Effect of free-stream turbulence on boundary layer transition

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This paper is concerned with the transition to turbulence in flat plate boundary layers due to moderately high levels of free-stream turbulence. The turbulence is assumed to be generated by an (idealized) grid and matched asymptotic expansions are used to analyse the resulting flow over a finite thickness flat plate located in the downstream region. The characteristic Reynolds number $R_\Lambda$ based on the mesh size $\Lambda$ and free-stream velocity is assumed to be large, and the turbulence intensity $\varepsilon$ is assumed to be small. The asymptotic flow structure is discussed for the generic case where the turbulence Reynolds number $\varepsilon R_\Lambda$ and the plate thickness and are held fixed (at $O(1)$ and $O(\Lambda)$, respectively) in the limit as $R_\Lambda \to \infty$ and $\varepsilon \to 0$. But various limiting cases are considered in order to explain the relevant transition mechanisms. It is argued that there are two types of streak-like structures that can play a role in the transition process: (i) those that appear in the downstream region and are generated by streamwise vorticity in upstream flow and (ii) those that are concentrated near the leading edge and are generated by plate normal vorticity in upstream flow. The former are relatively unaffected by leading edge geometry and are usually referred to as Klebanoff modes while the latter are strongly affected by leading edge geometry and are more streamwise vortex-like in appearance.

1. Introduction

This paper is concerned with the transition to turbulence in flat plate boundary layers due to moderately high levels of grid-generated turbulence—a subject that began with the work of Dryden [1] and Taylor [2], who showed that the unsteady boundary layer flow was dominated by chaotic low-frequency streaks. But these studies were largely ignored by subsequent researchers who focused
on the much more regular and (at the time) seemingly more interesting Tollmien–Schlichting waves that occur at very low free-stream turbulence levels (say less than 1% or so) until these earlier ideas were resurrected about two decades later by Klebanoff [3]. However, the research really took off in the late 1980s and 1990s. Klebanoff found that the measured hot wire signal that passed through a low-pass filter at 12 Hz was almost identical in magnitude to the signal measured over all frequencies, indicating that most of the energy was in frequencies below 12 Hz—a result that has now been reproduced many times. Kendall [4] later showed that the spanwise dimension of these low-frequency structures, which he referred to as Klebanoff modes, was of the order of the boundary layer thickness. Based on an earlier proposal that Bradshaw [5] introduced to explain some low free-stream turbulence observations, Klebanoff [3] suggested that these ‘modes’ (which, as has been pointed out many times, are not actually eigenmodes) could be interpreted as a local thickening and thinning of the Blasius boundary layer. Bradshaw [5] pointed out that perturbing the boundary layer thickness, say $\delta$, in the Blasius solution, $u = F_B'(\eta)$ (where the prime denotes differentiation with respect to $\eta = y/\delta$ and $F_B$ is the Blasius function) by a small amount, say $\delta^{(1)}$, and expanding in a Taylor series gives

$$u = F_B'(\eta) - \frac{\delta^{(1)}}{\delta} \eta F_B''(\eta) + \cdots,$$

(1.1)

which means that the difference between the actual streamwise velocity $u$ and the Blasius solution should be proportional to $\eta F_B''$—a result that is almost invariably found to be in excellent agreement with experimental observations.

A few months after Bradshaw came up with this proposal Crow [6], who had been visiting the National Physical Laboratory, Teddington, UK, at the time, put this result on a more rigorous foundation by analysing the flow over an infinitely thin flat plate produced by a small-amplitude spanwise-periodic velocity perturbation

$$w_\infty = \varepsilon \cos \left( \frac{2\pi z^*}{\Lambda} \right),$$

(1.2)

imposed on a uniform upstream flow $U_\infty$, where $\varepsilon$ is a measure of the disturbance amplitude (figure 1).

His analysis showed that

$$u = F_B'(\eta) - \frac{1}{2} \left[ \varepsilon \left( \frac{x^*}{\Lambda} \right) \sin \left( \frac{2\pi x^*}{\Lambda} \right) \right] \eta F_B''(\eta),$$

(1.3)
Figure 2. Generic flow configuration.

when $\delta \ll \Lambda$, which means that the boundary layer thickness perturbation $\delta^{(1)}$ in (1.1) should be given by $\delta^{(1)} = [\varepsilon \delta(x^*/\Lambda) \sin(2\pi z^*/\Lambda)]/2$. While this is a remarkably simple and elegant result, it does not agree with the experimentally observed growth rate, which seems to be more like $\sqrt{x^*}$ than like $x^*$.

There is obviously a lot more about Klebanoff modes and their subsequent breakdown into turbulence that still needs to be explained. This paper attempts to give an overview of a very general methodology that has been developed to provide this explanation. Most of the material has already appeared in the literature but some new material is included in §§2 (at the end) and 3. An excellent summary of the overall approach can be found in the Introduction of [7], which also provides a penetrating explanation of why the other methodologies proposed in the literature (optimal growth (see also Introduction of [8]), Orr–Sommerfeld theory, leading edge eigensolutions, etc.) do not provide viable alternatives to this approach.

2. Overall flow structure

The paper begins by considering the general situation shown in figure 2, but assumes, for simplicity, that the flow is incompressible [9–11]. The fundamental length-scale $\Lambda$ is set by the (idealized) upstream grid, which is presumed to generate weak free-stream turbulence with characteristic amplitude $O(\varepsilon U_\infty)$. It can, therefore, be locally represented by a convected disturbance, say $\varepsilon u_\infty(x - \hat{t} \delta; \bar{t})$, $x = \{x, y, z\}$ that satisfies the continuity equation $\nabla \cdot u_\infty = 0$ and can otherwise be specified as an upstream boundary condition [12]. All velocities are assumed to be normalized by the uniform free-stream velocity, $U_\infty$, all lengths by $\Lambda/U_\infty$ and, because of the important role played by the low-frequency disturbances, it seems appropriate to explicitly specify the parametric dependence of $u_\infty$ on the slow time variable $\bar{t} \equiv \varepsilon t/\sigma$, where $\sigma \gg \varepsilon$ is an additional gauge function. The technologically interesting cases, for which virtually all of the experiments were carried out, correspond to large values of the characteristic Reynolds number $R_\Lambda \equiv U_\infty \Lambda/\nu$. The generic scaling then corresponds to the double limit as $R_A \to \infty$, $\varepsilon \to 0$ with the turbulence Reynolds number $R_T \equiv \varepsilon R_A$ and plate thickness held fixed at $O(1)$ and $O(\Lambda)$, respectively; in which case, the additional gauge function $\sigma = \sigma (R_T)$ can be set equal to unity and the flow divides into the five asymptotic regions shown in figure 2.

The flow in region I, which corresponds to the vicinity of the leading edge (i.e. $|x| = O(1)$), is nearly two dimensional and steady with the three-dimensional and unsteady effects being an
$O(\varepsilon)$ perturbation of the two-dimensional potential flow, \(\{U_0, V_0, 0\}\). The latter is basically inviscid when the grid is sufficiently close to the leading edge, which is now assumed to be the case.

The viscous effects are then confined to a relatively thin boundary layer of thickness $O(R_A^{-1/2})$, which is denoted as region II in the figure and is again nearly two dimensional and steady with the three-dimensional and unsteady effects coming in as a $O(\varepsilon)$ perturbation of that flow. The flow in region I is governed by linear rapid distortion theory [13–15] about a two-dimensional potential mean flow \(U_0 - iV_0 = dW/d\xi, \xi = x + iy\), which is induced by the plate and the $O(R_A^{-1/2})$ displacement effects of the two-dimensional boundary layer flow. The latter can be accounted for by adding $-iR_A^{-1/2}\delta(\xi)$, where $\delta(\xi)$ is the analytic continuation into the complex $\xi$-plane of the scaled boundary layer displacement, to the complex potential $W = W_0$ of the plate itself. The rapid distortion theory solution is easily found by solving a Poisson equation [15].

Asymptotic solutions to the linearized boundary layer problem constructed in [6,11,16] show that the unsteady flow can be divided into a two-dimensional component that is driven by the streamwise velocity at the bottom of region I and remains small, i.e. $O(\varepsilon)$ for all $x$, and a three-dimensional component that is driven by the cross flow velocity at the bottom of region I and continues to grow with increasing $x$. Leib et al. [11] constructed a Wentzel–Kramers–Brillouin–Jeffreys solution for this component that is valid in the limit as $x \to \infty$ with $t = O(1)$ and has a single turning point at the outer edge of the boundary layer [7,11]. Since this solution must break down in the vicinity of that point a new asymptotic solution has to be constructed in this region, which is usually referred to as the ‘edge layer’. This result shows that the $O(1)$ frequency velocity perturbations move out into this ‘edge layer’ as $x \to \infty$, while the $1/t = O(\varepsilon/\sigma)$ low-frequency portion of the three-dimensional component of the boundary layer solution continues to grow with increasing $x$. The smaller transverse wavenumber components exhibit the most rapid growth, suggesting that the velocity fluctuations within the boundary layer will be dominated by the low-frequency small transverse wavenumber components at large downstream distances, where $y = O(1)$ and $x = O(\sigma/\varepsilon)$.

The boundary layer thickness also continues to increase with increasing $x$ and the classical linearized boundary layer solution (which only applies when the boundary layer thickness is much less than $\Lambda$ [8]) eventually becomes invalid. A new solution therefore has to be constructed in region III where

\[
\tilde{x} \equiv \frac{x}{\sigma}
\]

and $y$ are both $O(1)$. The flow is fully nonlinear in this region and the boundary layer thickness is of the same order as the spanwise wavelength when $R_T = O(1)$. It, therefore, scales like \(\{u, p + 1/2\} = (U, (\varepsilon/\sigma)V, (\varepsilon/\sigma)W, (\varepsilon/\sigma)^2 P)\) and is governed by the boundary region equations [17]

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = \frac{\sigma}{R_T} \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right),
\]

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} = \frac{\partial P}{\partial y} + \frac{\sigma}{R_T} \left( \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right),
\]

\[
\frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} + \frac{\sigma}{R_T} \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right),
\]

and

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,
\]

which are just the Navier–Stokes equations with the streamwise derivatives neglected in the viscous and pressure gradient terms, implying that spanwise ellipticity is important in region III. This is the region of primary interest since the spanwise wavelength is of the order of the boundary layer thickness in almost all Klebanoff mode experiments. It is worth noting that the spanwise ellipticity (which introduces additional viscous effects) decreases the streamwise growth rate of the wall layer streaks below the linear growth rate that occurs in the upstream boundary layer region II, which explains the discrepancy between the Crow [6] analysis and the
experimental observations. It is also worth noting that the definition of the scaled streamwise coordinate \( \tilde{x} \) is somewhat different from the one used in [11], but the two are asymptotically equivalent because the scale factors are of the same asymptotic order.

The thin edge layer that causes the high-frequency disturbances to move out of the unsteady boundary layer region at large downstream distances is expected to continue downstream and form a region V that lies between region III and region IV, where \( \tilde{x} = O(1) \) and \( \tilde{y} \equiv \epsilon \tilde{y} = O(1) \) when \( R_T = O(1) \). This layer (which was not included in some of the previous studies) is expected to act as a kind of buffer zone that prevents the high-frequency/small-scale disturbances in region IV from penetrating into region III (see [7]). It did not appear in the analyses in [8,18] because the region IV disturbances were of low frequency or completely steady in those papers, and it was not included in [11] because the low-frequency components could be treated independently of the high-frequency components in the linearized flow investigated in that paper.

### 3. Flow in region IV

The flow in this region expands like \([8,9,11,18]\)

\[
\mathbf{u} = \hat{\mathbf{u}} + \epsilon \left[ \frac{1}{\sigma} \mathbf{u}_0(\tilde{x}, \tilde{y}, \tilde{t}) + \tilde{\mathbf{u}}(\tilde{x}, \tilde{y}, \tilde{t}_0) \right] + \cdots ,
\]

(3.1)

where \( \tilde{u} \) depends on the slow moving frame variable \( \tilde{t}_0 \equiv \tilde{t} - \tilde{x} \) only parametrically, the complex conjugate velocity \( \mathbf{u}_0 = (u_0, v_0, 0) \) is given by [18]

\[
\mathbf{u}_0 = \tilde{u}_0(\tilde{x}, \tilde{y}, \tilde{t}_0),
\]

(3.2)

where \( \tilde{u}(\tilde{x}, \tilde{t}) \) is the analytic continuation of the displacement thickness

\[
\tilde{u} = \frac{1}{T} \int_0^T \int_0^\infty \left[ 1 - U(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) \right] d\tilde{y} d\tilde{z}
\]

(3.3)

into the complex \( \tilde{\xi} = \tilde{x} + i \tilde{y} \) plane, and \( \tilde{u} \) satisfies the full Navier–Stokes equations

\[
\frac{1}{\sigma} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{u} \cdot \nabla \tilde{u} = -\nabla \tilde{p}_\infty + \frac{1}{R_T} \nabla^2 \tilde{u},
\]

(3.4)

and

\[
\nabla \tilde{u} \cdot \tilde{u} = 0,
\]

(3.5)

in the slow variable \( \epsilon \tilde{x} \) (which plays the role of time) and the fast displaced moving coordinate system variable

\[
\tilde{x} = \{ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \} \equiv \{ \tilde{x} - t + \text{Im} \hat{\delta}(\tilde{\xi}, \tilde{t}), \tilde{y} - \text{Re} \hat{\delta}(\tilde{\xi}, \tilde{t}), \tilde{z} \},
\]

(3.6)

subjected to the upstream boundary condition

\[
\tilde{u}(\tilde{x}, 0, \tilde{t}_0) = \tilde{u}_\infty(\tilde{x}, \tilde{t}_0).
\]

(3.7)

The operator \( \nabla \tilde{x} \) is defined in the obvious way by

\[
\nabla \tilde{x} \equiv \left\{ \frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2}, \frac{\partial}{\partial \tilde{x}_3} \right\},
\]

(3.8)

the modified displacement \( \hat{\delta} \) is determined by the first-order partial differential equation

\[
\hat{\delta} + \hat{\delta}_t = \hat{\delta}_x,
\]

(3.9)

and \( p_\infty \) denotes the corresponding pressure, but the partial derivatives with respect to \( \tilde{x} \) and \( \tilde{x}_i, i = 1, 2, 3 \) are at constant \( \tilde{t}_0 \equiv \tilde{t} - \tilde{x} \), since the slow time variable \( \tilde{t}_0 \) enters only parametrically. So the actual time-dependent velocity at \( \tilde{x} \) is given by \( \tilde{u}(\tilde{x}, \tilde{t}, \tilde{t} - \tilde{x}) \). And as can be seen by direct substitution, the displaced streamwise coordinate \( \tilde{x}_1 \) (which did not enter the corresponding equations in [18] and [8] since the flow was of low frequency or completely steady in those papers) had to be introduced in order to balance the advection of streamwise momentum in the Navier–Stokes equation (3.4).
As region V is expected to filter out the $x-t$ dependence of the region IV solution, it is likely that the flow in the viscous wall layer (region III) will be mainly determined by the time-average (low-frequency) component

$$u_\perp(x_\perp, \tilde{x}; \tilde{t}_0) = \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} u(\tilde{x}, \tilde{x}; \tilde{t}_0) d\tilde{x}, \quad \tilde{t}_0 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(\tilde{x}, \tilde{x}; \tilde{t}_0) dt,$$

(3.10)

and

$$\bar{p}_\infty(x_\perp, \tilde{x}; \tilde{t}_0) = \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} p(\tilde{x}, \tilde{x}; \tilde{t}_0) d\tilde{x},$$

(3.11)

of the region IV solution, which satisfies

$$\frac{1}{\sigma} \frac{\partial \bar{u}}{\partial \tilde{x}} + \bar{u}_\perp \cdot \nabla_\perp \bar{u} = -\nabla_\perp \bar{p}_\infty + \frac{1}{R_T} \nabla^2_\perp \bar{u} - \nabla_\perp \cdot \bar{u}' u'$$

(3.13)

and

$$\nabla_\perp \cdot \bar{u}_\perp = 0,$$

(3.14)

subject to the inhomogeneous upstream boundary condition

$$\bar{u}(\tilde{x}_\perp, 0; \tilde{t}_0) = \bar{u}_\infty(\tilde{x}_\perp, \tilde{t}_0)$$

(3.15)

and is expected to remain relatively inhomogeneous and grid dependent even though the overall turbulence velocity $\bar{u}$ is expected to be approximately homogeneous. Appropriate transverse boundary conditions can probably be inferred by assuming that $\bar{u}$ has the same transverse periodicity as the imposed upstream flow. The remaining quantities in (3.13)–(3.15) are defined by

$$u' \equiv \bar{u} - \bar{u}, \quad \bar{u}_\perp \equiv \{0, \bar{u}_2, \bar{u}_3\}$$

(3.16)

(3.17)

and

$$\nabla_\perp \equiv \left\{0, \frac{\partial}{\partial \tilde{x}_2}, \frac{\partial}{\partial \tilde{x}_3}\right\},$$

(3.18)

Equations (3.13) and (3.14) show that $\bar{u}_\perp$ is decoupled from $\bar{u}_1$ and is determined by the two-dimensional Reynolds averaged Navier–Stokes equations, with $\tilde{x}$ playing the role of time when $u'u'$ is assumed to be independent of $\bar{u}_1$. The latter equations can be reduced to a single equation by using (3.14) to introduce the stream function $\psi$

$$\bar{u}_2 = \frac{\partial \psi}{\partial \tilde{x}_3} \quad \text{and} \quad \bar{u}_3 = -\frac{\partial \psi}{\partial \tilde{x}_2},$$

(3.19)

and then eliminating the pressure to obtain

$$\left(\frac{1}{\sigma} \frac{\partial}{\partial \tilde{x}} + \frac{\partial \psi}{\partial \tilde{x}_3} \frac{\partial}{\partial \tilde{x}_2} - \frac{\partial \psi}{\partial \tilde{x}_2} \frac{\partial}{\partial \tilde{x}_3}\right) \nabla^2_\perp \psi = \frac{1}{R_T} \nabla^4_\perp \psi - \frac{\partial}{\partial \tilde{x}_i} \left(\frac{\partial T_{i2}}{\partial \tilde{x}_3} - \frac{\partial T_{i3}}{\partial \tilde{x}_2}\right), \quad i = 2, 3,$$

(3.20)

where $T_{ij}$ denotes the Reynolds stress tensor

$$T_{ij} \equiv u'_i u'_j$$

(3.21)

produced by the small-scale relatively homogeneous components of the turbulence which (as noted above) are not expected to have a significant effect on the flow in region III. It would, therefore, not be unreasonable to close the system by introducing an appropriate model for this quantity.
Since $\psi(\tilde{x}_2, 0; \tilde{t}_0) = \psi(\tilde{x}_2, 2\pi/\gamma; \tilde{t}_0)$ when the upstream disturbance is spanwise periodic with wavelength $\gamma$, equation (3.19) implies that
\[
\int_0^{2\pi/\gamma} \tilde{u}_2 \, d\tilde{x}_3 = 0, \tag{3.22}
\]
which ensures that (3.10)–(3.18) will match the spanwise average transverse wall layer velocity as $\tilde{x}_2 \to 0$ (see (3.15) of [8]).

The streamwise velocity, $\bar{u}_1$, which can be determined after the fact from the formally linear equation
\[
\left( \frac{1}{\sigma} \frac{\partial}{\partial \tilde{x}} + \frac{\partial \psi}{\partial \tilde{x}_3} \frac{\partial}{\partial \tilde{x}_2} - \frac{\partial \psi}{\partial \tilde{x}_2} \frac{\partial}{\partial \tilde{x}_3} \right) \bar{u}_1 = \frac{1}{R_T} \nabla^2 \psi \bar{u}_1 - \frac{\partial T_{11}}{\partial \tilde{x}_i}, \quad i = 2, 3, \tag{3.23}
\]
when $T_{ij}$ is independent of $\bar{u}_1$, has no effect on the outer edge boundary condition for the flow in region III for the low-frequency and steady flows considered in [18] and [8], respectively. The streamwise component $\bar{\omega}_1 = \nabla^2 \psi$ of the time-averaged vorticity $\bar{\omega} = (\epsilon_{ijk} \bar{u}_j / \partial \tilde{x}_k)$ is then determined by stream function $\psi$ while the cross flow components are determined by $\bar{u}_1$. So it is not unreasonable to expect that the outer edge boundary condition for the flow in region III will only be affected by the streamwise vorticity at lowest order of approximation (but see comments at the end of this section): in which case the transverse vorticity components would only affect the flow in this region through the upstream boundary condition, i.e. through the leading edge interaction.

Equation (3.15) shows that the solution to (3.20) and (3.23) must satisfy the inhomogeneous upstream boundary condition
\[
\left\{ \bar{u}_1, \frac{\partial \psi(\tilde{x}_L, 0; \tilde{t}_0)}{\partial \tilde{x}_3}, -\frac{\partial \psi(\tilde{x}_L, 0; \tilde{t}_0)}{\partial \tilde{x}_2} \right\} = \bar{u}_\infty(\tilde{x}_L, \tilde{t}_0). \tag{3.24}
\]
Appropriate boundary conditions for (3.20) would then be to specify $\psi(\tilde{x}_L, 0; \tilde{t}_0)$ and require that the solution remain bounded as $|\tilde{x}_L| \to \infty$. The system (3.19) and (3.20) will be hyperbolic when any of the usual turbulence models are used for $T_{ij}$. This greatly simplifies the computation and it seems appropriate to take advantage of this simplification by modelling $T_{ij}$ since, as noted above, the flow in region III is expected to be relatively independent of the small-scale homogeneous component of the turbulence that contributes to this quantity.

The Reynolds stress terms obviously drop out of equations (3.20) and (3.23) when the small-scale turbulence is assumed to be homogeneous. They also drop out of (3.20) when the turbulence is assumed to be axisymmetric [19] but not necessarily homogeneous because
\[
T_{ij} = A(\tilde{x}, \tilde{\bar{x}}) \delta_{ij} + B(\tilde{x}, \tilde{\bar{x}}) \delta_{ij} \tag{3.25}
\]
in that case. This implies that
\[
\frac{\partial}{\partial \tilde{x}_i} \left( \frac{\partial T_{12}}{\partial \tilde{x}_3} - \frac{\partial T_{13}}{\partial \tilde{x}_2} \right) = 0, \quad i = 2, 3. \tag{3.26}
\]
Equation (3.20) becomes
\[
\left( \frac{1}{\sigma} \frac{\partial}{\partial \tilde{x}} + \frac{\partial \psi}{\partial \tilde{x}_3} \frac{\partial}{\partial \tilde{x}_2} - \frac{\partial \psi}{\partial \tilde{x}_2} \frac{\partial}{\partial \tilde{x}_3} \right) \nabla^2 \psi = \left( \frac{1}{R_T} + \bar{\mu} \right) \nabla^4 \psi, \tag{3.27}
\]
when $T_{ij}$ is approximated by the Boussinesq model
\[
T_{ij} = -\bar{\mu} \left( \frac{\partial \bar{u}_j}{\partial \tilde{x}_i} + \frac{\partial \bar{u}_i}{\partial \tilde{x}_j} \right). \tag{3.28}
\]
The boundary value problem corresponding to (3.24) together with the homogeneous versions of (3.20) and (3.23) was effectively solved in [18].

The flow in this region IV is independent of the slow transverse variable $\tilde{y}$ and can be determined independently of the flow in regions III and V. It can, therefore, be used as the outer
boundary condition for the flow in these regions, i.e. the problem can be solved in seriatim. In fact, the region IV flow should be well approximated by decaying homogeneous turbulence when \( \tilde{x} \) becomes sufficiently large. There are many calculations of this type of flow in the literature and the numerical procedures are highly developed. Simmons & Salter [20] showed that grid turbulence divides into two distinct regimes. (i) A near-field just downstream of the grid, which is strongly inhomogeneous and sheared. It consists of a multitude of interacting wakes and jets and is therefore likely to depend strongly on the grid properties. And (ii) a far-field, which is more or less homogeneous. The boundary between these two regimes is not abrupt and in any event is likely to depend on the grid in question. A general rule of thumb is that reasonable homogeneity is usually achieved after 50–80 integral scales.

The thin edge layer V occurs because the convection of the fast free-stream velocity \( \bar{u} \) by the \( O(1) \) streamwise wall layer velocity becomes large and must be balanced by an increase in the viscous terms (i.e. an increase in the transverse shear). This region is expected to act as a low-pass filter that filters out the fast (i.e. high frequency) \( x - t \) dependence of the region IV velocity \( \tilde{u}(\tilde{x}, \tilde{x}, \tilde{t}) \equiv \bar{u}(\tilde{x}, \tilde{x}; \tilde{t} - \tilde{x}) \). The resulting transmitted velocity field will then have \( O(1) \) and \( O(\epsilon) \) streamwise and transverse velocity components, respectively, that depend only on the slow variables \( \tilde{x}, \tilde{t} \) and the fast cross stream variables \( y, z \) as implied by equations (2.2)–(2.5).

Non-parallel flow effects are expected to remain important in this region (i.e. region V) because of the relatively rapid increase in the boundary layer displacement [7]. There is also the possibility that the nonlinear \( \bar{u} \bar{u} \bar{u} \) Reynolds stress terms (or other effects) can become large enough to cause a jump in the time-average velocity \( \bar{u} \) across this layer. This would, of course, impact the outer edge boundary condition for region III.

But when this jump in \( \bar{u} \) is neglected and the region IV Reynolds stress is modelled in the usual fashion the outer edge boundary condition for region III will be determined by the value of \( \bar{u} \) at the bottom of region IV and the complete problem, which, as noted earlier, can be solved in seriatim, is parabolic. However, the resulting flow field is still very complex and would probably require considerable numerical computation.

### 4. Limiting cases

But a lot of the physics can be understood by considering some limiting cases. These include the low- and high-turbulence Reynolds number limits, the zero plate thickness approximation and the steady or quasi-steady limits where the imposed upstream boundary condition \( u_{\infty}(x - \tilde{t}; \tilde{t}) \) depends only on the transverse component \( x_\perp \) of \( x - \tilde{t} \)—or some combination of these.

#### (a) Low-turbulence Reynolds number limit

The gauge function \( \sigma(R_T) \) can be set equal to \( R_T \) when \( R_T \ll 1 \), which means that \( \tilde{x} = x/R_A, \tilde{t} = t/R_A \) in this case. This limit is important because the flow is completely linear in this case. (The upstream flow is always linear in the rapid distortion region I and the unsteady boundary layer region II.) The solution can therefore be used to obtain a relationship between the spectrum of the imposed upstream turbulence and the spectrum of the Klebanoff modes in the boundary layer which can be directly compared with experiments. This was done by Leib et al. [11], who analysed the flow over a zero thickness flat plate and obtained reasonable agreement with the experiments (which at the time were mainly focused on the Klebanoff modes themselves and not so much on the turbulent spots that eventually developed) even though the turbulence Reynolds number was fairly large in most cases.

Leib et al. also found that the Klebanoff modes usually exhibited linear behaviour at the upstream boundary of the measuring region but eventually became nonlinear at the downstream end. However, the agreement between the predicted streamwise RMS levels and Kendall’s data [4], shown in figure 3, appears to be reasonably good over the entire measuring range.

Their results indicate that the Klebanoff mode amplitudes will only be large enough to agree with the experiments when they are continuously forced by the free-stream turbulence. They
also found that the Klebanoff modes were mainly produced by the streamwise vorticity in the upstream flow and not so much by the transverse (normal to the plate) vorticity—even though their computations show that the latter produced somewhat larger disturbance levels in the upstream region, where the Klebanoff mode amplitudes are relatively small.

(b) Upstream boundary conditions independent of $x_1 - t$

The main disadvantage of the linear solution is that it does not cause the streamwise velocity profiles to deviate from the Blasius profile by an $O(1)$ amount in region III and, therefore, does not provide the optimal base flow for studying the secondary instability. Wundrow & Goldstein [8] considered the simplest $O(1)$ turbulence Reynolds number case, which basically amounted to assuming zero plate thickness and choosing the upstream boundary condition $u_\infty(x - \hat{\imath}; \hat{t})$ to represent steady streamwise vortices in the upstream flow, i.e. putting $u_\infty(x - \hat{\imath}; \hat{t})$ equal to $\{0, v_\infty(x_\perp), w_\infty(x_\perp)\}$. The flow in region IV, $\tilde{u}(\tilde{x}, \tilde{x}; \tilde{t}_0)$, is therefore independent of $\tilde{t}_0 = \tilde{t} - \tilde{x}$ and satisfies (3.19) and a homogeneous version of (3.20) (figure 4).

Their calculations show that the boundary layer disturbances exhibit strong streamwise focusing and develop high-shear layers that become inflectional in the wall normal direction and may, therefore, be able to support rapidly growing inviscid instabilities which are governed by the generalized Rayleigh equation

$$ \left( \frac{\partial}{\partial \hat{t}} + U \frac{\partial}{\partial x} \right) \nabla^2 p' - 2 \frac{\partial}{\partial x} \left( U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z} \right) p' = 0, $$

(4.1)
because \( U = U(\bar{x}, y, z) \) changes on the slow streamwise length-scale \( \bar{x} \) and, therefore, varies more slowly with respect to \( x \) than it does with respect to \( y, z \). The subscripts on \( U \) denote partial derivatives, \( p' \) denotes the pressure fluctuation associated with the instability and, as usual, \( \nabla^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2) \) denotes the Laplacian.

Rayleigh’s theorem does not, of course, apply in this case and inflection of the streamwise velocity is not a necessary condition for instability, as shown, for example, by Hall & Horseman [21]—but there does seem to be some sort of empirical connection here. The Wundrow & Goldstein [8] calculations also show that the lateral high-shear layers move up towards the edge of the wall layer III with increasing downstream distance while the wall normal inflection points tend to lie on the fuller velocity profiles between the streaks.

A number of calculations of the secondary instability of low-frequency base flows have now appeared in the literature. These flows are governed by the fully unsteady boundary region equations (2.2)–(2.5) in the wall layer region III [18,22]. All of them require the flow to be spanwise periodic but consider more general upstream boundary conditions than the one considered in [8]. The unstable component of the flow is still governed by the generalized Rayleigh equation (4.1) but with \( U = U(\bar{x}, y, z, \bar{t}) \) now dependent on the slow time variable \( \bar{t} \) which, however, enters only parametrically. The instability is therefore intermittent and only occurs during part of the oscillation cycle [23–25]. Ricco et al. [18] consider the case where the imposed upstream velocity has the form \( \{0, v_\infty(x, \bar{t}), w_\infty(x, \bar{t})\} \). The flow, \( \bar{u}(\bar{x}, \bar{t}; \bar{t}_0) \), in region IV then depends on \( \bar{t}_0 \equiv \bar{t} - \bar{x} \) but satisfies the same equations as in [8].

The Vaughan & Zaki [22] calculations show that these flows can support two types of instability during at least part of the cycle (as first noted for a steady flow in [26]). They are now usually referred to as the varicose and sinuous modes because of their appearance when looking down at the plate surface. Swearingen & Blackwelder [26] investigated the instability of the streaks that appear downstream of the Gortler vortices on a concave wall. They found that only the varicose instability is related to the transverse shear while the varicose mode is a consequence of the lateral high-shear layers.

Ricco et al. [18] only consider the sinuous mode. They found that the wall layer flow is qualitatively similar to that of Wundrow & Goldstein—at least in so far as its outer portion is concerned, but did not seem to find the strong spanwise focusing reported in [8]. Their results suggest that the wall normal inflection points tend to lie on the fuller velocity profiles between the low-speed streaks while the lateral high-shear layers and the maximum of the sinuous mode shape functions appear to move up into the upper edge of the boundary layer during part of the oscillation cycle. Some typical results for the sinuous mode shape function are shown in figure 5.

It seems that both unsteady and nonlinear effects cause the lateral high-shear layers to move up into the outer edge of region III. But (as noted earlier) the edge layer is absent in these flows. It should, however, be present in the corresponding real flow where \( u_\infty(x - \bar{t} \bar{t}; \bar{t}) \) depends on \( x - t \) and therefore also contains high-frequency components. It is likely that the modal maxima of the region III instability waves will eventually approach this layer and, since the high-frequency free-stream disturbances may be able to penetrate, i.e. drive the unsteady flow in this region, they should be able to generate (the spatially growing) instability waves through a neutral point.
resonant mechanism similar to that considered in [27]. This could explain the experimentally observed wave packets which propagate downstream at about 0.8\(U_\infty\) and eventually turn into the turbulent spots that lead to transition.

Vaughan & Zaki [22] considered a similar problem from a more numerical point of view than Ricco et al. But they consider both varicose and sinuous instabilities and show that, while the sinuous mode tends to be concentrated in the outer edge of the boundary layer, the varicose mode tends to be localized in the inner region of the flow. They, therefore, refer to these instabilities as inner and outer modes and argue that, while the sinuous mode is generated locally by the free-stream disturbances, the varicose mode is generated near the leading edge.

But these instabilities are a collective phenomenon which depends on the entire spanwise structure, while the experiments suggest that the instability is local. However, both papers only consider flows that are time periodic in the slow variable. It may, therefore, be of some interest to consider the case where \(\tilde{u}(\tilde{x}, \tilde{x}; \tilde{t}_0)\) is a random function of \(\tilde{t}_0\) because that could produce some relatively rare streamwise distortions that are much larger than their RMS values and could help explain the random occurrence of the precursor wave packets that lead to transition at much smaller RMS values than predicted for time periodic disturbances.

\[x = 0.7 \quad \Phi = 25\pi/16\]

\[x = 1.2 \quad \Phi = \pi\]

**Figure 5.** Instability mode partially concentrates in the outer edge of the boundary layer during part of the oscillation cycle. (From [18].)

\[\bar{x} = 0.7 \quad \Phi = 25\pi/16\]

\[\bar{x} = 1.2 \quad \Phi = \pi\]

(c) **High-turbulence Reynolds number (the leading edge problem)**

The initial streak breakdown does seem to result from wave packets that occur at the outer edge of the boundary layer in most Klebanoff mode experiments. But these experiments were conducted at relatively low free-stream turbulence levels. Nagarajan et al. [28] carried out numerical simulations which suggest that a somewhat different mechanism might occur at the higher Tu levels that are typical of internal turbomachinery flows, say Tu > 4%, when the leading edge is sufficiently blunt. They found that linear wave packets seem to form in regions of high surplus velocity and appear to lie closer to the wall (since they move downstream at speeds in the 0.55\(U_\infty\) – 0.65\(U_\infty\) range), and, much more interestingly, that these streamwise wave packets tend to occur when a particularly energetic vortex extends downstream of the stagnation plane by virtue of being wrapped around the leading edge. These wave packets again appear to grow on the streak-like background disturbance inside the boundary layer and are again precursors to turbulent spots which eventually dominate the motion. But overlaying the spot precursors on streaks suggested that the wave packets were not located on the low-speed streaks.

This suggests, among other things, that distortion of turbulence by the mean flow around the leading edge causes wake-like disturbances in the free-stream turbulence to be transformed...
into streamwise aligned vortices that penetrate into the boundary layer to produce the localized thickening and thinning of the region characteristic of the Klebanoff modes.

It is therefore of some interest to consider the case where the upstream vorticity is normal to the plate even though the analysis by Leib et al. [11] shows that the Klebanoff modes are mainly determined by the streamwise vorticity in the upstream flow and not so much by the transverse (normal to the plate) vorticity when the plate has zero thickness. The stretching of the vorticity around the leading edge will not only selectively amplify the low-frequency component of this vorticity but will also produce the streamwise vorticity at the edge of the plate required to drive the spatially growing component of the boundary layer velocity in region II when the plate thickness is finite [29]. This is best illustrated by taking the upstream boundary condition \( u_\infty(x-i\tilde{t};\tilde{t}) \) to be \( \{u_\infty(x,\tilde{t}),0,0,0\} \) or even more simply by \( \{u_\infty(z,\tilde{t}),0,0\} \), which corresponds to the flow configuration shown in figure 6.

However, as indicated earlier, the finite plate thickness must now be taken into account. So, as in §2, the flow in region I consists of the two-dimensional potential flow about the plate corrected for the mean boundary layer displacement effect plus an \( O(\epsilon) \) perturbation flow that can be found from rapid distortion theory (RDT) by solving a simple boundary value problem for Poisson’s equation. Its solution shows that the cross flow velocity exhibits a logarithmic singularity at the surface of the plate, which would be absent in a similar solution for the completely unsteady case, where \( u_\infty(x-i\tilde{t};\tilde{t}) \) depends on \( x-\tilde{t} \) [31]. This justifies my previous assertion about the selective amplification of the low-frequency component of the motion.

The detailed solution has to be found numerically, but our interest here is in the solution in the downstream region III, and the upstream boundary conditions for this solution can again be found from the asymptotic solution to the RDT problem as \( x \to \infty \), which can be obtained in closed form independently of the upstream solution. The relatively mild singularity in this solution is smoothed out by viscous effects in the linearized boundary layer solution of region II. The mean boundary layer is of the Blasius type at large downstream distances and an analysis similar to the one carried out by Crow [6] shows that the streamwise velocity in this region behaves like [10]

\[
F_B(\eta) + \epsilon x \ln \left( \frac{\epsilon x}{R_T} \right) \tilde{u}_\infty''(z,\tilde{t})\eta F''_B + O(\epsilon x) \quad \text{as} \quad x \to \infty,
\]

Figure 6. Flow configuration considered in [10,15,30].
where (as in §1) \( \eta \) denotes the Blasius variable, \( F_B \) the Blasius function and \( \tilde{a} \) is an \( O(1) \) constant that can be determined from the mean potential flow around the plate. The main difference between this result and that of [6] is the logarithmic gauge function \( \ln(\varepsilon \chi/R_T) \) produced by the surface singularity in the outer RDT solution.

These results provide upstream matching conditions for the solution in region III. As noted in §2, the solution in this region still involves quite a bit of numerics. However, a lot can be learned by considering the high-turbulence Reynolds number limit \( R_T \equiv \varepsilon R_A \to \infty \). The additional gauge function \( \sigma(R_T) \) is determined by the leading edge bluntness effects in this limit and is given by

\[
\sigma(R_T) \equiv \frac{1}{\tilde{a} \ln \delta_0}, \tag{4.3}
\]

with \( \delta_0 \) determined by \( \delta_0 = [R_T \tilde{a} \ln(1/\delta_0)]^{-1/2} \). The flow in region III where \( \tilde{x} = O(1) \) then splits into three layers. These include an outer region where the upstream linear rapid distortion theory solution continues to apply, and a viscous wall layer in which \( Y \equiv y/\delta_0 = O(1) \), the solution expands like [31]

\[
u = \{U(\tilde{x}, Y, z, \tilde{t}), (\varepsilon/\sigma)\delta_0 V(\tilde{x}, Y, z, \tilde{t}),(\varepsilon/\sigma)W(\tilde{x}, Y, z, \tilde{t})\} + \cdots, \tag{4.4}
\]

and (since the thickness, \( A\delta_0 \), of this layer is small compared with \( A \)) the flow is governed by the three-dimensional boundary layer equations

\[
\frac{\partial U}{\partial \tilde{t}} + U \frac{\partial U}{\partial \tilde{x}} + V \frac{\partial U}{\partial Y} + W \frac{\partial U}{\partial z} = \frac{\partial^2 U}{\partial Y^2}, \tag{4.5}
\]

\[
\frac{\partial W}{\partial \tilde{t}} + U \frac{\partial W}{\partial \tilde{x}} + V \frac{\partial W}{\partial Y} + W \frac{\partial W}{\partial z} = \frac{\partial^2 W}{\partial Y^2}, \tag{4.6}
\]

and

\[
\frac{\partial U}{\partial \tilde{x}} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial z} = 0, \tag{4.7}
\]

with no pressure gradient term. Equations (2.1) and (4.3) show that region III now lies closer to the leading edge than in the \( R_T = O(1) \) case discussed in §4b, which implies that the Klebanoff modes will be generated closer to the leading edge when the turbulent Reynolds number is large.

The third layer is an inviscid vorticity layer induced by the log singularity in the upstream RDT solution. The scaled variable \( \tilde{y} \equiv -\sigma \ln y \) is \( O(1) \) in this region and the scaled cross flow velocity \( W_\infty(\xi, z; \tilde{t} - \tilde{x}) = w/\tilde{a} \ln y = wo/\tilde{a}_0 \tilde{y} \), which depends on \( \xi \equiv \tilde{x} \ln y/\ln \delta_0 = \tilde{x} \tilde{a}/\varepsilon, z \) and parametrically on the slow variable \( \tilde{t} - \tilde{x} \), is determined by the inviscid Burger equation

\[
\frac{\partial W_\infty}{\partial \xi} + W_\infty \frac{\partial W_\infty}{\partial z} = 0, \tag{4.8}
\]

which can of course be solved analytically to obtain

\[
W_\infty(\xi, z; \tilde{t} - \tilde{x}) = f(z - \xi W_\infty(\xi, z; \tilde{t} - \tilde{x}); \tilde{t} - \tilde{x}). \tag{4.9}
\]

Matching with the upstream solution shows that the arbitrary function \( f \) in this solution is determined by \( f(Z; \tilde{t} - \tilde{x}) = -u'_\infty(Z; \tilde{t} - \tilde{x}) \), where the prime denotes differentiation with respect to the dummy variable \( Z \).

This solution provides an outer edge boundary condition for the solution in the viscous wall layer whose thickness \( \delta_0 \Lambda \) is now smaller than \( \Lambda \) by a factor of the square root of the turbulence Reynolds number \( R_T \) times \( \ln(1/\delta_0) \). Since \( \xi = \tilde{x}(1 - \tilde{a}_0 \sigma \ln Y) \to \tilde{x} \) for \( Y = O(1) \), matching inner and outer solutions shows that the outer edge boundary conditions for the scaled streamwise and cross flow velocities in the viscous wall layer are

\[
U \to 1, \ W \to W_\infty(\tilde{x}, z; \tilde{t} - \tilde{x}) \quad \text{as} \quad Y \equiv \frac{y}{\delta_0} \to \infty. \tag{4.10}
\]

The scaling (4.4) suggests that the flow structures in the viscous wall layer will be more vortex-like and less streak-like than the Klebanoff modes discussed in §4b.
In the completely steady case where \( u_\infty \) is independent of the slow variable, \( \bar{t} - \bar{x} \), the solution to this boundary value problem is given in terms of the Blasius solution by [10]

\[
U = U_B(\bar{\eta})
\]

and

\[
W = W_\infty U(\bar{\eta}),
\]

where (as can be seen by direct substitution) \( \bar{\eta} \) is given by

\[
\bar{\eta} = \frac{(1 + \bar{x}^f)Y}{[(1/3f^2)((1 + \bar{x}^f)^3 - 1)]^{1/2}},
\]

with \( f(z, \bar{x}) \) related to \( W_\infty \) by

\[
f = f(z - \bar{x}W_\infty) = W_\infty
\]

(rather than the usual Blasius variable). Since \( \bar{\eta} \) now depends on \( z \), \( U \) also exhibits this dependence and the solution is directly interpretable as a local thickening and thinning of the Blasius boundary layer as originally proposed in [3,5].

The local inviscid instability of the more general, i.e. unsteady, wall layer flow is now determined by the usual Rayleigh equation

\[
\left( \frac{\partial}{\partial T} + U \frac{\partial}{\partial X} \right) \left( \frac{\partial^2 p'}{\partial X^2} + \frac{\partial^2 p'}{\partial Y^2} \right) - 2U \frac{\partial^2 p'}{\partial X \partial Y} = 0,
\]

since \( U(\bar{x}, Y, z, \bar{t}) \) changes much more rapidly with respect to the fast space variable \( Y \) than it does relative to the slow variables \( \bar{x}, z, \bar{t} \) and, therefore, behaves like a one-dimensional steady parallel shear flow to lowest approximation. Rayleigh’s theorem now applies, i.e. wall normal inflection is now a necessary condition for instability. The sinuous mode is not present and the instability is a limiting form of the varicose mode discussed in the previous section. Figure 7 is a plot of the instantaneous streamwise velocity contours (calculated from (4.5)–(4.7) and (4.8)–(4.10)) for the, more or less randomly chosen, upstream velocity perturbation

\[
u'_\infty(z; \bar{t} - \bar{x}) = -0.267[\sin \omega(\bar{t} - \bar{x}) + \sin 3\omega(\bar{t} - \bar{x})]\left[\sin \left( \frac{z + \pi}{3} \right) + \sin 3z \right].
\]

It is easy to show from (4.11) and the Blasius solution that the velocity profiles are completely non-inflectional in the steady case—except, of course, at the wall—and cannot, therefore, support convective Rayleigh instabilities. But figure 7 shows that the velocity profiles can be inflectional during part of the oscillation cycle in the unsteady case, which means that unsteadiness is necessary for instability of this flow. The calculations also show that the wall normal inflection points can even occur in the fuller velocity region between the low-speed streaks where they tend to be closer to the wall. These profiles can therefore support the streamwise growing wave packets observed in [28] as suggested by the rudimentary two-dimensional stability calculations carried out in [30].

Of course, the Burger equation solution \( W_\infty = f(z - \bar{x}W_\infty) \) eventually develops a spanwise array of singularities at discrete downstream positions say \( (\bar{x}, z) = (\bar{x}_n(\bar{t}), z_n(\bar{t})) \), for \( n = 0, \pm 1, \ldots \) and is discontinuous along lines extending downstream of these points. This introduces lines of singularities in the inviscid vorticity layer solution \( \omega = -\bar{\varepsilon} \ln y f(z - \zeta W) \) and surfaces of discontinuity downstream of those lines. New inviscid solutions can be constructed to eliminate these singularities and an inviscid solution that brings in pressure gradient effects could probably be constructed to eliminate the surface discontinuities downstream of these lines. However, the wall layer solution continues to apply on both sides of these lines.

Equation (4.12) implies that the limiting (or surface) streamlines satisfy \( dz/d\bar{x} = W_\infty \) for the steady solution (4.11), which means that these streamlines have discontinuous slope across these lines of discontinuity and produce a streamline pattern similar to that shown in figure 8.
This, in turn, suggests that there is a collisional separation along these lines of discontinuity [32]. It can probably be thought of as a kind of bursting of the boundary layer that is usually associated with unsteady flows. This separation is of course steady, but the computations in [30] show that similar but wholly unsteady types of separation occur when low-frequency unsteadiness is added back into the upstream flow.

Numerical simulations of a similar type of flow in [33], which studies the impulsively started flow produced by a spanwise-periodic array of vortices adjacent to a flat plate, show that the expected eruptive separation (i.e. the formation of an eruptive spike) is prevented from occurring.

**Figure 7.** Contours of constant streamwise velocity and its second derivatives at two different instants of time (calculated by Dr Adrian Șescu of Mississippi State University).

**Figure 8.** Wall streamline pattern for Goldstein, Leib and Cowley problem [15].
by a localized instability in the cross flow motion when the Reynolds number is sufficiently high. Brinkman et al. claim that their extensive numerical computations prove that the instability is physical and not numerical and present evidence to suggest that it is associated with Rayleigh instability of the spanwise velocity profiles as opposed to the streamwise profiles. However, Cowley [34] suggests that there may be some issues with these calculations and provides an analytical argument that appears to show that this type of instability can arise spontaneously.

The experiments seem to show that the turbulent spots appear spontaneously without any precursor wave packets when the free-stream turbulence level is greater than 5% or so. The Brinkman & Walker [33] instability is of the absolute or global type that grows in time rather than in space and appears to develop spontaneously in unsteady flows without external forcing. It may, therefore, explain the near spontaneous turbulent spot formation that occurs in these experiments [34]. This is of course fairly speculative and needs additional analysis before it can be accepted as fact.

While the Nagarajan et al. [27] results imply that leading edge bluntness is extremely important at the higher free-stream turbulence levels, Kendall [4] and Whatmuff [35] carried out some experiments at relatively low free-stream turbulence levels which seem to show that changing the leading edge aspect ratio changes the transition location but not the Klebanoff mode RMS amplitude and spacing. However, the turbulence was pretty low in their experiments and their plates were fairly thin, and more importantly the screens were located upstream of a large wind tunnel contraction in their experiments, which would tend to suppress the normal vorticity relative to the streamwise component.

5. Summary and conclusion

There appears to be three somewhat distinct spot generation mechanisms that can occur when the free-stream turbulence level is greater than 1%:

(i) The breakdown of streak-like structures due to sinuous mode instability generated by disturbances in the free stream. These streaks are primarily generated by streamwise vorticity in the upstream flow and appear relatively far downstream from the leading edge whose exact geometry seems to be unimportant.

(ii) Breakdown of more vortex-like structures due to varicose mode instability. These structures are primarily generated by plate normal vorticity in the upstream flow and appear to be concentrated near the leading edge [22]. The leading edge geometry seems to play an important role here.

(iii) Instability arising from collision of vortex-like structures generated by plate normal vorticity.

It is also worth noting that the edge layer problem has largely been ‘swept under the rug’ in this paper and further analysis of the flow in this region could turn out to be highly beneficial.

It is also worth noting that, while the focus of this paper has been on incompressible flows, the turbomachinery flows mentioned in §4c are highly compressible and could contain shock waves. The Klebanoff modes could also be strongly affected by acoustic disturbances emanating from turbulent boundary layers. It may, therefore, be worth considering the effect of shock waves on the Klebanoff modes, as recommended by one of the referees.

Acknowledgements. The author would like to thank Dr Adrian Sescu of Mississippi State University for carrying out the computations in figure 7.

Funding statement. This work was supported by the NASA Aerosciences Project.

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