Nonlinear Schrödinger equation on graphs: recent results and open problems

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In this paper, an introduction to the new subject of nonlinear dispersive Hamiltonian equations on graphs is given. The focus is on recently established properties of solutions in the case of the nonlinear Schrödinger (NLS) equation. Special consideration is given to the existence and behaviour of solitary solutions. Two subjects are discussed in some detail concerning the NLS equation on a star graph: the standing waves of the NLS equation on a graph with a δ interaction at the vertex, and the scattering of fast solitons through a Y-junction in the cubic case. The emphasis is on a description of concepts and results and on physical context, without reporting detailed proofs; some perspectives and more ambitious open problems are discussed.

1. Introduction

In recent decades, a large amount of work has been done concerning the existence and behaviour of solutions of nonlinear dispersive equations of Hamiltonian type. This is in part a consequence of the fact that many fundamental physical models belong to this family. In particular, nonlinear Klein–Gordon equations and their relatives are milestones of classical and quantum field theory; but in fact an important boost to these developments comes from more phenomenological models, such as the Korteweg–de Vries (KdV) equation describing shallow-water waves in certain approximations or the ubiquitous nonlinear Schrödinger (NLS) equation which describes electromagnetic pulse propagation in nonlinear (Kerr) media, Langmuir plasma waves, or in the quantum realm Bose–Einstein condensates (BECs), where it is better known under the name of the Gross–Pitaevskii equation. All the above Hamiltonian equations share a common
characteristic: they admit solitary solutions or briefly solitons. As it is well known, solitons are solutions emerging owing to a balance between nonlinearity and dispersion, and they are related to symmetries of the equations; in some relevant examples (such as the one-dimensional KdV and cubic NLS equation) they also have a strong relation to the complete integrability of the infinite-dimensional Hamiltonian system to which they refer, but their existence and main interesting properties are by no means restricted to integrable equations. Solitons are non-dispersive and to some extent particle-like solutions of certain PDEs; similar to the equilibrium points of finite-dimensional dynamical systems, they are an essential point of departure in the description of the phase portrait of the model equation which they solve. In this paper, we will demonstrate the existence and role of solitons for some model PDEs on ramified structures, in particular the simplest kind of them, the so-called star graphs. The model equation described in some detail is the NLS equation on graphs, about which a rigorous mathematical activity is developing. Before describing mathematical models, let us look at the literature on the physical origin which motivates the study. The two main fields where the NLS equation enters as a preferred model are the optics of nonlinear Kerr media and the dynamics of BECs. Both of these quite different physical situations have potential or actual application to graph-like structures. In nonlinear optics, we can mention arrays of planar self-focusing waveguides, and propagation in variously shaped fibre-optic devices, such as Y-junctions, H-junctions and more complex examples can be considered. Papers relevant to these items are [1,2], where also symmetry-breaking phenomena related to geometry are studied. In [3,4], experimental evidence of the interaction of solitons with inhomogeneities and defects is given, in particular scattering and capture of solitons in photonic traps and escaping of solitons from potential wells. These last phenomena are studied on one-dimensional media but they suggest a natural generalization to simple graph-like structures, such as Y-junctions. In [5], after an analysis of general issues discussed later in this paper, an example of a potential application to signal amplification in resonant scattering on networks of optical fibres is given. In the field of BECs and more generally of nonlinear guidance of matter waves there has been increasing interest in one-dimensional or graph-like structures. Boson liquids or condensates can be treated in the presence of junctions and defects in analogy with the Tomonaga–Luttinger fermionic liquid theory, with applications to boson Andreev-like reflection, beam splitter or ring interferometers (see [6,7] and references therein). Apart from experimental activity, some other theoretical and numerical studies should be mentioned. The NLS equation on graphs has discrete analogues [8–11] in which several dynamical behaviours can be studied analytically and numerically; other discrete models are spin models on star graphs related to the Kondo model [12]. The possible integrability of the cubic NLS on star graphs with special boundary conditions is discussed in [13], and finally quantum field theory on star graphs with a consideration of symmetry, integrability and analysis of special models is studied in [14].

(a) Preliminaries and the mathematical model

To pose the models, we briefly recall some preliminary notions (see [15–20] for a more systematic account). By a metric graph $G$ it is meant a set of edges $\{e_j\}_{j=1}^n$ and vertices, with a metric structure on any edge. Every edge of the graph is identified with a (bounded or unbounded) oriented segment, $e_j \sim I_j$. A function on a graph is a vector

$$\Psi = (\psi_1, \ldots, \psi_N) \quad \text{with} \quad \psi_j \equiv \psi_j(x_j); \quad x_j \in I_j.$$

In the following, as above, we will denote the elements of $L^2(G)$ by capital Greek letters, while functions in $L^2(\mathbb{R}^+)$ are denoted by lowercase Greek letters. The spaces $L^p(G)$ are defined in the natural way by

$$L^p(G) = \bigoplus_{j=1}^N L^p(I_j); \quad \|\Psi\|_p = \left(\sum_{j=1}^N \|\psi_j\|^p_{L^p(I_j)}\right)^{1/p}.$$
Only for the $L^2$-norm, we drop the subscript and simply write $\| \cdot \|$. Accordingly, we denote by $(\cdot, \cdot)$ the scalar product in $L^2(\mathcal{G})$. When an element of $L^2(\mathcal{G})$ evolves in time, we use in notation the subscript $t$: for instance, $\Psi_t$. Sometimes we shall write $\Psi(t)$ in order to emphasize the dependence on time, or whenever such a notation is more understandable. In a similar way, one can define the Sobolev spaces

$$H^p(\mathcal{G}) = \bigoplus_{j=1}^{N} H^p(I_j).$$

The definition, however, should be used with caution, because as such, without specification of behaviour at the vertices, these spaces have not the usual properties of Sobolev spaces, in particular continuous or compact embeddings. The main example of a metric graph that will be considered here is a star graph, which is characterized by a single vertex $v$ and $N$ infinite edges, and we put $e_j \sim (0, +\infty)$, $v \equiv 0$.

After functional spaces, differential operators can be given on graphs. In particular, we are interested in operators connected with the Laplacian and some variants. On the Hilbert space

$$\mathbb{H} = \bigoplus_{j=1}^{N} L^2((0, \infty)),$$

with elements

$$\psi = (\psi_1, \ldots, \psi_N) \in \mathbb{H},$$

endowed by a norm

$$\| \psi \| = \left( \sum_{j=1}^{N} \| \psi_j \|^2_{L^2(\mathbb{R}^+)} \right)^{1/2},$$

we consider the operator $H_\mathcal{G}$

$$H_\mathcal{G} \psi = \left( -\frac{d^2 \psi_1}{dx_1^2}, \ldots, -\frac{d^2 \psi_N}{dx_N^2} \right)$$

on a suitable domain

$$\mathcal{D}(H_\mathcal{G}) = \bigoplus_{j=1}^{N} H^2((0, \infty))$$

and self-adjoint conditions at the vertex.

The choice of the boundary condition of course qualifies the operator, and different boundary conditions give rise to different dynamics when the operator $H_\mathcal{G}$ is the generator of an evolution, e.g. Schrödinger or Klein–Gordon.

As is well known, unitary $N \times N$ matrices $U$ parametrize the family of self-adjoint Laplacians on $\mathcal{G}$, and the relation between $U$ and the specific boundary condition at the vertex is given by

$$(U - 1) \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix} + i(U + 1) \begin{pmatrix} \psi'_1(0) \\ \vdots \\ \psi'_N(0) \end{pmatrix} = 0.$$

However, we are not interested here in the greatest generality, and we restrict ourselves to the following simple cases:

— $\delta$ condition: $[U_{jk} = 2(N + i\alpha)^{-1} - \delta_{jk}]$

$$\psi(v) \equiv \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0), \quad \sum_{j=1}^{N} \psi'_j(0) = \alpha \psi(0) \quad \alpha \in \mathbb{R}.$$
The δ condition includes the so-called Kirchhoff condition \( \alpha = 0 \) as a special case, but it is convenient to distinguish between them. Note that for a graph with two edges, i.e. the line, continuity of wave function and its derivative for an element of the domain makes the interaction disappear; this fact justifies the name of free Hamiltonian.

From now on \( H_\alpha \) is the operator \( H_G \) with delta condition in the vertex with ‘strength’ \( \alpha \). To simplify the notation, the Kirchhoff case will be indicated with the symbol \( H_0 \). Another more singular interaction, to which we will refer occasionally, is given by the

\[
- \delta'_s \text{ condition:}
\frac{1}{n} \sum_{j=1}^{n} \psi'_j(0) = 0, \quad \psi_j(0) - \psi_k(0) = \frac{\beta}{n} (\psi'_j(0) - \psi'_k(0)), \quad j, k = 1, 2, \ldots, n,
\]

which coincides, in the case of the line, with a \( \delta' \) interaction of strength \( \beta \).

The operator \( H_\alpha \) has simple spectral characteristics. The absolutely continuous spectrum coincides with \([0, \infty)\). As regards the point spectrum,

- for \( \alpha < 0 \), \( \sigma_p(H_\alpha) = -\alpha^2 / N^2 \) and
- for \( \alpha \geq 0 \), \( \sigma_p(H_\alpha) = \{0\} \).

The presence/absence of the (negative) eigenvalue is the motivation for the name of attractive/repulsive delta vertex given to \( H_\alpha \) in correspondence with the cases \( \alpha < 0 / \alpha > 0 \). More specifically, a \( \delta \) vertex with \( \alpha < 0 \) can be interpreted as modelling an attractive potential well or attractive impurity. In fact, as in the case of the line, the operator \( H_\alpha \) is a norm resolvent limit for \( \epsilon \) vanishing of a scaled Hamiltonian \( H_\epsilon = H + \alpha V_\epsilon \), where \( V_\epsilon = (1/\epsilon)V(x/\epsilon) \) and \( V \) is a positive normalized potential defined on the graph in the natural way and \( H \) is the free Hamiltonian (see [17] and references therein).

Finally, the quadratic form associated with \( H_\alpha \) is

\[
Q[\Psi] = \frac{1}{2} \|\psi'\|^2 + \frac{\alpha}{2} |\psi(0)|^2
\]

\[
\mathcal{D}(Q) = \{\Psi \in H^1 \text{ s.t. } \psi(0) = \cdots = \psi_N(0) \} \equiv \mathcal{E}
\]

and

\[
\psi' \equiv (\psi'_1, \ldots, \psi'_N)^T.
\]

Note that \( \mathcal{D}(Q) \), the form domain of \( Q \), is independent on \( \alpha \). In the following, we will use the convenient notation \( \mathcal{E} \), calling it the energy domain (it is often considered the ‘true’ Sobolev space of order 1 on \( G \)).

To introduce the nonlinearity, we define a vector field \( G = (G_1, \ldots, G_N) : \mathbb{C}^n \to \mathbb{C}^n \) acting ‘componentwise’ as

\[
G_i(\zeta) = g(|\zeta_i|)\zeta_i \quad \text{with } g : \mathbb{R}^+ \to \mathbb{R} \quad \text{and} \quad \zeta = (\zeta_i) \in \mathbb{C}^n.
\]

The vector field \( G \) enjoys the important property of gauge (U(1)) invariance, i.e. \( G(e^{i\theta}\zeta) = e^{i\theta}G(\zeta) \). After this preparation, the more common evolution equations on the graph can be defined in the obvious way. The main examples are

(i) the nonlinear Schrödinger equation

\[
\frac{d}{dt} \psi_t = H_\alpha \psi_t + G(\psi_t) \quad \text{(NLS equation)}; \tag{1.1}
\]

(ii) the nonlinear Klein–Gordon equation

\[
- \frac{d^2}{dt^2} \psi_t = H_\alpha \psi_t + m^2 \psi_t + G(\psi_t) \quad \text{(NLKG equation).} \tag{1.2}
\]
Note that, from a mathematical point of view, the nature of a PDE on a graph amounts to a system of PDEs on suitable finite or infinite intervals (in the case of a star graph $N$ half-lines) in which the coupling is given exclusively through the boundary conditions at the vertices. In the following paragraph, we will see several illustrations of this simple remark in the case of our main example, the NLS equation.

A further specification of the model is given, specializing the study to the important case of a power nonlinearity
\[ g(z) = \pm |z|^{2\mu}, \quad \mu > 0. \]

In the case of the NLS equation, the minus sign corresponds to the so-called focusing nonlinearity and the plus sign to the defocusing one, both meaningful in the applications. In the NLKG equation, the minus sign is the most relevant. The cubic case is especially important both for the cubic-quintic nonlinearity encountered in nonlinear optics and other fields (e.g. \cite{21}), can be considered as well. In the following, we almost exclusively refer to the example of the NLS equation with focusing power nonlinearity and an attractive $\delta$ vertex. After fixing the model, the first preliminary and essential information regards its well-posedness, i.e. its existence (local or global) and uniqueness of solution in suitable functional spaces. For our preferential model, the NLS equation, the classical line of attack to well-posedness is through the integral form of the equation, given by
\[ \psi_t = e^{-iH_t^L} \psi_0 + i \int_0^t e^{-iH_t(t-s)} |\psi_s|^{2\mu} \psi_s \, ds. \] (1.3)

A formal analysis shows that for the NLS on graphs there exist conserved quantities, similar to the case of the line or of an open set in $\mathbb{R}^n$. These are the mass
\[ M[\psi] = ||\psi||^2 \] (1.4)
and the energy
\[ E[\psi] = \frac{1}{2} ||\psi'||^2 - \frac{1}{2\mu+2} ||\psi||^{2\mu+2} + \frac{\alpha}{2} |\psi(0)|^2; \] (1.5)
this last quantity for $\mu \in (0, 2)$ is well defined on the domain $\mathcal{D}(E) = \mathcal{E}$ of the quadratic part of the energy, i.e. the energy of the linear Schrödinger dynamics on the graph.

A sample rigorous result giving the well-posedness of the NLS equation on star graphs and conservation of mass and energy is the following theorem.

**Theorem 1.1 (local and global well-posedness in $\mathcal{E}$).** For any $\psi_0 \in \mathcal{E}$, there exists $T > 0$ such that equation (1.3) has a unique solution $\psi \in C^0([0, T), \mathcal{E}) \cap C^1([0, T), \mathcal{E}^*)$.

Moreover, equation (1.3) has a maximal solution $\psi^{\text{max}}$ defined on an interval of the form $[0, T^*)$, and the following ‘blow-up alternative’ holds: either $T^* = \infty$ or
\[ \lim_{t \to T^*} ||\psi^{\text{max}}_t||_\mathcal{E} = +\infty, \]
where we denoted by $\psi^{\text{max}}_t$ the function $\psi^{\text{max}}$ evaluated at time $t$.

Moreover, in the same hypotheses, the following conservation laws hold at any time $t$:
\[ M[\psi_t] = M[\psi_0], \quad E[\psi_t] = E[\psi_0]. \]

Finally, for $0 < \mu < 2$ and any $\psi_0 \in \mathcal{E}$, equation (1.3) has a unique solution $\psi \in C^0([0, \infty), \mathcal{E}) \cap C^1([0, \infty), \mathcal{E}^*)$.

In the previous theorem, $\mathcal{E}^*$ is the dual space of $\mathcal{E}$. As in the case of the NLS equation on the line, we will call the range of nonlinearities $0 < \mu < 2$, where existence for all times is guaranteed, the subcritical case, in contrast to the supercritical nonlinearities ($\mu > 2$) where blow-up occurs.
(in the form described in the theorem), or the threshold critical ($\mu = 2$) nonlinearity where global existence depends on the size of the initial datum. For details of the proof and generalization to other couplings at the vertex of the star graph, see [22,23]. The same theorem holds true for more general nonlinearities with essentially the same proof.

The previous examples are modelled starting from generators which are nonlinear perturbations of ‘quantum graphs’.

On the other hand, in principle, other equations could be considered. Without embarking here on a general theory we limit ourselves to mentioning some special models. The first one is given by the Benjamin–Bona–Mahony equation (BBM), which describes the unidirectional shallow-water flow under the long-wave and small-amplitude approximation,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3 \partial t} = 0.$$  

Of course here the unknown $u$ is real. We can immediately extend such an equation on a graph. To simplify the exposition, consider the case of a star graph with three edges, a Y-junction. Let us define a vector $u = (u_1, u_2, u_3)$, where $u_i: (0, \infty) \to \mathbb{R}$, and suppose that at the vertex the three components $u_i$ satisfy the Kirchhoff boundary conditions. These conditions are rather natural in the context of water (and other fluids) waves, corresponding to continuity of the flow and flux balance.

Now orienting the edges of the Y-junction in the outgoing direction as above and setting $\sigma_1 = -1, \sigma_2 = \sigma_3 = 1$, let us consider the evolution problem

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x} + u_i \frac{\partial u_i}{\partial x} - \frac{\partial^3 u_i}{\partial x^3 \partial t} = 0 \quad x \in \mathbb{R}^+, \ t > 0;$$

$$u_i \in H^2(\mathbb{R}^+), \ u_1(0) = u_2(0) = u_3(0), \ \frac{\partial u_1}{\partial x}(0) + \frac{\partial u_2}{\partial x}(0) + \frac{\partial u_3}{\partial x}(0) = 0.$$

This is a system of scalar BBM equations on the half-line, coupled through the Kirchhoff boundary condition at the origin. In [24], a generalization of this system to trees (including star graphs) is studied as a model of the cardiovascular system, and in particular it is shown that BBM on a tree is well-posed. It is interesting that [25] travelling waves for the BBM equation on a tree have recently been constructed for particular vertex conditions. Also, a result of well-posedness for the NLS equation on trees appears as a consequence of relevant dispersive estimates proved in [26,27]. Other nonlinear models of quite different nature are the reaction–diffusion equations on networks, about which some literature exists, in particular regarding pulse propagation in axons and neural networks according to the FitzHugh–Naguno–Rall model (see [28] and references therein). Concerning the mathematical setting, we conclude with a comment about the one-dimensional approximation given by the NLS equation on a graph. The graph should be a limit in some sense of a more realistic system of thin tubes (or guides) connecting at junctions. The first problem is getting the limit of a certain dynamical model, e.g. the NLS equation defined on the system of thin guides when the transversal size of tube vanishes. It is reasonable to conjecture that this limit should be an NLS on a graph as defined before, but with which boundary conditions? How might the boundary conditions at the vertex depend on the limit process of shrinking the tubes, guides and junction size? A further difficulty could be the dependence of the limit process on the conditions at the boundary of the thin tube, e.g. Neumann or Dirichlet. These problems have been tackled, with partial solutions, in the linear case (see [29–31] and references therein). They remain open for nonlinear models, where there is no literature, with the only exception (to the knowledge of the author) being a series of papers about the reduction of the Ginzburg–Landau equation and its stationary counterpart from thin tubes to graphs (see [32–34] and references therein), where some special boundary conditions at the junction appear in the limit. The second point of view is less phenomenological, and is related to the deduction of the evolution equation for BECs from first principles. The dimensional reduction of BECs using scaling trapping potentials is a well-understood process in several limiting regimes; in particular a Gross–Pitaevskii energy functional on the line describes the so-called cigar-like BECs under...
certain conditions (see [35,36] and references therein). Being confident that a Gross–Pitaevskii
equation realizes the correct quasi-one-dimensional limit, a similar procedure could be attempted
on a graph-like structure, for example on a Y-junction; to simplify the analysis, one could start
directly from an $N$-body theory on the graph, and attempt to take the $N \to \infty$ limit with suitable
scalings. Note that the main problem, i.e. the treatment of the boundary condition, is open also in
the simplest case of a graph with two edges, i.e. a line with a defect.

2. The nonlinear Schrödinger equation on star graphs: rigorous results

After setting the mathematical model and giving the main physical premises and possible
applications, we turn to a description of some of its dynamical features. Again, we refer to the
case of the focusing NLS equation where more information is known. Two main topics have been
rigorously investigated in recent years: the existence and characterization of standing waves, and
the scattering of solitons through a junction. To introduce the two subjects, let us preliminarily
recall some properties of NLS solitons on the line. Let us consider the usual focusing NLS equation
with power nonlinearity in one space dimension

\[ i \frac{\partial u}{\partial t}(x,t) = -\frac{\partial^2 u}{\partial x^2}(x,t) - |u(x,t)|^{2\mu}u(x,t) \quad x \in \mathbb{R}, \quad t > 0; \]  

(2.1)

as it is well known, it admits a special solution $u(x,t) = e^{i\phi}$ with

\[ \phi(x) = [(\mu + 1)]^{1/2\mu} \text{sech}^{1/\mu}(\mu x). \]  

(2.2)

A richer family of solutions is obtained through application of Galilean and scaling symmetries
of the NLS equation

\[ u_{x_0,v,\omega}(x,t) := e^{i(v/2)x} e^{-it(v/4)} e^{i\omega t} \omega^{1/2\mu} \phi(\sqrt{\omega}(x - x_0 - vt)). \]  

(2.3)

Note that frequency $\omega$ of the oscillation and amplitude of the solitary wave $u_{x_0,v,\omega}$ are nonlinearly
related, and in particular the greater the amplitude, the greater the oscillation frequency.

(a) Standing waves of the nonlinear Schrödinger equation on star graphs

Let us begin with the first subject, standing waves. On the line, they appear by putting $v = 0$ in
the previous family of solitary solutions, and they have the character of localized solutions (or
‘pinned’ solitons, in the physical literature) around a certain centre $x_0$. In particular, they are the
only soliton solutions when travelling waves are excluded by the presence of inhomogeneities;
these can be represented, for example, by external potentials, magnetic fields or, as in our case,
by a boundary condition at the junction. In these cases, the localization of the standing wave
is around stationary points of external potentials, or at the location of singular interactions in
the case of junctions, point defects, etc. Here, we will define standing waves as a finite energy
solution to an NLS equation (for other models with $U(1)$ symmetry, such as the NLKG equation,
the definition is exactly the same) of the form

\[ \Psi_\omega(x) = e^{i\omega t} \Phi_\omega(x). \]  

(2.4)

The function $\Phi_\omega$ is the amplitude of the standing wave. In particular, we are interested in standing
waves of the NLS equation on a graph, equation (1.1). A regularity argument shows that the
standing waves belong in fact to the operator domain of $H_\alpha$. Correspondingly, there should exist
a frequency $\omega$ and an amplitude $\Phi_\omega$ that satisfy in the strong sense the stationary equation

\[ H_\omega \Phi_\omega - |\Phi_\omega|^{2\mu} \Phi_\omega = -\omega \Phi_\omega. \]  

(2.5)

The analysis of this equation on a star graph is simple (see [23,37] for details).
On every edge the operator $H_\alpha$ coincides with the second derivative, and so it holds
\[ -\phi'' - |\phi|^{2\mu} \phi = -\omega \phi; \]
the most general solution with $\phi \in L^2(\mathbb{R}^+)$ is (introducing explicitly the dependence on parameters)
\[ \phi(\sigma,a;x) = \sigma(\mu + 1)\omega)^{1/2\mu} \text{sech}^{1/\mu}(\mu \sqrt{\omega}(x-a)), \quad |\sigma| = 1, \ a \in \mathbb{R}, \]
so that the components of the amplitudes are
\[ (\Phi_\omega)_i = \phi(\sigma_i, a_i), \]
where $\sigma_i, a_i$ have to be chosen to satisfy $\Phi_\omega \in \mathcal{D}(H_\alpha)$. Note that on every edge $i$ the stationary state has an amplitude which is ‘bump-like’ or ‘tail-like’ in shape, according to the position of the centre $a_i$, within the edge $i$ or not.

Continuity at the vertex implies the following conditions on the parameters of the amplitude:
\[ \sigma_j = 1, \quad a_j = \varepsilon_j a, \quad \varepsilon_j = \pm 1, \ j = 1, \ldots, N, \quad a \geq 0. \]
Being $\sigma_1 = 1 \ \forall j$ we will drop its mention in the following.

Moreover, imposing the $\delta$ vertex boundary condition to $\Phi_\omega$ gives
\[ \tanh(\mu \sqrt{\omega} a) \sum_{i=1}^{N} \varepsilon_i = \frac{\alpha}{\sqrt{\omega}}, \]
so that $\sum_{i=1}^{N} \varepsilon_i$ must have the same sign as $\alpha$ if $\alpha \neq 0$.

The immediate qualitative consequences of the above limitations are

- $\alpha > 0$ strictly more bumps than tails,
- $\alpha < 0$ strictly more tails than bumps,
- $\alpha = 0$ same number of tails and bumps or $a = 0$,
- for any value of $\alpha \neq 0$, there are $[(N + 1)/2]$ states ($[s]$ is the integer part of $s$) and
- lower bound on the allowed frequencies: $a^2/N^2 < \omega$.

We index the stationary states $\Phi_j^{\omega}$ with the number $j$ of bumps. With the above limitations, this identifies completely the state. More explicitly, the $j$-bumps state $\Phi_j^{\omega}$ is given by
\[ (\Phi_j^{\omega})_i (x) = \begin{cases} \phi(a^j; x) & i = 1, \ldots, j \\ \phi(-a^j; x) & i = j + 1, \ldots, N \end{cases} \]
and
\[ a^j = \frac{1}{\mu \sqrt{\omega}} \arctanh \left( \frac{\alpha}{(2j - N) \sqrt{\omega}} \right). \]

Concluding, solutions of (2.5) for $\alpha > 0$ are given by $\Phi_j^{\omega}$ with $j = [N/2 + 1], \ldots, N$ and for $\alpha < 0$ by $\Phi_j^{\omega}$ with $j = 0, \ldots, [(N - 1)/2]$.

The situation is shown in figure 1 for the $N = 3$ star graph.

So for $\alpha \neq 0$, for every $N$ and $\omega > \alpha^2/N^2$ there exist branches $\{\Phi_j^{\omega}\}$ of stationary states (the branch is unique only in the case $N = 2$, i.e. the line). More precisely, the state with $j = 0$ arises for $\omega > \alpha^2/N^2$, while to have states with $j > 0$ higher frequencies are needed, according to the general relation $\omega > \alpha^2/(N - 2j)^2$.

Now let us consider the Kirchhoff case, $\alpha = 0$.

From an analysis of the boundary conditions, it follows that star graphs with an odd or even number of edges behave differently.

For $N$ odd, the only value of $a$ compatible with the boundary conditions is $a = 0$. So the stationary state is unique
\[ (\Phi_0^{\omega})_i (x) = \phi(0, x) \quad i = 1, \ldots, N, \]
and it is composed by $N$ half-solitons continuously joined at the vertex.
Figure 1. \( N = 3 \) stationary states. (a) \( \alpha < 0 \); (b) \( \alpha > 0 \).

Figure 2. The Kirchhoff case \( \alpha = 0 \) for an odd (a) and even (b) number of edges.

For \( N \) even, every real value of \( a \) is compatible with the boundary conditions and there is the same number of tails and bumps on the graph. A one-parameter family of stationary states exists and is given by (note the slight change of notation used for the Kirchhoff solitary wave only)

\[
(\Phi^a_\omega)_i(x) = \begin{cases} 
\phi(-a, x) & i = 1, \ldots, \frac{N}{2} \\
\phi(+a, x) & i = \frac{N}{2} + 1, \ldots, N 
\end{cases} \quad a \in \mathbb{R}.
\]

These stationary states could be thought of as \( N/2 \) identical solitons on \( N/2 \) lines. The situation is depicted in figure 2. As a consequence, there exist travelling waves on a Kirchhoff graph with an even number of edges: simply translate the complete solitons on every fictitious line by the same amount \( vt \)

\[
\Phi^a_\omega(t) = e^{i((v/2)x-(v^2/4)t+\theta)}\Phi^a_\omega(t) \quad a(t) = a + vt.
\]

Note that in the Kirchhoff case stationary states exist for every positive \( \omega \).

A similar construction can be performed for NLKG standing waves with different restrictions on parameters, i.e. \( |\omega| < \sqrt{m^2 - (\alpha^2/N^2)} \). Details will be given elsewhere.

(b) Variational properties of standing waves

After constructing the stationary states, the natural problem is to identify the ground state and, if possible, to order the states in energy, i.e. to describe the nonlinear spectrum of the NLS equation on the graph. This is a variational problem, which is also relevant for the analysis of stability of standing waves. A difficulty immediately arises, in that NLS energy (1.5) for the focusing NLS equation is unbounded from below, as easily recognized (this is not the case for the defocusing nonlinearity). So, the seemingly natural definition of the ground state as the minimizer of the energy is meaningless. Nevertheless, the physics of the problems behind the NLS equation suggests that a possible relevant variational problem is a constrained variational problem: to minimize the energy at fixed mass. With this constraint and for subcritical nonlinearities \( \mu < 2 \)
the energy is shown bounded from below for every finite energy state, as in the case of the NLS equation on $\mathbb{R}^n$. In fact, the following result holds true (see [38] for details and proofs) for the focusing NLS on a star graph with an attractive $\delta$ interaction, $\alpha < 0$.

**Theorem 2.1 (minimizers for the energy functional).** Let $m^*$ be defined by

$$m^* = 2 \frac{(\mu + 1)^{1/\mu}}{\mu} \left| \frac{\alpha}{N} \right|^{2-\mu/\mu} \int_0^1 (1 - t^2)^{(1/\mu) - 1} dt.$$  \hfill (2.10)

Let $\alpha < 0$ and assume $m \leq m^*$ if $0 < \mu < 2$ and $m < \min(m^*, \sqrt{\frac{3}{\pi}} \cdot \frac{\sqrt{3N}}{4})$ if $\mu = 2$, where $\tilde{c}$ is the constant in the Gagliardo–Nirenberg inequality $\|\Psi\|_6^6 \leq \tilde{c} \|\Psi\|^2 \|\nabla\Psi\|^4$ for any $\Psi \in H^1$, and set

$$-\nu = \inf \{ E[\Phi] \text{ s.t. } \Phi \in \mathcal{E}, M(\Phi) = m \}.$$  

Then $0 < \nu < \infty$ and there exists $\hat{\Phi}$ such that $M(\hat{\Phi}) = m$ and $E[\hat{\Phi}] = -\nu$.

Moreover, the minimizer $\hat{\Phi}$ coincides with the N-tail state $\Phi_0^0$ where $\omega_0$ is such that $M(\Phi_0^0) = m$.

So, for every mass above a certain threshold $m^*$ (which however is not optimal) the problem of minimizing the NLS energy on the graph at constant mass has a solution if the vertex carries an attractive $\delta$ interaction. More precisely, if the mass constraint coincides with the mass of the N-tail state, the minimizer is exactly the N-tail state. Some comments are in order.

After some calculation, the mass of the stationary states as a function of $\omega$ turns out to be

$$M(\Phi_{\omega}^j) = \frac{(\mu + 1)^{1/\mu}}{\mu} \omega^{(1/\mu) - (1/2)} \left( N - 2 \right)^{1/2} \int_{|\omega|/(N-2)^{1/2}}^{1} (1 - t^2)^{(1/\mu) - 1} dt + 2jI, \quad (2.11)$$

where $I = I(\mu)$ is a certain constant depending on $\mu$ only. We recall that $\Phi_{\omega}^j$ is defined for $\omega \in (|\omega|^2/(N-2)^2, \infty)$. So from (2.11), one easily concludes that the functions $M(\Phi_{\omega}^j)$ are increasing in $\omega$ and the minimum value is in correspondence with the threshold $|\omega|^2/(N-2)^2$. As a consequence, the N-tail state can have an arbitrarily small mass, whereas the other stationary states have a minimal mass separated from zero. Stated otherwise, the manifold $M(\Psi) = m$ for $m < m^*$ may not contain all the stationary states, owing to the fact that some of them could have too large masses. On the contrary, the N-tail state always belongs to the constraint manifold because its mass has a vanishing lower bound. Taking into account the dependence of $m^*$ on $\alpha$, one concludes that for small $|\omega|$ the constraint manifold contains only the N-tail state while for large $|\omega|$ all the stationary states belong to the constraint manifold. Moreover from the expression of $m^*$, it follows that to guarantee the existence of the ground state for a given mass constraint one has to have a sufficiently deep $\delta$ well. So alternative statements and proofs of the above theorems are obtained by fixing $m$ and requiring $\alpha$ to be sufficiently negative. Analogous remarks also apply to the critical case $\mu = 2$, the quintic NLS.

Taking into account the above preliminary comments, a well-defined order in energy exists for the stationary states: the energies of the stationary states $\Phi_{\omega_j}^j$, where $\omega = \omega_j$, such that $M(\Phi_{\omega_j}^j) = m$, are increasing in $j$, i.e. they can be ordered in the number of the bumps [38].

At least in one case things are simple: the cubic case. If $\mu = 1$, then $\omega_j$ is independent of $j$ and

$$\omega_j \equiv \omega^* = \frac{(m + 2|\omega|)^2}{4N^2}.$$  

So, in the cubic case the energy spectrum at fixed mass can be explicitly computed

$$E[\Phi_{\omega_j}^j] = -\frac{N}{3} \omega_j^{3/2} + \frac{1}{3} \frac{|\omega|^3}{(2j - N)^2} = -\frac{1}{24} \frac{(m + 2|\omega|)^3}{N^2} + \frac{1}{3} \frac{|\omega|^3}{(2j - N)^2}.$$  

The energy of the ground state is given by

$$E[\Phi_{\omega_j}^0] = -\frac{1}{24N^2} m(m^2 + 6m|\omega| + 12|\omega|^2).$$

As a final remark, note that the ground state of the system is the only stationary state which is symmetrical with respect to permutation of the edges.
The second variational problem which has both a physical and mathematical relevance is related to the minimization of the action functional. The action for an NLS equation is obtained by adding the nonlinear term to the usual action of the linear Schrödinger equation

$$A[\psi] = \int_{t_1}^{t_2} \left( \frac{1}{2} \int_G (\bar{\psi} \Box \psi) dx + E(\psi) \right) dt.$$  

This expression, for a standing wave $\psi_t = e^{i\omega t} \Phi_\omega$, takes the form

$$A[e^{i\omega t} \Phi_\omega] = (t_1 - t_0) \left( E[\Phi_\omega] + \frac{\omega}{2} \| \Phi_\omega \|^2 \right).$$

This fact suggests that we should consider the ‘reduced’ action

$$S_\omega[\Phi] = E[\Phi] + \frac{\omega}{2} M[\Phi] = \frac{1}{2} \| \Phi' \|^2 + \frac{\omega}{2} \| \Phi \|^2 - \frac{1}{2\mu + 2} \| \Phi \|^{2\mu+2} + \frac{\alpha}{2} |\Phi(0)|^2,$$

which we continue to call action.

Apart from this Lagrangian origin the action $S_\omega$ has, at least in the context of the BEC, the physical interpretation of the grand-potential functional of the condensate corresponding to the chemical potential $\omega$.

Whatever the theoretical interpretation, the action functional just introduced enjoys the important property that its stationarity condition $S'_\omega[\Phi] = 0$ coincides with stationary equation (2.5). As for the energy, it is easy to see that the action is unbounded from below. Nevertheless, a solution of equation (2.5) satisfies necessarily (just take the scalar product of the stationarity equation with $\Phi$) to the constraint

$$I_\omega[\Phi] = \| \Phi' \|^2 - \| \Phi \|^{2\mu+2} + \omega \| \Phi \|^2 + \alpha |\Phi(0)|^2 = S'_\omega[\Phi] \Phi = 0.$$  

So, it is expedient to search for minima of the action restricted to the above natural constraint, also called in mathematical literature the Nehari manifold. It contains all the stationary states by definition. Now an immediate calculation shows that restricted on the natural constraint the action is

$$S_\omega[\Phi] = \frac{\mu}{2\mu + 2} \| \Phi \|^{2\mu+2},$$

and so it is bounded from below and non-negative (this is true for every power nonlinearity $\mu$, note the difference from the constrained energy). The absolute minimum of the action constrained to the Nehari manifold, if it exists, is called the ground state of the action, as for the energy. Note that, by the Lagrange multiplier theorem, the stationary points of the energy at fixed mass are stationary points of the action with the Lagrange multiplier $\omega$. In fact the ground states of the two problems coincide, as a consequence of the following result (see [23] for a complete discussion and proof).

**Theorem 2.2 (minimizers for the action functional).** Let $\mu > 0$. There exists $\alpha^* < 0$ such that for $-N\sqrt{\omega} < \alpha < \alpha^*$ the action functional $S_\omega$ constrained to the Nehari manifold admits an absolute minimum, i.e. a $\hat{\Phi} \neq 0$ such that $I_\omega[\hat{\Phi}] = 0$ and $S_\omega[\hat{\Phi}] = \inf(S_\omega[\Phi]) = 0$. Moreover, for $-N\sqrt{\omega} < \alpha < \alpha^*$ the ground state is $\hat{\Phi} = \Phi^0_\omega$.

The threshold $\alpha^*$ in the above theorem is known as a function of $N$, $\mu$ and $\omega$. The proof of theorem 2.2 is quite different from that of theorem 2.1. Nevertheless, the origin of thresholds $\alpha^*$ and $\alpha^*$ is analogous, and it relies on the fact that the action of the NLS on a Kirchhoff junction has no ground state: the infimum exists, but is not attained at any finite energy state. To explain, let us consider a Y-junction, i.e. a $N = 3$ star graph. There exist sequences of ‘runaway’ states (figure 3) on the Kirchhoff graph given by a complete soliton on a couple of half-lines plus a correctly joined soliton tail on the third half-line; Kirchhoff boundary conditions are easily verified, so we have a sequence of domain elements. Now, the more the big soliton shifts to infinity and the tail extinguishes, the less is the action, which can be shown to converge towards its infimum, strictly lower than the action of any stationary state. The same phenomenon occurs at the constrained energy level as shown in [39]. The bad behaviour of the Kirchhoff junction ($\alpha = 0$) prevents the
action with $\alpha$ small, or the energy with a big $m$, to have a constrained absolute minimum. It is a conjecture that the action has a \textit{local} constrained minimum that is larger than the infimum when the condition 
$$-N\sqrt{\omega} < \alpha < \alpha^*$$
fails, and similarly for the constrained energy with a corresponding condition on mass.

\section*{(c) Orbital stability of standing waves}

Stability is an important requisite of a standing wave, because unstable states are rapidly dominated by dispersion, drift or blow-up, depending on dynamics, and so are undetectable (stability and instability of the NLS with a $\delta$ potential on the line are studied, partly numerically, in [40]; see also references therein). Owing to gauge or $U(1)$ invariance of the action and dynamics, a standing wave is not stable in the usual Lyapunov sense. This is a general fact in the presence of symmetries, and it is well known in the example of relative equilibria for finite-dimensional mechanical systems. In our case to introduce the main concept of orbital stability, let us consider the special solution $e^{it}\phi(x)$ to equation (2.1), where $\phi$ is the initial datum given in (2.2); by scaling and $U(1)$ invariance, $u(x,t) = e^{i\omega t}\frac{m}{\mu}\phi(\sqrt{\omega}x)$ is the solution corresponding to the initial datum $\frac{m}{\mu}\phi(\sqrt{\omega}x)$. Choosing $\omega$ close to 1 the two initial data are close. But their time evolutions are not, because of different frequencies which make the distances of solutions vary with time:

$$\sup_{t \in \mathbb{R}} \|u(x,t) - e^{it}\phi(x)\|_{H^1} > \|\phi\|_{H^1}.$$ 

The same phenomenon occurs on $\mathbb{R}$ for Galilean invariance: slightly different velocities make travelling solitary waves separate from each other. In the case of graphs, Galilean invariance is not relevant because it is broken by the junction, travelling waves do not exist and so we concentrate on standing waves. The point of the previous discussion is that the stability has to be defined up to symmetries: solutions remain close to the orbit $e^{i\theta\Phi_\omega}$ of a ground state for all times if they start close enough to it (see [41,42] for a general discussion). The orbit of $\Phi_\omega$ is

$$\mathcal{O}(\Phi_\omega) = \{e^{i\theta\Phi_\omega}(x), \theta \in \mathbb{R}\}.$$

\textbf{Definition 2.3.} The state $\Phi_\omega$ is orbitally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{cases}
\Psi(0) \in \mathcal{E} \\
d(\Psi(0), \mathcal{O}(\Phi_\omega)) < \delta
\end{cases} \Rightarrow d(\Psi(t), \mathcal{O}(\Phi_\omega)) < \epsilon \quad \forall t > 0,$$

where

$$d(\Psi, \mathcal{O}(\Phi_\omega)) = \inf_{u \in \mathcal{O}(\Phi_\omega)} \|\Psi - u\|_{\mathcal{E}}.$$

The stationary state $\Phi_\omega$ is orbitally unstable if it is not stable.

A general theory of orbital stability was established in the 1980s in the classical papers by Weinstein [43] and Grillakis–Shatah–Strauss [44,45], and developed in a number of subsequent papers by many authors; it applies to infinite-dimensional Hamiltonian systems (such as the abstract NLS equation) when a regular branch of standing waves $\omega \mapsto \Psi_\omega$, not necessarily ground states, exists (see [46,47] for recent surveys and results about stability and instability in the
The first step is to give the NLS equation on a graph a Hamiltonian structure. This is achieved in a standard way, considering an element of $L^2(G, \mathbb{C})$ as the couple of its real and imaginary part, $\psi = U + iV \equiv (U, V)$ and endowing the Hilbert space $L^2(G, \mathbb{C}) \equiv L^2(G, \mathbb{R}) \oplus L^2(G, \mathbb{R})$ so obtained with the real scalar product $\langle \psi_1, \psi_2 \rangle = \Re \int_G \bar{\psi}_1 \psi_2 = \int_G U_1 U_2 + V_1 V_2$, and analogously decomposing the higher Sobolev spaces. The NLS on a graph turns out to be a Hamiltonian system

$$\frac{d}{dt} \begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{J} E'[U, V], \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix},$$

(2.13)

where $E[U, V] \equiv E[\psi]$ and the functional derivative is defined as usual

$$E'[U, V](X, Y) = \frac{d}{d\epsilon} E[(U, V) + \epsilon (X, Y)]_{\epsilon = 0} \quad \forall (X, Y) \in H^1(G, \mathbb{R}) \oplus H^1(G, \mathbb{R}).$$

Now linearize the Hamiltonian system around the ground state, setting

$$\psi(t) = (\phi_\omega + W + iZ) e^{i\omega t}.$$  

(2.14)

Note that the previous definition amounts to passing to a rotating coordinate system, co-moving with the ground state. Then the fluctuations $W$ and $Z$ satisfy

$$\frac{d}{dt} \begin{pmatrix} W \\ Z \end{pmatrix} = \mathcal{L} \begin{pmatrix} W \\ Z \end{pmatrix},$$

(2.15)

where

$$\mathcal{L} \begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1 W \\ \mathcal{L}_2 Z \end{pmatrix}$$

and $\mathcal{L}_1$ and $\mathcal{L}_2$ are matrix s.a. operators: their domain coincides, with a slight abuse of notation, with $\mathcal{D}(H_\omega)$ and the action is given by

$$(\mathcal{L}_1)_{i,k} = \left( -\frac{d^2}{dx^2} + \omega - (2\mu + 1)|\phi_{\omega,k}^0|^2 \right) \delta_{i,k}$$

(2.16)

and

$$(\mathcal{L}_2)_{i,k} = \left( -\frac{d^2}{dx^2} + \omega - |\phi_{\omega,k}^0|^2 \right) \delta_{i,k}.$$  

(2.17)

Owing to definition (2.14), it turns out that the linearized operator $\mathcal{L}$ coincides with the second derivative of the action: $\mathcal{L} = S''_\omega (\phi_\omega^0)$, after identification of the sesquilinear form with the operator via the scalar product.

The first information about stability is given by the spectrum of the operator $\mathcal{L}$. When the linearized operator $\mathcal{L}$ admits at least one eigenvalue with non-vanishing real part the stationary state $\phi_\omega^0$ is said to be spectrally unstable; otherwise, it is said to be spectrally stable. In fact, owing to the presence of conservation laws, in particular of the mass $\|\psi\|^2$, the system can be spectrally unstable without being orbitally unstable. Precisely, according to the general theory of Weinstein and Grillakis–Shatah–Strauss, for a solitary solution $\phi_\omega$ being orbitally stable it is sufficient that

(i) spectral conditions hold:

(1) $\ker \mathcal{L}_2 = \{ \phi_\omega \}$ and the remaining part of the spectrum is positive

(2) $\ker \mathcal{L}_1 = \{ 0 \}$

(3) the number of negative eigenvalues (the Morse index) of $\mathcal{L}_1$ is equal to 1;

(ii) the Vakhitov–Kolokolov condition $(d/d\omega)\|\phi_\omega\|^2 > 0$ holds.

These conditions can be verified and one obtains the following theorem.

**Theorem 2.4.** Let $0 < \mu \leq 2, \alpha < \alpha^* < 0, \omega > \alpha^2/N^2$. Then the ground state $\phi_\omega^0$ is orbitally stable in $E$. Moreover, if $\mu > 2$ there exists $\omega^*$ such that $\phi_\omega^0$ is orbitally stable for $\omega \in (\alpha^2/N^2, \omega^*)$ and orbitally unstable for $\omega > \omega^*$. 
The theorem gives the orbital stability of the ground state for every $\omega$ also for the critical nonlinearity $\mu = 2$. The case $\omega = \omega^2$ is undecided. The proof is a calculation for the Vakhitov–Kolokolov condition; and concerning the spectral conditions, the Morse index of $L_1$ is 1 as a consequence of the fact that the action has a minimum at $\Phi_\omega$ restricted to the co-dimension 1 Nehari manifold. The other details are in [23].

We end this section with some comments. There is a second strategy to show orbital stability, which again makes use of a variational property. A stationary state which minimizes the energy at constant mass is orbitally stable (see the classical paper [42], where several examples are treated). So a direct consequence of theorem 2.1 is the orbital stability of ground states. Nevertheless, some remarks are in order. The first is that while theorem 2.1 and in general the concentration compactness technique developed in [42] give information only if absolute constrained energy minima (i.e. ground states) exist, the above Weinstein and Grillakis–Shatah–Strauss theory is more general, and it is in principle applicable to every stationary state of the action, for example excited states of the NLS equation on star graphs described above. For excited states, the expectation is orbital instability, which is in fact the case, as will be shown elsewhere. The difficult part of the analysis is the calculation of the Morse index of operator $L_1$ and the use of general results in [45] and their recent refinements in [47]. One could ask whether a similar analysis could be performed on more complex graphs, for example trees or graphs with loops. Of course the fact that stationary states on star graphs are completely known, which is a rare case, is a strong facilitation in obtaining precise results. In the case of less trivial graphs, it is generally impossible to obtain explicitly standing waves, but some simple non-trivial graphs can probably be treated along the lines discussed. On more general grounds, when the linear part of the model, i.e. the underlying quantum graph, has an eigenvalue, for example corresponding to the linear ground state, bifurcation theory suggests that a branch of nonlinear standing waves exists and it bifurcates from the vanishing solution in the direction of the linear ground state (see [48] for a classical application to the case of the NLS with external potential). But there are several problems that arise at this point. The first is that a direct analysis of the conditions guaranteeing orbital stability or instability of standing waves, in particular counting of negative eigenvalues of $L_1$ and verification of the Vakhitov–Kolokolov condition, becomes in general impossible or at least very difficult. The second problem is that bifurcation theory allows us to identify branches of nonlinear stationary states which have a linear counterpart, but how can we obtain branches of states without a linear counterpart, which exist as we know from the example of nonlinear excited states in star graphs? A guess is that excited or in general bound states without a linear counterpart bifurcate from (not small) solitary waves turning on the external potential. A different extension could be in the direction of different boundary conditions at the vertices. In such a case, one expects new dynamical effects, for example bifurcation and symmetry breaking of ground states as proved for the NLS on the line with $\delta$ interaction [49]. A final open and difficult problem is the so-called asymptotic stability of standing waves. In our case, a standing wave $\Phi_\omega e^{i\alpha t}$ is said to be asymptotically stable if, for every solution $u(t)$ starting near $\Phi_\omega$ in the energy norm, one has the representation

$$u(t) = e^{i\omega t}\Phi_{\omega_{\infty}} + U_t*\Psi_{\infty} + R_{\infty}(t), \quad (2.18)$$

where $U_t$ is the unitary evolution of the linear s.a. operator $H_\omega$, and $\Psi_{\infty}, R_{\infty}(t) \in L^2(G)$, with $\|R_{\infty}(t)\| = O(t^{-\beta})$ as $t \to +\infty$, for some $\beta > 0$ and $\omega_{\infty} > a^2/N^2$. So every solution starting near an asymptotically stable standing wave is asymptotically a standing wave (not necessarily the original one) up to a remainder which is a sum of a dispersive term (a solution of the linear Schrödinger equation) and a tail small in time. The physical interpretation of the concept is that dispersion, or radiation at infinity, provides the mechanism of stabilization or relaxation, towards the asymptotic standing wave or more generally solitons (see [50], which treats the NLS equation with a potential on the line, and references therein). The problem is very hard and there is also only partial information in the case of an NLS equation on the line with a $\delta$ potential [51]. Part of the difficulty arises because of the fact that the Hamiltonian structure plays a role in the analysis and this makes it unavoidable to study the spectrum of the not self-adjoint
and not skew adjoint operator $\mathcal{J}\mathcal{L}$ (the Hamiltonian linearization) and to get dispersive estimates about its evolution $\exp(t\mathcal{J}\mathcal{L})$; this introduces some interesting and new mathematical problems about operators on graphs, already at the linear level. Moreover, the proof of possible asymptotic stability requires a control of the decay of nonlinear remainders; this control depends in a critical way on several analytical tools (in particular dispersive Strichartz estimates) which at present work only for restricted classes of nonlinearities; for example, for subcritical power nonlinearities the procedure fails.

(d) Scattering of fast solitons on junctions

In the preceding section, the behaviour of localized standing soliton solutions of the NLS equation and of the solutions in their vicinity has been studied. Here, concentrating on the case of the cubic NLS equation on a three-edge star graph, we explore a different region of state space of this model—that of the asymptotically travelling waves. As recalled before among the family of solitary solutions of the NLS equation on the line given by the action of the Galilean group on the elementary function

$$
\phi(x) = \sqrt{2} \cosh^{-1} x, \quad x \in \mathbb{R},
$$

there are the translating waves

$$
\phi_{x_0,v}(x,t) = e^{i(v/2)x} e^{-i(t(v^2/4)} e^{i t} \phi(x - x_0 - vt) \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad v \in \mathbb{R}.
$$

This special state, when put on a single edge and pushed to infinity, could be reasonably considered as an asymptotic soliton travelling on the graph. Of course, the presence of the vertex breaks Galilean invariance and the soliton cannot rigidly translate in the course of evolution, also for the simplest graph, i.e. the star graph with the Kirchhoff boundary condition at the vertex. The classical and well-known algebraic and analytic techniques to construct exact solutions of the (integrable) cubic NLS on the line fail on a graph. The interaction with the Y-junction could be in general quite complex, and from a mathematical point of view essentially nothing is known, if not in the special case where the asymptotic solitary wave is a fast soliton, in a sense that will be made precise later. In such a case, after the collision of a soliton with the vertex there exists a time lapse during which the dynamics can be described as the scattering of three split solitary waves, one reflected on the same edge where the soliton was resident asymptotically in the past, and two transmitted solitary waves on the other edges. The amplitudes of the reflected and transmitted solitary waves are given by the scattering matrix of the linear dynamics on the graph. This behaviour takes place with a small error along a time scale of the order $\ln v$ after the collision, where $v$ is the asymptotic velocity of the impinging soliton. On this time scale, figure 4 is a realistic approximation of the process.

The results in [22] and those discussed here are inspired by the analogous analysis for the NLS equation on the line with a repulsive $\delta$ potential in [52]. It should be said that a graph with two edges is equivalent to a line with a point interaction, so the treatment in [22] where several types of vertices are considered (Kirchhoff and repulsive $\delta$ and $\delta'$, but the analysis could be extended
to more general boundary conditions) shows how to generalize the results of [52] to other point interactions on the line and star graphs. To simplify the exposition, let us consider the following setting:

— cubic NLS (this is essential: see later)
— Kirchhoff vertex \( H_\alpha = H \) (more general boundary conditions are allowed)
— initial state \( (v \gg 1); x_0 \geq v^{1-\delta}, \) with \( 0 < \delta < 1; \chi \) a smooth cut-off of the tail at the vertex

\[
\psi_0(x) = (\sqrt{2}\chi(x) e^{-i(v/2)x} \cosh^{-1}(x - x_0), 0, 0).
\]  

(2.21)

We are interested in the evolution \( \psi_t \) of this initial condition. To this end, we will find an approximate solution of the equation

\[
\psi_t = e^{-iH} \psi_0 + i \int_0^t e^{-iH(t-s)} |\psi_s|^2 \psi_s \, ds.
\]  

(2.22)

The dynamics can be divided into three phases.

The first phase is the approach to the vertex in the time interval \( t \in [0, t_1] \), where \( t_1 = x_0/v - v^{-\delta} \). In this phase, the incoming (quasi) soliton moves from \( x_0 \) to \( x_0 - vt_1 = v^{1-\delta} \) and ends the run at a distance of order \( v^{1-\delta} \) from the vertex. During this phase, only a small tail of the pulse touches the vertex, the solution \( \psi_t \) behaves much as the solitary solution of the NLS in \( \mathbb{R} \) and it remains supported on the edge \( e_1 \) with an exponentially small error. Choosing as the approximating function \( \Phi_t(x) = (\phi_{x_0,-v}(x,t), 0, 0) \) the following estimate (in \( L^2 \) norm: control of masses) holds true.

**Lemma 2.5.** For any \( t \in [0, t_1] \), \( \| \psi_t - \Phi_t \| \leq C e^{-v^{1-\delta}} \) for \( v \gg 1 \) and \( \delta \in (0, 1) \).

The proof consists in the accurate use of the well-known and already cited Strichartz estimates (a ubiquitous tool in the study of nonlinear dispersive equations; see [41]) to control the distance between the unperturbed NLS flow and the NLS flow on the graph.

The second phase is the interaction phase, when the ‘body’ of the soliton crosses the vertex. This occurs during the time \( t \in [t_1, t_2] \), where \( t_2 = x_0/v + v^{-\delta} \). The time interval \( t_2 - t_1 \) is small (of order \( v^{-\delta}, 0 < \delta < 1 \)), and the effect of the nonlinear term is demonstrably small. The soliton is fast and the pulse travels for a large distance (of order \( v(t_2 - t_1) = v^{1-\delta}, 0 < \delta < 1 \)). This allows the linear dynamics to be described by using a scattering approximation. Let \( \mathcal{T} \) and \( \mathcal{R} \) be the transmission and reflection coefficients of the Kirchhoff interaction \( H \), \( \mathcal{T} = 2/N \) and \( \mathcal{R} = -(N-2)/N \). This means that the function \( \psi(k,x_1,x_2,x_3) = (e^{-ikx_1} + \mathcal{R} e^{ikx_3}, \mathcal{T} e^{ikx_1}, \mathcal{T} e^{ikx_3}) \) satisfies the Kirchhoff boundary conditions, and it is a solution of \( H\psi = k^2 \psi \) in a distributional sense. The approximating function is chosen then as (recall (2.20))

\[
\Phi^S_t = (\phi_{x_0,-v}(t) + \mathcal{R} \phi_{-x_0,v}(t), \mathcal{T} \phi_{-x_0,v}(t), \mathcal{T} \phi_{-x_0,v}(t)).
\]

The choice of this reference approximate dynamics is explained in figure 5. Reflected and transmitted contributions are represented as tails (with \( \mathcal{R} \) and \( \mathcal{T} \) factors) of travelling solitons on ‘ghost’ half-lines. This trick is useful and moreover introduces fictitious lines where some representations and known properties of solitary waves are at our disposal (see [22] for details). In any case, the true solution is compared with this approximate solution with an error small in an inverse power of velocity:

**Lemma 2.6.** For any \( t \in [t_1, t_2] \) \( \| \psi_t - \Phi^S_t \| \leq Cv^{-b/2} \) for \( v \gg 1 \) and \( \delta \in (0, 1) \).

The proof is technical and makes use of various well-chosen representations for the linear time evolution in the interaction phase, an integral equation for the difference between the true solution and the approximate one, and an iteration of Strichartz estimates for evaluating errors.

Finally there is the post-interaction phase, in the time interval \( t \in [t_2, t_3] \), with \( t_3 = t_2 + T \ln v \) where the free NLS dynamics dominates again; however, now the initial data are not asymptotic solitary waves, but waves with approximately soliton-like profiles and wrong amplitudes, owing
to the presence of scattering coefficients. Precisely, for $t \geq t_2$ and $x \in \mathbb{R}$ we define the functions $\phi^{tr}$ and $\phi^{ref}$ by

$$\begin{cases}
\frac{\partial}{\partial t} \phi^{tr} = -\frac{\partial^2}{\partial x^2} \phi^{tr} - |\phi^{tr}|^2 \phi^{tr} \\
\phi^{tr}(x,t_2) = T \phi_{-\infty,0}(x,t_2)
\end{cases}$$

$$\begin{cases}
\frac{\partial}{\partial t} \phi^{ref} = -\frac{\partial^2}{\partial x^2} \phi^{ref} - |\phi^{ref}|^2 \phi^{ref} \\
\phi^{ref}(x,t_2) = \mathcal{R} \phi_{-\infty,0}(x,t_2)
\end{cases}$$

The approximate solution is defined as

$$\Phi^{out}_t := (\phi^{ref}(t), \phi^{tr}(t), \phi^{tr}(t))$$

and it satisfies the following lemma, which is the main result.

**Lemma 2.7.** Fix $T > 0$, then for any time $t \in [t_2, t_2 + T \ln v]$, there exists $0 < \eta < 1/2$ such that

$$\|\Psi_t - \Phi^{out}_t\| \leq C_0 v^{-\eta}.$$ 

To prove the lemma, one has to use in an essential way the fact that the cubic NLS equation is integrable on the line, and thanks to this it is possible to get large-time behaviour of initial ‘lowered solitons’ along the nonlinear evolution (see [52], which is the source of the idea). Again with due estimates of the errors these results can be translated on the graph, ending with the above result.

We end with some remarks and comments.

The linear Hamiltonians in (2.22) to which the theorem refers, when different from the Kirchhoff one, have to be rescaled in order to give a non-trivial scattering matrix in the regime of high velocity; for example, in the case of a delta potential one has to set $\alpha \to v \alpha$.

The estimates are in $L^2$-norm. So the present analysis of soliton scattering is rigorous for what concerns mass transmission and reflection. On the contrary, the result proposed in [52] is in $L^\infty$-norm and the control is on the profile of the outgoing pulses.

In the time interval $t_2 < t < t_2 + T \ln v$, the last lemma implies that $\|\Phi^{out}_t\|/\|\Psi_t\| = |\mathcal{T}| + \mathcal{O}(v^{-\sigma})$ ($\sigma > 0$) holds for a certain $\mathcal{T}$ is the transmission coefficient. So, in the limit of fast solitons, i.e. $v \to \infty$, the ratio which defines the nonlinear scattering coefficient converges to the corresponding linear scattering coefficient. Analogously for the reflection coefficient.

The results discussed in this section are given and proved for a repulsive interaction (no negative eigenvalues) at the vertex. In the presence of eigenvalues, a more refined analysis has been performed in [53] for the NLS equation on the line with an attractive $\delta$ potential, giving results qualitatively similar to the repulsive case. Notice that this is the case in which stable standing waves exist. In view of the previous remarks, the directions in which the scattering of solitons on star graphs could be extended with a certain amount of technical work, but without introducing new ideas, are: the (straightforward) case of the NLS on star graphs with more than three edges; the case in which the underlying linear quantum graph with a $\delta$ or $\delta'$ vertex has bound states; with some caution, the case of general self-adjoint boundary conditions. Nothing is presently known about the possible existence of multi-solitons, or about the collision of several solitons at the vertex.
Scattering of solitons on graphs having a less trivial topology is an open problem. In particular, it could be interesting to study the propagation of solitons on trees, about which something has been said before in relation to the BBM equation. Note that among the main technical ingredients are Strichartz estimates, and every generalization needs their validity. Recent advances in this direction are given in \cite{26,27}. To give an idea of the interplay between nonlinearity and scattering on complex networks, let us briefly discuss the interesting paper \cite{5}. The authors consider a (possibly complex) graph with bounded edges where NLS dynamics is posed, and to this localized nonlinear network two external legs are attached where linear propagation occurs. After some general remarks, numerical results are given and discussed concerning the scattering of stationary wave $a_{\text{in}} e^{i k x}$ incoming from one of the external edges, entering the nonlinear network and outgoing from the second external edge: numerical experiments are done for a large spectral range of $k$’s and intensities $I_{\text{in}} = |a_{\text{in}}|^2$. It turns out that, also for relatively simple examples of networks, scattering is dominated by sets of sharp resonances of the underlying linear model; these tend to sensibly amplify the effect of nonlinearity and they prevent it being considered as a small perturbation. This moreover gives rise to typical effects in nonlinear dynamics, such as multi-stability and hysteresis. The resonances correspond to long-living states captured in the networks, and they are a result of the non-trivial topology of the graph. Another important problem concerns the possibility of extending the time scale of the validity of the approximation by the solitary outgoing waves. In a different model (scattering of two solitons on the line) in \cite{54}, some considerations are given about the possibility of longer time scales of outgoing soliton approximation depending on the initial data and external potential, but it is unclear whether similar considerations can be applied to the present case.

As regards the limitation to the cubic NLS, the fundamental asymptotics proved in \cite{52} and used to obtain the analogous result on graphs depend on the integrability of the cubic NLS. One can conjecture that for nonlinearities close to integrable the outgoing waves are close to solitons over time scales similar to those given above. In this respect, the recent result of Perelman \cite{55} on the asymptotics of colliding solitons on the line is interesting.

As a final remark, let us note that the two main dynamical features described here and about which a description, if partial, has been achieved, i.e. standing waves and their neighbourhood, on the one hand, and scattering of fast solitons, on the other hand, correspond to states and regimes far apart in the energy space of the system. Many interesting physical phenomena experimentally well tested are probably in a different or intermediate region. In particular, capture and generation of solitons are not yet understood from a rigorous point of view and they represent a challenge to mathematical methods and theoretical interpretation where perhaps the simple but non-trivial model of the NLS equation on graphs could offer some insight.

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