Some observations regarding steady laminar flows past bluff bodies

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Steady laminar flows past simple objects, such as a cylinder or a sphere, have been studied for well over a century. Theoretical, experimental and numerical methods have all contributed fundamentally towards our understanding of the resulting flows. This article focuses on developments during the past few decades, when mostly numerical and asymptotical advances have provided insights also for steady, although unstable, high-Reynolds-numbers flow regimes.

1. Introduction

Viscous flows past blunt bodies become unstable at relatively low Reynolds numbers (\(Re\)). The first calculations that reliably continued steady solutions into such regimes were carried out about 30 years ago, revealing some unexpected trends. In the case of \(Re = \infty\) (Euler flows), different computational and analytical tools show a much larger number of solution options than those that correspond to plausible limits of \(Re \to \infty\).

In this article, we summarize computational results for these two cases, and quote some associated analysis. We conclude by noting some unresolved issues.

A detailed survey of the problems that we touch upon would require a much longer report than we are attempting here. However, by highlighting select results obtained by different methodologies, we hope to facilitate the exchange of ideas between specialists on different perspectives of the topic. Previous surveys specializing on numerical and asymptotic aspects of this topic include Fornberg [1] and Chernyshenko [2], respectively.
2. Calculations of steady flows past a cylinder and a sphere at finite Re

(a) Governing equations

Viscous, incompressible flows are described by the Navier–Stokes (NS) equations. In two dimensions, these are most easily expressed in terms of stream function $\psi$ and vorticity $\omega$. With $u$ and $v$ denoting velocities in the $x$- and $y$-directions, respectively, it will hold that $u = \partial \psi / \partial y$ and $v = -\partial \psi / \partial x$. With vorticity defined by $\omega = \partial v / \partial x - \partial u / \partial y$, the steady-state NS equations become

$$\Delta \psi + \omega = 0 \quad (2.1)$$

and

$$\Delta \omega = \frac{Re}{2} \left\{ \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial x} \right\}, \quad (2.2)$$

where $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$. The Reynolds number $Re$ has here been defined according to the diameter of a cylinder with unit radius—hence the division by 2 in (2.2). In the special case of steady axisymmetric three-dimensional flows, (2.1) and (2.2) can be replaced by

$$\Delta \psi - \frac{1}{y} \frac{\partial \psi}{\partial y} + y \omega = 0 \quad (2.3)$$

and

$$\Delta \omega + \frac{1}{y} \frac{\partial \omega}{\partial y} - \frac{\omega}{y^2} = \frac{Re}{2y} \left\{ \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} + \frac{\omega}{y} \frac{\partial \psi}{\partial x} \right\}. \quad (2.4)$$

Very low-Re flows are known as Stokes’ flows or creeping flows. For these, inertial terms become insignificant compared to viscous ones, and the flow field past a sphere approaches front-to-back symmetry when $Re \to 0$. For flow past a cylinder, Stokes’ paradox tells that flows in this limit cannot match free stream at large distances, making the symmetry only approximate. These symmetries are rapidly lost when $Re$ is increased. The flows first develop downstream recirculation regions (RRs) and shortly thereafter temporal instabilities, the latter for the cylinder and the sphere at $Re \approx 40$ and $Re \approx 105$, respectively. As noted above, this article focuses on steady (although unstable) flows well past these $Re$ numbers.

(b) Computational issues

Several computational issues arise when computing steady high-Re flows. Some of these are discussed next.

(i) Far-field boundary conditions

Focusing on the cylinder case, we will see in §2c(i) that the vorticity will be highest along the edge of the RR and downstream following it, decaying exponentially fast when moving away from these regions. Imposing $\omega = 0$ just outside these regions is entirely acceptable. The situation for $\psi$ is different, as (2.1) closely couples the flow at the body surface (and, with it, the vorticity generation) to $\psi$-values far out in all angular directions. As figure 1a,b illustrates, boundary conditions for either $\psi$ or $\partial \psi / \partial r$ that do not somehow possess knowledge about the actual wake structure will have to be imposed extremely far out, suggesting computations throughout a large region in which the governing equations (2.1) and (2.2) have simplified to $\Delta \psi = 0, \omega = 0$. The two uppermost curves in figure 1c illustrate more desirable outermost computational grid lines. As first used in [6], one can analytically create any number of ‘sample’ solutions that obey $\Delta(\psi - \psi_{FS}) = 0$ outside the computational domain, together with the appropriate boundary condition at infinite distance, e.g. by placing point vortices in turn at the locations marked by circles (and their antisymmetric counterparts). Each case gives a linear relation that has to hold between the $\psi$-values at the outermost and next-to-outermost grid lines. These relations, reminiscent of Robin-type boundary conditions, remove the need for any numerical computations.
outside the domain sketched in figure 1c. It will transpire that the downstream boundary is not critical, since errors in these will barely penetrate backwards against the flow.

(ii) **Spatial discretization**

Several discretization options are available. Once the computational domain has been limited to a region similar to the one shown in figure 1c, finite elements (using unstructured meshes) and radial basis function-generated finite differences (RBF-FD; which are mesh-free) [7, 8] can be applied directly. It has however been more common to conformally map the domain to a rectangle, and then apply nonlinear stretchings in the two directions to resolve the boundary layer, etc. Between the further choices of finite differences (FD) of different orders and pseudospectral approximations, the second-order centred FD approximations used successfully in [5, 6, 9] were nevertheless quite certainly sub-optimal with regard to computational efficiency. Furthermore, given the vast improvements in computer power during the past 30 years, it is surprising that these early works have not yet been far surpassed.

(iii) **Newton’s method for obtaining steady solutions**

Newton’s method is well known both for scalar equations and for nonlinear systems, in both cases featuring quadratic convergence in the vicinity of simple roots. It was not fully appreciated until around 1980 that no adverse issues would arise when the system sizes were scaled up to the 10s or 100s of thousands of nonlinear equations that arise for challenging partial differential equation discretizations. Compared to other fixed point iterations, the quadratic convergence provides some major advantages: (i) guaranteed low number of iterations, (ii) no possibility of time instabilities carrying over into the artificial time of successive iterations and (iii) no problem with the fact that, on a solid boundary, there are two conditions for \( \psi \) and none for \( \omega \), since the total count of equations and unknowns remains correct. Homotopy (continuation) methods are readily available to effectively step in a parameter (such as \( Re \)). Also, with Newton’s method, numerical procedures to handle bifurcation points are well established. As it happened, no such

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**Figure 1.** (a) Difference between free stream \( \psi_{FS} = y \) and the stream function within the region \( 1 \leq r \leq 60 \) according to the leading term in the Oseen expansion [3, 4] at \( Re = 200 \) (assuming the correct value of the drag coefficient \( C_d \) at this \( Re \)). (b) Actual difference. (c) Concept illustration for obtaining a boundary condition for \( \psi \) that is accurate already when used immediately outside the downwind vorticity streak. The different parts of the figure are copied from [5, 6], with permission.
Figure 2. Stream function and vorticity in steady flows past a cylinder. The figure is adapted from [5], with permission.

Figure 3. Surface display of the vorticity fields shown in figure 2. The figure is adapted from [5], with permission.

were encountered for either flow past a single cylinder or axisymmetric flow past a sphere, but cases where such arise are noted in §5c.

(c) Some steady-state results

(i) Flow past a cylinder

Noteworthy trends seen when Re is increased up to around Re = 200 include (i) the length L of the RR increases and (ii) although the vortex streak gets thinner, dissipation is reduced and the streak ‘survives’ much further downstream. As seen in figures 2 and 3, the latter trend overtakes the former around Re = 400, in the sense that vorticity then starts to be convected back into the RR from its back, and soon fills up its interior with nearly constant vorticity. Surprisingly, this transition has hardly any effect on the linear growth of L, but the width W of the RR then also starts to grow linearly with Re (cf. figure 4). The sequence of dots labelled ‘0.17Re’ corresponds to (slender) wake analysis in [11], whereas the significance of the dots labelled ‘length/1.6691’ will
become apparent after the discussion of the Sadovski vortices in §3. We note that the slope of this dot sequence closely matches that for $W$.

In the case of a cascade of infinitely many side-by-side cylinders, an interesting further effect occurs in that there is a critical cylinder separation (with cylinder centres around 20 radii apart) below which this transition never occurs, no matter how high the $Re$ is chosen. Above it, significant amounts of vorticity enters the RR from its rear, causing it to ‘bulge’ (although with $W$ constrained by the cylinder separation). Cascade flows were considered in [10,12–15]. The fact that cascade configurations simplify far-field boundary conditions is advantageous for analysis, but not essential for computation (cf. §2b(i)).

(ii) Flow past a sphere

Figure 5 illustrates the evolution of the RR for increasing $Re$, as computed in [6]. Differences from the cylinder case include (i) starting shortly before $Re = 1000$, a secondary RR emerges inside the primary one, (ii) in agreement with the Prandtl–Batchelor theorem, the vorticity inside the RR becomes proportional to $y$ (rather than to a constant) and (iii) the shape of the RR approaches the form of a Hill’s spherical vortex (cf. §4).

3. Euler solutions for flows past a cylinder and a sphere

In seeking to understand high-$Re$ flows, it is natural to consider Euler flows, formally corresponding to $Re = \infty$. Only those in which the vorticity somewhere is non-zero are of interest, since otherwise the action of viscosity in the history of the flow cannot be felt, and deductions of non-physical results such as d’Alembert’s paradox can be made. A discussion of relevant ideas pre-dating many of the results surveyed here is given in [16] (written for the 25th anniversary of the Journal of Fluid Mechanics). In the case of a circular cylinder in two dimensions, the present authors believe that [17] identifies all single vortex flow options. These flows are naturally catalogued in terms of equilibrium point vortex flows as indicated in figures 6 and 7. There is a natural continuation from point vortices with a given circulation to attached, Batchelor-type vortices [18] (of finite size for increasing $Re$), and to families of vortices in which the support extends to infinity. All of the shown solutions were obtained by solving nonlinear free-boundary Poisson equations for the stream function on an FD grid.

Equilibrium positions for point vortices played a central role in the works just quoted and also in studies by Zannetti and co-workers [19,20]. The point vortices seem to indicate where less singular solutions also may exist. In general, steady Euler flows are too large a class for describing possible $Re \to \infty$ limits, and some criterion must be used to single out a solution. For example,
Figure 5. Stream function and downstream wake vorticity for steady flow past a sphere, at $Re = 100, 1000, 5000$. The figure is adapted from [6], with permission.

Figure 6. Sadovski-type vortices (a) without any body and (b) behind a cylinder. These different vortices feature constant levels of vorticity inside the shown contours, and zero vorticity outside them.

the use of a Kutta condition at corners or cusps on the physical boundary can be used and, in other problems, separation of the boundary layer can play a role.

The catalogue of flows found in [17] extend to axisymmetric flows past a sphere. Although there is then no analogue of the point vortex, some possible flows are shown in figure 8. Flows including swirl around the axis, as described by the Bragg–Hawthorne equation, can also be found (cf. figure 9). More illustrations and further discussion can be found in [22].

Of particular interest for large-$Re$ limits of the NS equations are the Sadovski vortex in two dimensions (the curve reaching the centre line in figure 6$a$) and Hill’s spherical vortex in three dimensions. The Sadovski vortex does not have a closed-form expression, but it has been computed to high accuracy [23,24]. It features the aspect ratio $L/W \approx 1.6691$. This is the case that is relevant for steady high-$Re$ flows past a cylinder (cf. §2 and figure 4). In a generalized form, the Sadovski vortex can also have a jump in Bernoulli constant across its boundary. This vortex family has been computed in [25].
4. Some asymptotic analysis results

The first building block in creating Euler solutions that might play a useful role in realistic flow models was given by Helmholtz [26], introducing vortex sheets separating from the body. Brodetsky [27] showed how this could be done also in the case of flow past a circular cylinder. A geometric condition determined the separation point, and a wake opening up parabolically to infinity was found. A series of theoretical studies starting with Squire [28] supported the idea of a ‘slender’ elliptic wake tending for increasing $Re$ towards Brodetsky’s free streamline model.

A detailed asymptotic theory of separation for the NS equations was given by Smith [29], further developing results in [30]. These works effectively incorporated the ‘triple deck’ model for local separation (see [31] for its history). Batchelor [18] had earlier introduced the idea of a
cyclic boundary layer around a recirculating eddy, which was analysed further in [32,33]. Various attempts were made to model the scale of the RR with Re, often predicting the length L to increase linearly with Re and the width W to increase as Re^{1/2}, consistent with the infinite Re parabolic wake of Brodetsky.

The first to propose a theory in which both L and W increased linearly with Re was Taganov [34]. The computations of Fornberg described earlier confirmed this concept by showing a change in RR character at Re around 400, after which this wide wake is observed. The first complete self-consistent theory was subsequently given by Chernyshenko [35], in excellent agreement with these numerical results. In this model, the asymptotic separation analysis was matched to a boundary layer surrounding a steady Sadovski vortex, which grows linearly with Re. There are several distinguished limits in the matching process, and it is remarkable that a complete determination of the relevant parameters is possible. In the case of cascades of bodies, the corresponding asymptotic theory [36] is again in good agreement with the numerical results cited in §2c(i). At least as far as flows past symmetric bodies in two dimensions are concerned, it can be said [2, p. 526] that ‘... the high Reynolds number asymptotics of steady plane flow past a bluff body is known’.

For the analogous axisymmetric flow past a sphere, the calculation in [6], described in §2c(ii), revealed a large wake in the shape of a scaled Hill’s spherical vortex. Some asymptotic results similar to the cylinder case can again be given [37], but the secondary separation complicates the analysis. The problem of high-Re asymptotics is still partly open, including how the RR radius scales with Re.

5. Open questions and challenges

The following is just a small collection of issues that seem especially intriguing, given the discussion and observations above.
(a) Navier–Stokes solutions

Given that flows past asymmetric obstacles necessarily become asymmetric, it becomes natural to ask about the properties also of such solutions for large $Re$. If the obstacle shrinks to a point, it might seem that its geometric details should not have a significant effect on the scale of the wake. On the other hand, present asymptotic theory makes essential use of symmetry assumptions. It has been conjectured (SI Chernyshenko, 2013 personal communication) that, for a pair of counter-rotating vortices attached to an obstacle, one of these may persevere and the other shrink as $Re$ grows. Suggestive results supporting this conjecture are obtained in [38] for an ellipse at 45° angle of attack. However, the largest $Re$ used there is $Re = 40$, and this is too low to draw definite conclusions. It would be interesting to obtain results also for significantly larger $Re$.

(b) Euler flows

The authors are not aware of any asymmetric Prandtl–Batchelor flows with pairs of counter-rotating vortices attached to the body. It seems likely that they may exist, at least if the geometry is a mild perturbation of symmetry, but none seem to have been described. There are solutions with a single vortex, for example the ‘trapped vortices’ used in the project VortexCell2050 (as documented at several websites most easily found by searching on the project name).

(c) Existence and uniqueness

The flows described earlier had symmetry about the centre line built into the solution process. It is natural to ask whether there can be more than one symmetric solution at a given $Re$. This is especially interesting in view of the wealth of possible Euler solutions and various conjectures about large-$Re$ limits [16]. The solution branches described in [5,6] were monitored for bifurcation points, but none were found. This does of course not rule out altogether separate solution branches, or bifurcation points outside the considered parameter regimes. Instances of non-unique symmetric cascade flow solutions were reported in Castro [39]. However, the author of that work also noted that their physical reality was unclear, since they could only be seen on coarse-grid and not on fine-grid calculations. When the assumption of symmetry in the flow is dropped, asymmetric steady solutions have been found both for flow past a single sphere [8,40] and for flow past a pair of cylinders [41,42]. Many issues with regard to solution existence and uniqueness deserve further study.

(d) Time instabilities and flow control

As noted earlier, steady symmetric flow past both a cylinder and a sphere appear to be unique for all values of $Re$. Without the constraints of ‘steady’ and ‘symmetric’, a variety of instability modes enter when $Re$ is increased. In the cylinder case, the first is a Hopf bifurcation, leading to vertical oscillations towards the end of the RR, producing a Kármán vortex street in the wake. Because the end of a steady RR is a stagnation point, a small vertical plate at this location will have very little effect on the flow, apart from being highly effective in suppressing this particular mode [43]. However, such ‘passive’ flow control approaches will be unable to control subsequent instability modes (with the initial laminar ones analysed in [44]). Flow past a two-dimensional cylinder and over backwards steps have served as test problems for several active flow control strategies, both for the finite-$Re$ case [45,46] and for Euler (point vortex) flows [47].

Extending to three dimensions, figure 10 compares the drag for steady versus unsteady flows in the case of a sphere. This illustrates one reason why it can be very desirable to operate in steady, although unstable, flow regimes—a blunt body can then move through a fluid as easily as a streamlined one. The ‘drag crisis’ (sudden drop in drag by an order of magnitude) seen around $Re = 4 \times 10^5$ corresponds to the unsteady wake changing character from being dominated by large laminar vortices to taking the form of a narrower turbulent streak. An example of passive
flow control for this case is provided by the dimples on a golf ball, shifting this transition to slightly lower $Re$, thereby making the lower drag regime available in actual play.

Active flow control has been developed successfully for several ‘real life’ applications, most notably allowing military aircraft to fly in unstable regimes, offering lowered drag as well as increased manoeuvrability. However, both cost and complexity of such systems can be high.

Beyond sucking away boundary layers on the surface of an aerofoil (potentially beneficial for delaying or eliminating separation as well as reducing some turbulence effects), attempting to control turbulent instabilities by direct feedback mechanisms encounters fundamental limitations already in two dimensions [48]. Nevertheless, the possibility that steady flows can feature much better characteristics than unsteady ones provides a major motivation for their investigation.

6. Conclusion

We have reviewed some computational results and algorithms for the steady NS equations and related Euler flows. For flow past a cylinder, the RR starts around $Re = 400$ to grow also in width proportionally to $Re$. Both numerical solutions and asymptotic analysis indicate that the RR thereafter can be described by a scaled Sadovski vortex. There are similar results for flow past a sphere, with the Sadovski vortex replaced by Hill’s spherical vortex, but here there is a secondary separation and asymptotic results are not complete.

There are rich families of Euler flows past a cylinder and a sphere. In some cases, physically significant solutions can be singled out with extra conditions, such as Kutta conditions at corners. Some Euler flows can only occur as quasi-steady limits of solutions of the NS equations, approximating flows for some period of time.

When the body is symmetric, there is no firm evidence of non-uniqueness for symmetric solutions, but certain asymmetric solutions have been found. When the body is not symmetric, as, for example, for flow past an ellipse at angle of attack so that the flow cannot be symmetric, little is known about solutions for large $Re$.

Although steady flows become unsteady as $Re$ grows, there is substantial motivation for finding ways to stabilize them, primarily in order to reduce drag.

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