldots, n.$$  

Suppose we take $\beta \in (0, \pi)$, then (4.18) tells us that we must choose $\beta \in (0, \psi_m)$, where $\psi_m = \min_{j=2}^{n}(\psi_j)$. Thus, the numerators of the $\rho_j^k$ take their smallest positive value with $k = 0$ and the smallest of these will have $\psi_j = \psi_m$. Thus,

$$\hat{\rho}_0 = \frac{\psi_m - \beta}{\theta \tau_0 \sin \beta}.$$  

The necessary condition required to be able to choose $\rho_0$ to satisfy (4.19) thus becomes

$$\frac{\psi_m - \beta}{\theta \tau_0 \sin \beta} > -\frac{1}{\tau_0 \theta (\cos \beta + c \sin \beta)}.$$  

which we can rewrite as

$$g(\beta) < -c, \quad (4.20)$$

where

$$g(\beta) = \cot \beta + \frac{1}{\psi_m - \beta}. \quad (4.21)$$

Recall that $c$ is the ratio of the imaginary to the real parts of the coefficient of the cubic term in the normal form equations in the subcritical case (2.1), and so we have no control over its value.

Note that, for any $\beta$ and $\psi_m$, equation (4.20) is satisfied for sufficiently negative $c$. However, $g(\beta)$ has a minimum at $\beta = \beta^\star$, where $g(\beta^\star) > 0$. Thus, if $-c < g(\beta^\star)$, it is not possible to choose a $\beta \in (0, \psi_m)$ which satisfies the required conditions.

Similar calculations follow if we instead suppose that we choose $\beta \in (\pi, 2\pi)$, giving the result that we can choose an appropriate $\beta$ so long as $c > h(\beta^\dagger)$ where

$$h(\beta) = \frac{-1}{\psi_M + \pi - \beta} - \cot \beta \quad (4.22)$$

has a maximum at $\beta^\dagger$, and $\psi_M = \max_{j=2}^n(\psi_j)$.

Together these conditions give a range of $c$, which includes $c = 0$, for which we are not able to choose any $\beta$ to satisfy all the required conditions for stabilizing the targeted periodic orbit.

(d) Summary

We now give a summary of the conditions under which stabilization is possible, and how to choose the symmetry element $s$ and gain parameter $\rho_0 e^{i\beta}$.

First, a symmetry element $s = (\sigma, \theta) \in \Sigma \subset \Gamma \times S^1$ must be chosen so that $\text{Fix } s = \text{Fix } \Sigma$; that is, the fixed-point subspace of $s$ is two-dimensional. One should then compute the eigenvalues $\phi_j$ of the representation of the spatial part of $s$ and $\sigma$.

In the case where the target orbit bifurcates supercritically, it is sufficient to choose $\beta = 0$ and any $\rho_0 > 0$ in order to obtain stabilization.

If the target orbit bifurcates subcritically, the parameter $c$ gives restrictions on which gain parameters can be chosen for stabilization. If $-g(\beta^\star) < c < h(\beta^\dagger)$ (for $g$ and $h$ as given in equations (4.21) and (4.22); a range which includes $c = 0$), then there is no choice of $\beta$ and $\rho_0$ for which stabilization is possible. If $c$ is outside of this range, then choosing $\beta = \beta^\star$ or $\beta = \beta^\dagger$ (depending on which side of the range $c$ falls) means that $\rho_0^\star < \hat{\rho}_0$ and hence we can choose any $\rho_0 \in (\rho_0^\star, \hat{\rho}_0)$ to obtain stabilization of the target orbit.

Note that $g$ and $h$ are functions that depend on the $\phi_j$, and hence on the choice of $s$. Thus, if stabilization is not possible for one symmetry element it may be possible for a different choice. We discuss this choice for our $\mathbb{D}_N$ example in §5.

5. $\mathbb{D}_N$ example continued

In this section, we apply the general results computed above to the $\mathbb{D}_N$ example obtained in §3. Recall that our results show that it is always possible to stabilize this branch in the supercritical case, so we consider in detail the subgroup $\mathbb{Z}_N$ in the challenging case where the branch bifurcates subcritically. The other subcritical cases follow in a similar manner so we do not give the details here.

(a) Subcritical $\mathbb{Z}_N$ subgroup

Recall that, for this subgroup, we have a choice of which symmetry element we use in the feedback. The possible elements are $\zeta^m$, for $m = 0, \ldots, N - 1$, and each element has eigenvalues
with $\phi_1 = 2\pi m/N$ and $\phi_2 = -2\pi m/N$, and the twist is $\theta = m/N$. Thus, following the calculations above, we find

$$
\psi_m = \psi_M = \psi_2 = \pi \theta - \frac{\phi_2}{2} = 2\pi \frac{m}{N} \mod \pi.
$$

Recall that we are not allowed to have $e^{i\phi_2} = e^{i\phi_1}$, so this restricts our possible choices of $m$; for example, for $N$ even we cannot choose $m = N/2$.

Completing the calculations from the section above, we find that either we can choose $\beta \in [0, \psi_m)$, and require that $g_1(\beta) < -c$, or we can choose $\beta \in (\psi_m + \pi, 2\pi)$, requiring that $g_2(\beta) > -c$, where

$$
g_1(\beta) = \cot \beta + \frac{1}{\psi_m - \beta} \quad \text{and} \quad g_2(\beta) = \cot \beta + \frac{1}{\psi_m + \pi - \beta}.
$$

It is clear that, for some $c$, it is not possible to choose a symmetry element and angle $\beta$ so that it is possible to satisfy all our conditions. However, let us suppose that we wish to maximize the range of $c$ for which it is possible. If $c < 0$, then we need to choose $\beta \in [0, \psi_m)$, and we can minimize the minimum of $g_1(\beta)$ by choosing $\psi_m$ to be as large as possible. That is, we choose $m = (N - 1)/2$ if $N$ is odd, and $m = N/2 - 1$ if $N$ is even. If $c > 0$, we need to choose $\beta \in (\psi_m + \pi, 2\pi)$, and we can maximize the maximum of $g_2(\beta)$ by choosing $\psi_m$ to be as small as possible. That is, we choose $m = (N + 1)/2$ if $N$ is odd, and $m = N/2 + 1$ if $N$ is even (equivalently, we can choose $m = 1$).

These results are demonstrated in figure 2 for the case when $N = 5$. The four plots show $g(\beta)$ in the four cases $m = 1, 2, 3, 4$, and $\beta$ must be chosen so that $-c$ is above the black curve (if $\beta \in (0, \psi_m)$) or below the black curve (if $\beta \in (\psi_m + \pi, 2\pi)$). It is clear that if $c < 0$, then in order to maximize the allowed range of $c$, we should choose $m = 2$ and $\beta \approx 1.51$, which means that for $-c > c^*$, where $c^* \approx 1.057$, we can choose a $\rho_0$ so that we can stabilize the targeted orbit. If $c > 0$, then we should choose $m = 3$, and $\beta \approx 4.77 (= 2\pi - 1.51)$, so that we can stabilize the orbit if $c > c^*$.

Note that, as $N$ increases, $\psi_m$ tends to $\pi$, and hence $c^*$ decreases, towards a minimum equal to zero, which occurs at $\beta = \pi$. 

**Figure 2.** Plots of $g_1(\beta)$ and $g_2(\beta)$ (black solid curves) in the case $N = 5$, for $m = 1, 2, 3, 4$ (reading from (a) to (d)). The blue dotted curves are $\cot(\beta)$, and the red dashed curves show $1/(\psi_m - \beta)$ in the range $\beta \in (0, \psi_m)$ and $1/(\psi_m - \beta)$ in the range $\beta \in (\psi_m + \pi, 2\pi)$. The vertical dashed lines are at $\beta = \psi_m$ and $\beta = \psi_m + \pi$; $\beta$ is not allowed to be between these lines. (Online version in colour.)
(b) Numerical example

We now consider an example system of three coupled oscillators with \( \mathbb{D}_3 \) symmetry, adapted from an example in [7]. We show that we are able to stabilize target periodic orbits by adding feedback directly to the oscillator equations, even though the results computed above were for the normal form equations on the centre manifold. This follows from our previous results in [4].

The equations for the oscillators are given by

\[
\dot{x} = Lx + N(x) \quad \text{and} \quad x = (x_1, y_1, x_2, y_2, x_3, y_3)^T \in \mathbb{R}^6, \quad \tag{5.1}
\]

where

\[
L = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \quad A = \begin{pmatrix} -4 & 1 \\ -1 & -4 \end{pmatrix}, \quad B = -\left( 1 + \frac{\lambda}{4} \right) \begin{pmatrix} 4 & -2 \\ 2 & 4 \end{pmatrix}
\]

and

\[
N(x) = p \begin{pmatrix} x_1(x_1^2 + y_1^2) \\ y_1(x_1^2 + y_1^2) \\ x_2(x_2^2 + y_2^2) \\ y_2(x_2^2 + y_2^2) \\ x_3(x_3^2 + y_3^2) \\ y_3(x_3^2 + y_3^2) \end{pmatrix} + q \begin{pmatrix} -y_1(x_1^2 + y_1^2) \\ x_1(x_1^2 + y_1^2) \\ -y_2(x_2^2 + y_2^2) \\ x_2(x_2^2 + y_2^2) \\ -y_3(x_3^2 + y_3^2) \\ x_3(x_3^2 + y_3^2) \end{pmatrix} + r \begin{pmatrix} x_1^3 \\ y_1^3 \\ x_2^3 \\ y_2^3 \\ x_3^3 \\ y_3^3 \end{pmatrix}.
\]

Note that our adaption from the example in [7] is the addition of the final nonlinear term. We do this so that the equations do not contain an extra phase-shift symmetry; hence, the \( S^1 \) symmetry is only present after a normal form transformation (and truncation) of the equations restricted to the centre manifold. The symmetries act on these coordinates as

\[
\xi(x_1, y_1, x_2, y_2, x_3, y_3) = (x_2, y_2, x_3, y_3, x_1, y_1)
\]

and

\[
\kappa(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, x_3, y_3, x_2, y_2).
\]

In order to estimate the period of the periodic orbits, we perform a centre manifold reduction of system (5.1) using the coordinate transformation

\[
\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ -i \\ e^{2\pi i/3} \\ -i e^{2\pi i/3} \\ e^{2\pi i/3} \\ -i e^{2\pi i/3} \end{pmatrix} + z_2 \begin{pmatrix} 1 \\ -i \\ e^{-2\pi i/3} \\ -i e^{-2\pi i/3} \\ e^{-2\pi i/3} \\ -i e^{-2\pi i/3} \end{pmatrix} + w + c.c.,
\]

and use \( w = \bar{w} = 0 \) as the approximation to the centre manifold since there are no quadratic nonlinearities in (5.1), to obtain the equations

\[
\dot{z}_1 = \left( \lambda + i \left( 1 + \frac{\lambda}{2} \right) \right) z_1 + (4p + 3r + 4i\eta)(|z_1|^2 + 2|z_2|^2)z_1 + 3rz_1\bar{z}_2^2 \quad \tag{5.2a}
\]

and

\[
\dot{z}_2 = \left( \lambda + i \left( 1 + \frac{\lambda}{2} \right) \right) z_2 + (4p + 3r + 4i\eta)(2|z_1|^2 + |z_2|^2)z_2 + 3rz_1^2\bar{z}_2, \quad \tag{5.2b}
\]

which can be transformed via a near-identity coordinate transformation (see [7] for details) to the truncated normal form

\[
\dot{z}_1 = \left( \lambda + i \left( 1 + \frac{\lambda}{2} \right) \right) z_1 + (4p + 3r + 4i\eta)(|z_1|^2 + 2|z_2|^2)z_1 \quad \tag{5.3a}
\]

\[
\dot{z}_2 = \left( \lambda + i \left( 1 + \frac{\lambda}{2} \right) \right) z_2 + (4p + 3r + 4i\eta)(|z_1|^2 + 2|z_2|^2)z_2. \quad \tag{5.3b}
\]
and
\[
\dot{z}_2 = \left( \lambda + i \left(1 + \frac{\lambda}{2}\right) \right) z_2 + (4p + 3r + 4iq)(2|z_1|^2 + |z_2|^2)z_2. \tag{5.3b}
\]

We consider a parameter regime for which the \( \mathbb{Z}_3 \) solution branches subcritically, and use feedback in the original equations (5.1) to stabilize this solution. This branch corresponds to a solution with \( z_2 = 0 \) (table 1). Then, comparing with the normal form equations (2.1) given in §2a, and using equation (2.2), we have
\[
\omega_0 = 1, \quad \omega_1 = \frac{1}{2}, \quad c = 4q/(4p + 3r).
\]
Thus, using (2.3), we can estimate the period of the periodic solutions as
\[
\tau(\lambda) = \frac{2\pi}{1 + (1/2 - 4q/(4p + 3r))\lambda}.
\]

We then add feedback to equations (5.1) to obtain
\[
x = Lx + N(x) + K\left(\sigma x\left(t - \frac{\tau(\lambda)}{3}\right) - x(t)\right),
\tag{5.4}
\]
where we choose the gain matrix \( K \) so that the feedback acts only in the centre directions, and the matrix \( \sigma \) is the representation of the symmetry element \( \zeta \) given above.

We use parameter values
\[
p = 0, \quad q = -5, \quad r = 1, \quad \lambda = -0.05, \quad \beta = 0.54,
\]
and compute \( \rho_0^c \approx 0.186 \) and \( \dot{\rho}_0 \approx 1.444 \).

We use the function
\[
x(t) = 2\sqrt{\frac{-\lambda}{3r}} \begin{pmatrix}
\cos(\omega t) \\
\sin(\omega t) \\
\cos(\omega t + \frac{2\pi}{3}) \\
\sin(\omega t + \frac{2\pi}{3}) \\
\cos(\omega t + \frac{4\pi}{3}) \\
\sin(\omega t + \frac{4\pi}{3})
\end{pmatrix} + 0.01 \cos(20t), \quad t \in \left[ -\frac{\tau(\lambda)}{3}, 0 \right),
\]
where \( \omega = 2\pi/\tau(\lambda) \), as initial conditions (the first term is the approximate target solution derived from the normal form, and the second term is a small perturbation) and integrate equations (5.4) for \( \rho_0 = 0.3 \) for \( t \in [0, 100] \), and then with \( \rho_0 = 0 \) for \( t \in [100, 160] \), using the Matlab delay-differential equation solver \( \text{dde23} \). In figure 3, we show plots of both the time series of the solution and the feedback term. It can be clearly seen that the solution is initially stable, and has the correct symmetry properties (each cell is one-third of a period out of phase with the next), and when the feedback is turned off the solution becomes unstable. The feedback term is several orders of magnitude smaller than the solution when the feedback is on, but it is non-zero (it is approx. \( 3 \times 10^{-4} \)) since our estimate of the period of the target solution from the normal form equations is slightly wrong.

6. Discussion

In this paper, we have considered using a modified version of the Pyragas time-delayed feedback control to stabilize UPOs, which appear in an equivariant Hopf bifurcation. The feedback exploits a spatio-temporal symmetry of the bifurcating orbits so that it vanishes on the targeted orbit and is thus non-invasive. We are able to determine conditions under which stabilization is possible using a single linear feedback term. In particular, we determine how to choose the complex gain parameter so that the feedback will be effective.
Figure 3. The figures show results from a numerical integration of equations (5.4) with parameter values as given in the text. The parameter $\rho_0 = 0.3$ for $t < 100$ and $\rho_0 = 0$ for $t > 100$. (a) The time series of the $x_1, x_2$ and $x_3$ components of the solution and (b) the feedback terms of the same solution components. Note the difference in vertical scale showing that the amplitude of the feedback terms is much smaller (although non-zero) than the solution. The dotted horizontal line in (a) shows the predicted amplitude of the target solution. Stability of the targeted solution is lost when the feedback is turned off at $t = 100$. (Online version in colour.)

The success of the single feedback term method requires that there is a single symmetry group element $s \in \Sigma$, where $\Sigma$ is the symmetry group associated with the target orbit, for which $\text{Fix } s = \text{Fix } \Sigma$. This places quite a big restriction on the types of groups for which this method will work. However, it should be noted that this restriction could be lifted if we allowed multiple feedback terms, each using a different symmetry element. We expect that it would be possible to obtain criteria under which stabilization would be ensured in this case, although this is beyond the scope of this paper.

With the above restriction on the groups, if the targeted orbit bifurcates from a stable equilibrium into $\lambda > 0$, we show that stabilization is always possible if the feedback gain is real and positive. Essentially, in this supercritical case, it is enough to destroy some of the symmetry of the underlying Hopf bifurcation problem so that only a single pair of complex conjugate eigenvalues cross the imaginary axis at $\lambda = 0$. If, on the other hand, the targeted orbit bifurcates into $\lambda < 0$ there are further restrictions on the parameter regions for which stabilization is possible; namely, the magnitude of the imaginary part of the complex coefficient associated with the normal form equation must be greater than some value determined by the properties of the group representation. This is an extension of a restriction noted in similar calculations for a generic subcritical Hopf bifurcation [3] in which the same coefficient had merely to be non-zero.

The calculations we have shown here were on the centre manifold, but using results of [4] we can extend these two systems that are not in normal form and include additional dimensions. In the normal form, the $S^1$ symmetry has a linear action (a rotation) as all orbits are circular. This phase-shift symmetry is thus equivalent to a shift in time. In the original equations, the
orbits will not in general be circular, yet we still interpret the normal form symmetry as a time shift [7]. In our numerical example, we demonstrate that our methods work in practice. Here, we have chosen equations that are not in normal form and have higher dimension than the centre manifold. We have to approximate the period of the orbit using the normal form reduction, but we add feedback terms directly to the original equations and stabilization of the unstable orbit is easily seen in numerical simulations.

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