We report on a new vector model of an erbium-doped fibre laser mode locked with carbon nanotubes. This model goes beyond the limitations of the previously used models based on either coupled nonlinear Schrödinger or Ginzburg–Landau equations. Unlike the previous models, it accounts for the vector nature of the interaction between an optical field and an erbium-doped active medium, slow relaxation dynamics of erbium ions, linear birefringence in a fibre, linear and circular birefringence of a laser cavity caused by in-cavity polarization controller and light-induced anisotropy caused by elliptically polarized pump field. Interplay of aforementioned factors changes coherent coupling of two polarization modes at a long time scale and so results in a new family of vector solitons (VSs) with fast and slowly evolving states of polarization. The observed VSs can be of interest in secure communications, trapping and manipulation of atoms and nanoparticles, control of magnetization in data storage devices and many other areas.

1. Introduction

Mode-locked fibre laser as a source of femtosecond pulse train with the states of polarization (SOPs) either locked or evolving at different time scales from nanoseconds to microseconds (vector solitons, VSs [1–10]) is an attractive system in the context of applications in metrology [11], high-precision spectroscopy [12] and high-speed fibre-optic communication [13]. Achieving high flexibility in the generation and control of dynamic SOPs is also of interest for trapping and manipulation of atoms and...
nanoparticles [14,15], control of magnetization [16] and secure communications [17]. Our recent experimental study [7–10] revealed a variety of VSs which cannot be characterized in terms of previously used coupled either nonlinear Schrödinger or Ginzburg–Landau equations [1–5]. Some of the VSs with slowly evolving SOPs, including chaotic double scroll polarization attractor, have been characterized theoretically based on a new vector model accounting for the vector nature of the interaction between an optical field and an erbium-doped active medium and light-induced anisotropy caused by the pump field [8]. To explain transition from VSs with slowly evolving SOPs to VSs with fast-evolving SOPs observed in our previous papers [7–10], we study herein interplay between fibre birefringence and birefringence caused by an in-cavity polarization controller, polarization hole burning and light-induced anisotropy caused by an elliptically polarized pump [7–10].

2. Vector model of erbium-doped fibre laser mode locked by carbon nanotubes

To obtain a new vector model of erbium-doped fibre laser with a carbon nanotube (CNT) as a saturable absorber, we start from semi-classical equations for a unidirectional laser [18–20]

\[
\begin{align*}
\frac{\partial E_x}{\partial t} + c \frac{\partial E_x}{\partial z} &= -kE_x + ik \int (e_x P(g)) \, dg, \\
\frac{\partial E_y}{\partial t} + c \frac{\partial E_y}{\partial z} &= -kE_y + ik \int (e_y P(g)) \, dg, \\
\frac{\partial P(g)}{\partial t} &= (-\gamma_P + i\Delta_0)P(g) - i\gamma_P D(g) m^e_0(E_x(e_x m_e) + E_y(e_y m_e)) \\
\frac{\partial D(g)}{\partial t} &= \gamma_d \left( D_0 - D(g) + i \frac{4}{\Delta^2} P(g)[E_x e_x + E_y e_y] - P(g)[E^*_x e_x + E^*_y e_y] \right). 
\end{align*}
\]

(2.1)

Here, \( P(g) \) and \( D(g) \) are medium polarization and normalized gain angular distributions, respectively; \( g = (\theta, \varphi, \psi) \) are Euler angles describing the orientation of the local reference frame \((X', Y', Z')\) related to the orientation of the dipole moments of Er\(^{3+}\) ion with respect to the laboratory reference frame \((X, Y, Z)\) related to the orientation of the cross-polarized components of the electric field \( e_x \) and \( e_y \). Hence, \( \int \ldots \, dg = (1/8\pi)^2 \int_0^{2\pi} \int_0^{\pi} \sin \theta \, d\theta \, d\varphi \, d\psi \) [21], \( E = E_x e_x + E_y e_y \) is a lasing electric field, \( m_e \) is a unit vector along the dipole moment of the transition with emission[19,20], \( D_0 \) is the scaled parameter of the pumping, \( \Delta_0 \) is detuning of the lasing wavelength with respect to the maximum of the gain spectrum, the vector \( m^e_0 \) denotes complex conjugation of \( m_e \), \( \gamma_P \) and \( \gamma_d \) are relaxation rates for photons in the cavity, medium polarization and gain. In view of the relaxation rate of the medium polarization in erbium-doped silica matrix \( \gamma_P = 4.75 \times 10^{14} \, s^{-1} \gg \gamma_d, k (\gamma_d = 100 \, s^{-1}, k = 10^7 - 10^8 \, s^{-1}) \) [22], we can consider the limit \( \frac{\partial P(g)}{\partial t} = 0 \) and so equation (2.1) can be rewritten as follows:

\[
\begin{align*}
\frac{\partial E_x}{\partial t} + c \frac{\partial E_x}{\partial z} &= -kE_x + \frac{1 + i\Delta}{1 + \Delta^2} (D_{xx}E_x + D_{xy}E_y), \\
\frac{\partial E_y}{\partial t} + c \frac{\partial E_y}{\partial z} &= -kE_y + \frac{1 + i\Delta}{1 + \Delta^2} (D_{yx}E_x + D_{yy}E_y), \\
\frac{\partial D(g)}{\partial t} &= \gamma_d \left( D_0 - D(g) - \frac{D(g)}{2} R(E_x, E_y, g) \right),
\end{align*}
\]

(2.2)

and

\[
R(E_x, E_y, g) = \frac{1}{1 + \Delta^2} \left[ |E_x|^2 |e_x m_e|^2 + |E_y|^2 |e_y m_e|^2 \\
+ E_x E^*_y (e_x m_e)(e_y m_e^*) + E_y E^*_x (e_y m_e)(e_x m_e^*) \right].
\]
Here, $\Delta = \Delta_0/\gamma_p$ and

\[
D_{xx} = k \left[ D(g)|e_x m_e|^2 \right] \text{d}g, \\
D_{yy} = k \left[ D(g)|e_y m_e|^2 \right] \text{d}g, \\
D_{xy} = k \left[ D(g)(e_x m_e)(e_y m^*_e) \right] \text{d}g \\
D_{yx} = k \left[ n(g)(e_x m_e)(e_y m^*_e) \right] \text{d}g.
\]

By adding a saturable absorber (single-wall CNTs), accounting for fibre birefringence, Kerr nonlinearity, chromatic dispersion [1–5], absorption from the ground state at the lasing wavelength for erbium ions [23] (figure 1) and pump wave state of polarization, we can modify equation (2.1) as follows [8]:

\[
\frac{\partial E_x}{\partial z} = i \beta E_x - \eta \frac{\partial E_x}{\partial t} - \frac{\partial^2 E_x}{\partial t^2} + i \gamma \left( |E_x|^2 E_x + \frac{2}{3} |E_y|^2 E_x + \frac{1}{3} E_y^2 E_x^* \right) + D_{xx} E_x + D_{xy} E_y, \\
\frac{\partial E_y}{\partial z} = -i \beta E_y - \eta \frac{\partial E_y}{\partial t} - \frac{\partial^2 E_y}{\partial t^2} + i \gamma \left( |E_y|^2 E_y + \frac{2}{3} |E_x|^2 E_y + \frac{1}{3} E_x^2 E_y^* \right) + D_{yx} E_x + D_{yy} E_y, \\
D_{xx} = \frac{\sigma_1 (1 - i) \Delta}{2(1 + \Delta^2)} \left[ \chi \int [n(g)|e_x m_e|^2 \text{d}g - 1] - \alpha_2 \int [n(g)|e_y m_a|^2 \text{d}g - \alpha_4, \\
D_{xy} = \frac{\sigma_1 \chi (1 - i) \Delta}{2(1 + \Delta^2)} \int [n(g)(e_x m_e)(e_y m^*_e) \text{d}g - \alpha_2 \int [n(g)(e_y m_a)(e_x m^*_a) \text{d}g], \\
D_{yx} = \frac{\sigma_1 \chi (1 - i) \Delta}{2(1 + \Delta^2)} \int [n(g)(e_y m_e)(e_x m^*_e) \text{d}g - \alpha_2 \int [n(g)(e_x m_a)(e_y m^*_a) \text{d}g], \\
D_{yy} = \frac{\sigma_1 (1 - i) \Delta}{2(1 + \Delta^2)} \left[ \chi \int [n(g)|e_y m_e|^2 \text{d}g - 1] - \alpha_2 \int [n(g)|e_y m_a|^2 \text{d}g - \alpha_4,
\right] \\
\frac{\partial n(g)}{\partial t} = \gamma_d \left( \frac{I_p}{I_{ps}} (1 - n(g)) |e_p m_a|^2 - n(g) - (\chi n(g) - 1) R_{Er}(E_x, E_y, g) \right),
\]

\[
N(g) = \frac{1}{1 + \alpha_3 R_{CNT}(E_x, E_y, g)(1 + \Delta^2)},
\]

\[
R_{Er}(E_x, E_y, g) = \frac{1}{1 + \Delta^2} \left[ \frac{|E_x|^2}{I_{ss}} |e_x m_e|^2 + \frac{|E_y|^2}{I_{ss}} |e_y m_e|^2 \right. \\
+ \left. \frac{E_x E_y^*}{I_{ss}} (e_x m_e)(e_y m^*_e) + \frac{E_y E_x^*}{I_{ss}} (e_y m_e)(e_x m^*_e) \right]
\]

and

\[
R_{CNT}(E_x, E_y, g) = \frac{1}{1 + \Delta^2} \left[ \frac{|E_x|^2}{I_{ss}} |e_x m_a|^2 + \frac{|E_y|^2}{I_{ss}} |e_y m_a|^2 \right. \\
+ \left. \frac{E_x E_y^*}{I_{ss}} (e_x m_a)(e_y m^*_a) + \frac{E_y E_x^*}{I_{ss}} (e_y m_a)(e_x m^*_a) \right].
\]

Here, $n(g)$ and $N(g)$ is the angular distributions of the erbium ions at the first excited level and CNT in the ground state, $\alpha_2$ is the CNT absorption at the lasing wavelength, $\alpha_3$ is the ratio of saturation powers for CNT and EDF, $\alpha_4$ represents the normalized losses, $\beta$ is the birefringence strength ($2\beta = 2\pi/L_b, L_b$ is the beat length), $m_a$ and $\mu_a$ are unit vectors along the dipole moment of the transition with absorption for erbium ions and CNT, $V_g$ is the group velocity, $\eta = \beta \lambda/(2\pi c)$ is the inverse group velocity difference between the polarization modes, $\alpha_1 = \sigma_a f_{L,\rho}$ is the EDF.
absorption at the lasing wavelength, $I_{ps} = \gamma_{ps} A \nu_p / (\sigma_a(r^p) \Gamma_p)$ and $I_{ss} = \gamma_{ss} A \nu_s / (\sigma_a(L) \Gamma_L)$ are saturation powers for pump and lasing ($h$ is Planck’s constant, $\nu_p$ and $\nu_s$ are pump and lasing frequencies, respectively), $\chi = (\sigma_a(L) + \sigma_e(L)) / \sigma_a(L) + \sigma_e(e)$ and $\sigma_a(p)$ are absorption and emission cross sections at the lasing wavelength and absorption cross section at the pump wavelength, $\Gamma_L$ and $\Gamma_p$ are the confinement factors of the EDF fibre at the lasing and pump wavelengths, $\rho$ is the concentration of erbium ions, $A$ is the fibre core cross-sectional area.

To simplify equations (2.8) further, we use an approximation introduced in [8,24] the validity of which has been justified in [25], i.e. we suggest that dipole moments of the transition with absorption and emission for erbium-doped silica $(m_a, m_e)$ and the dipole moment of the transition with absorption for CNT $\mu_a$ are located in the plane defined by the orthogonal components of the lasing field $E = E_x e_x + E_y e_y$ transverse to the direction of propagation [8] (figure 1c).

In addition, we use the approximation $m_a = m_e$ and consider an elliptically polarized pump $e_p = (e_x + i \delta e_y) / \sqrt{1 + \delta^2}$ (here $\delta$ is the ellipticity of the pump wave) [8].

$$
(m_e e_x) = \cos(\theta), \quad (m_e e_y) = \sin(\theta), \quad (m_a e_p)^2 = \frac{\cos^2(\theta) + \delta^2 \sin^2(\theta)}{1 + \delta^2}
$$

(2.5)

and

$$
(m_a e_x) = \cos(\theta_1), \quad (m_a e_y) = \sin(\theta_1).
$$

The angular distributions $n(\theta)$ now depends only on $\theta$ and so can be expanded into a Fourier series as follows [8,24]:

$$
n(\theta) = \frac{n_0}{2} + \sum_{k=1}^{\infty} n_{1k} \cos(k \theta) + \sum_{k=1}^{\infty} n_{2k} \sin(k \theta).
$$

(2.6)
As a result, we find a complete set of equations for $E_x, E_y, n_0, n_{12}, n_{22}$ as follows [8]:

\[
\begin{align*}
\frac{\partial E_x}{\partial z} &= i \beta E_x - \eta \frac{\partial E_x}{\partial t} - \frac{\beta_2}{2} \frac{\partial^2 E_x}{\partial t^2} + \gamma \left( |E_x|^2 E_x + \frac{2}{3} |E_y|^2 E_x + \frac{1}{3} E_y^2 E_x^* \right) + D_{xx} E_x + D_{xy} E_y, \\
\frac{\partial E_y}{\partial z} &= -i \beta E_y + \eta \frac{\partial E_y}{\partial t} - \frac{\beta_2}{2} \frac{\partial^2 E_y}{\partial t^2} + \gamma \left( |E_y|^2 E_y + \frac{2}{3} |E_x|^2 E_y + \frac{1}{3} E_x^2 E_y^* \right) + D_{yy} E_y + D_{yx} E_x, \\
D_{xx} &= \left( \frac{\alpha_1 (1 - i \Delta)}{1 + \Delta^2} I_{xx}(n_0, n_{12}, n_{22}) - f_{xx} - \alpha_4 L \right), \\
D_{xy} &= \frac{\alpha_1 (1 - i \Delta)}{1 + \Delta^2} I_{xy}(n_0, n_{12}, n_{22}) - f_{xy}, \\
D_{yx} &= \frac{\alpha_1 (1 - i \Delta)}{1 + \Delta^2} I_{yx}(n_0, n_{12}, n_{22}) - f_{yx} - \alpha_4 L, \\
D_{yy} &= \frac{\alpha_1 (1 - i \Delta)}{1 + \Delta^2} I_{yy}(n_0, n_{12}, n_{22}) - f_{yy} - \alpha_4 L, \\
I_{xx}(n_0, n_{12}, n_{22}) &= \left( \chi \frac{n_0}{2} - 1 \right) + \chi \frac{n_{12}}{2}, \\
I_{xy}(n_0, n_{12}, n_{22}) &= \left( \chi \frac{n_0}{2} - 1 \right) - \chi \frac{n_{12}}{2}, \\
I_{yx}(n_0, n_{12}, n_{22}) &= \chi \frac{n_{12}}{2}, \\
J_{xx} &= \alpha_2 \left( \frac{1}{2} - \alpha_3 \frac{1}{8} \left[ 3 |E_x|^2 + |E_y|^2 \right] \right), \\
J_{xy} &= -\frac{\alpha_3 \alpha_2}{8} \left[ E_x E_y^* + c.c. \right], \\
\frac{dn_0}{dt} &= \gamma_d \left[ I_p + 2 R_{10} - \left( 1 + \frac{I_p}{2} + \chi R_{10} \right) n_0 \right. \\
& \quad \left. - \left( \chi R_{11} + \frac{I_p (1 - \delta_2)}{2 (1 + \delta_2)} \right) n_{12} - \chi n_{22} R_{12} \right], \\
\frac{dn_{12}}{dt} &= \gamma_d \left[ \frac{(1 - \delta_2)}{(1 + \delta_2)} \frac{I_p}{2} + R_{11} - \left( \frac{I_p}{2} + 1 + \chi R_{10} \right) n_{12} \right. \\
& \quad \left. - \left( \frac{(1 - \delta_2)}{(1 + \delta_2)} \frac{I_p}{2} + \chi R_{11} \right) \frac{n_0}{2} \right], \\
\frac{dn_{22}}{dt} &= \gamma_d \left[ R_{12} - \left( \frac{I_p}{2} + 1 + \chi R_{10} \right) n_{22} - \chi R_{12} \frac{n_0}{2} \right], \\
\text{and} \\
R_{10} &= \frac{1}{2(1 + \Delta^2)} (|E_x|^2 + |E_y|^2), \\
R_{11} &= \frac{1}{2(1 + \Delta^2)} (|E_x|^2 - |E_y|^2), \\
R_{12} &= \frac{1}{2(1 + \Delta^2)} [E_x E_y^* + c.c.].
\end{align*}
\]  

Here, we use approximation $(\alpha_3/4)[3|E_x|^2 + |E_y|^2] \ll 1$. Though CNT and output coupler are lumped elements, we use distributed form for saturable absorption and losses in equation (2.7). Further, we consider evolution of the lasing SOP averaged over the pulsewidth in terms of the round-trip numbers and so approximation of distributed saturable absorption and losses in equation (2.7) is justified.

We introduce a new slow-time variable $t_s = z/(V_g t_r)$, where $t_r = L/V_g$ is the photon round-trip time, $L$ is the cavity length, and assume an ansatz in the form [8]:

\[
E_x(t, t_s) = u(t_s) \text{sech} \left( \frac{t}{T_p} \right) \quad \text{and} \quad E_y(t, t_s) = v(t_s) \text{sech} \left( \frac{t}{T_p} \right).
\]  

(2.8)
Here, $T_p$ is the pulse width. After substitution of equation (2.8) into equation (2.7) and averaging over the time $T_p \ll t \ll t$, we obtain the following equations [8]:

\[
\begin{align*}
\frac{du}{dt} &= i\beta Lu + \frac{yL_{ss}}{2}|u|^2u + \frac{2}{3}|v|^2u + \frac{1}{3}v^2u^* + D_{xx}u + D_{xy}v, \\
\frac{dv}{dt} &= -i\beta Lv + \frac{yL_{ss}}{2}|v|^2v + \frac{2}{3}|u|^2v + \frac{1}{3}u^2v^* + D_{xy}u + D_{yy}v, \\
\frac{dn_0}{dt} &= \varepsilon \left[ I_p + 2R_{10} - \left( 1 + \frac{I_p}{2} + \chi R_{10} \right) n_0 - \left( \chi R_{11} + \frac{I_p}{2} (1 - \delta^2) \right) n_{12} - \chi n_{22}R_{12} \right], \\
\frac{dn_{12}}{dt} &= \varepsilon \left[ \frac{1}{1 + \Delta^2} + R_{11} - \left( \frac{I_p}{2} + 1 + \chi R_{10} \right) n_{12} - \left( \frac{1 - \delta^2}{1 + \Delta^2} \right) + \chi R_{11} \right] n_{0}, \\
\frac{dn_{22}}{dt} &= \varepsilon \left[ R_{12} - \left( \frac{I_p}{2} + 1 + \chi R_{10} \right) n_{22} - \chi n_{12}R_{12} \right], \\
\end{align*}
\]

and $R_{10} = \frac{1}{1 + \Delta^2}(|u|^2 + |v|^2)$, $R_{11} = \frac{1}{1 + \Delta^2}(|u|^2 - |v|^2)$, $R_{12} = \frac{1}{1 + \Delta^2}(uv^* + vu^*)$.

Coefficients $D_{ij}$ can be found as follows:

\[
\begin{align*}
D_{xx} &= \frac{\alpha_2L(1 - i\Delta)}{1 + \Delta^2} (f_1 + f_2) - \left( \frac{\alpha_2L}{2} - \frac{2\alpha_2\alpha_3L}{8\pi} k_1 \right) - \alpha_4L, \\
D_{yy} &= \frac{\alpha_2L(1 - i\Delta)}{1 + \Delta^2} (f_1 - f_2) - \left( \frac{\alpha_2L}{2} - \frac{2\alpha_2\alpha_3L}{4\pi} k_2 \right) - \alpha_4L, \\
D_{xy} &= D_{yx} = \frac{\alpha_2L(1 - i\Delta)}{1 + \Delta^2} f_3 - \frac{2\alpha_2\alpha_3L}{8\pi} k_3,
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= \chi \frac{n_0}{2} - 1, \quad f_2 = \chi \frac{n_{12}}{2}, \quad f_3 = \chi \frac{n_{22}}{2}, \\
k_1 &= \frac{3|u|^2 + |v|^2}, \quad k_2 = \frac{|u|^2 + 3|v|^2}{} \quad \text{and} \quad k_3 = uv^* + vu^*.
\end{align*}
\]

Here, $\varepsilon = t_R\gamma_d$, and $u$ and $v$ are normalized to the saturation power $I_{ss}$, and $I_p$ is normalized to the saturation power $I_{ps}$. We have also neglected the inverse group velocity difference corresponding to $\eta \approx 0$. For a cavity length $L_c = 7.8 \text{ m}$, beat length $L_b = 5 \text{ m}$ and $\lambda = 1.56 \mu \text{m}$ the time delay between cross-polarized pulses over the length of the cavity can be found based on notations to equation (2.6) as $T_d = 8 \text{ fs}$. The time delay is much less than the pulse duration of $600 \text{ fs}$ and the CNT relaxation time of $300 \text{ fs}$ and so the group velocity difference can be ignored in equation (2.6).

We have also used the following notations [8]:

\[
\int_{-T_p}^{T_p} \frac{(\cosh(x)^2 - 2)}{\cosh(x)^3} dx \to 0, \quad \int_{-T_p}^{T_p} \frac{\cosh(x)^3}{\cosh(x)} dx \approx \frac{1}{2}, \quad \int_{-T_p}^{T_p} \frac{\cosh(x)^3}{\cosh(x)} dx \approx \frac{2}{\pi}.
\]

We have neglected the dynamics of the absorption in CNT that holds when saturable absorber relaxation time $\tau_a$ time is smaller than the pulse width $T_p$. In our experiments, $\tau_a \sim 300 \text{ fs}$ and $T_p \sim 600 \text{ fs}$ and so approximation of fast saturable absorber will be still valid if we make change of variables $\alpha_2 \to \alpha_2(1 - \exp(-T_p/\tau_a))$ for the case of $\alpha_2 \ll 1$. Though $\text{Er}^{3+}$ ion is usually described as a four-level system shown in figure 1b, we reduce this model to two-level one by neglecting excited state absorption from $^4I_{11/2}$ and population of this level that is justified for pump powers $I_p < 200 \text{ mW}$, viz. for the case considered herein and in our previous publications [8,9]. It has been shown by Sergeyev et al. that migration assisted upconversion (MAUP) in high-concentration erbium-doped fibre results in decreasing first excited level lifetime more than 10 times [26–28] and so we mimic MAUP herein by decreasing the lifetime at the first excited level [8]. The time scale for MAUP is microseconds, and so these processes have no effect on pulse shape in
3. In-cavity birefringence control

Our recent experimental study demonstrated fast pulse-to-pulse SOP evolution that is caused by fibre and in-cavity polarization controller birefringence [9] and so we modify our previous model by accounting for aforementioned factors. To study VSs with fast-evolving SOPs, we account for birefringence caused by in-cavity polarization controller. First, we rewrite equation (2.9) for $\psi = (u, v)^T$ as follows:

$$
\psi(t_s + 1) = B \exp(G) \psi(t_s), \quad (3.1)
$$

where

$$
G = \begin{bmatrix}
\int_{t_s}^{t_{s+1}} D_{xx}(t_s) \, dt_s & \int_{t_s}^{t_{s+1}} D_{xy}(t_s) \, dt_s \\
\int_{t_s}^{t_{s+1}} D_{yx}(t_s) \, dt_s & \int_{t_s}^{t_{s+1}} D_{yy}(t_s) \, dt_s
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
\exp \left( \frac{i\pi L}{L_b} \right) & 0 \\
0 & \exp \left( -\frac{i\pi L}{L_b} \right)
\end{bmatrix} \quad (3.2)
$$

If an in-cavity polarization controller is installed, then equation (3.1) can be modified as follows:

$$
\psi(t_s + 1) = T B \exp(G) \psi(t_s), \quad (3.3)
$$

where $T$ is the transfer matrix of the polarization controller [29]

$$
T = \begin{bmatrix}
A + iB & C + iD \\
-C + iD & A - iB
\end{bmatrix}, \quad A = -\cos(\psi_1) \cos(\psi_2), \\
B = -\sin(\psi_3) \sin(\psi_1), \quad C = -\cos(\psi_1) \sin(\psi_2), \\
D = -\sin(\psi_1) \cos(\psi_3)
$$

$$
A^2 + B^2 + C^2 + D^2 = 1, \quad \psi_1 = \xi - \frac{\Delta}{2}, \quad \psi_2 = \frac{\phi}{2}, \quad \psi_3 = \frac{\xi}{2} + \nu,
$$

Here, $\nu/2$, $\Delta/2$ and $(\nu + \xi)/2$ are the orientations of the first quarter-wave plate (QWP), half-wave plate and the second QWP with respect to the vertical axis $Y$.

As follows from equations (3.2) and (3.3):

$$
T_1 = T B, T_1 = \begin{bmatrix}
A_1 + iB_1 & C_1 + iD_1 \\
-C_1 + iD_1 & A_1 - iB_1
\end{bmatrix}, \quad A_1^2 + B_1^2 + C_1^2 + D_1^2 = 1. \quad (3.5)
$$

Normalized Stokes parameters can be found as functions of the output powers of a two linearly cross-polarized SOPs $|u|^2$, $|v|^2$ and the phase difference between them $\Delta \phi$ as follows:

$$
S_0 = |u|^2 + |v|^2, \quad S_1 = |u|^2 - |v|^2, \quad S_2 = 2|u||v| \cos \Delta \phi, \quad S_3 = 2|u||v| \sin \Delta \phi,
$$

$$
s_i = \frac{S_i}{\sqrt{S_1^2 + S_2^2 + S_3^2}}, \quad (i = 1, 2, 3). \quad (3.6)
$$

As follows from equations (3.3) and (3.6), reproducibility of SOP in $n$-round trips results in the following equation:

$$
(T_1 \exp(G))^n = aI, \quad I = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad |a| = 1, \quad \arg(a) = \pi k, k = 0, 1 \ldots \quad (3.7)
$$

For example, if we neglect SOP rotation caused by an active medium, i.e. set $\exp(G) = I$, reproducibility of SOP for two round trips results in condition $A_1 = C_1 = D_1 = 0$ and $B_1 = 1$ that means $L_b = 2L$, whereas for reproducibility for three round trips results in condition $C_1 = D_1 = 0$ and $A_1 = 3^{1/2}/2, B_1 = 1/2$, i.e. $L_b = 3L$. 

picosecond scale. Unlike previously used models based on either coupled nonlinear Schrödinger or Ginzburg–Landau equations [1–5], we account for slow gain dynamics that leads to slowly evolving VSs that have been found theoretically and experimentally in our previous papers [6–10].
Finally, to calculate pulse-to-pulse evolution of SOP numerically, we transform equation (3.3) into the distributed form as follows:

$$\frac{d\Psi}{dt_s} = (G + \ln(T_1))\Psi,$$

(3.8)

In view of condition $C_1 = D_1 = 0$ for the case of SOP reproducibility in $n$-round trips, equation (3.8) take the form

$$\frac{d\Psi}{dt_s} = G\Psi + \left(\begin{array}{cc} i\pi L/b_1 & 0 \\ 0 & -i\pi L/b_1 \end{array}\right)\Psi + NL,$$

(3.9)

where $L_{b1}$ is the beat length for combined fibre-POC birefringence, NL describes contribution of the Kerr nonlinearity as follows:

$$NL = i\frac{\gamma LI_{ss}}{2} \left(\frac{|u|^2 u + \frac{2}{3}|v|^2 u + \frac{1}{3}v^2 u^*}{|v|^2 v + \frac{2}{3}|u|^2 v + \frac{1}{3}u^2 v^*}\right).$$

(3.10)

For an analytical study of SOP evolution, we substitute $\Psi = (|u| \exp(i\varphi_x), |v| \exp(i\varphi_y))^T$ in equation (3.9) and find equation for the phase difference $\Delta\phi = \phi_x - \phi_y$

$$\frac{d\Delta\phi}{dt_s} = -\frac{2\pi L}{L_{b1}} + \frac{\gamma LI_{ss}}{12}(|v|^2 - |u|^2)(1 - 2\cos(2\Delta\phi)) + \text{Im}(D_{yy}) - \text{Im}(D_{xx})$$

$$+ \frac{(|v|^2 - |u|^2)}{|u||v|} \text{Im}(D_{xy}) \cos(\Delta\phi) - \frac{(|v|^2 + |u|^2)}{|u||v|} \text{Re}(D_{xy}) \sin(\Delta\phi).$$

(3.11)

4. Results and discussion

By tuning in-cavity polarization controller, we can change combined fibre-POC birefringence from isotropic case to the case of high birefringent cavity. For the case of weak birefringence, i.e. $L_{b1} \gg L$, cylindrical symmetry results in SOP degeneration when the pump power for the laser is below the threshold. However, when the pump power is above the threshold value, instability of the symmetric steady-state solution with the Stokes vector $S = (S_0, 0, 0 \pm 1)$ results in emergence of a double scroll attractor located at the Poincaré sphere [8]. As follows from equation (3.11), phase difference and SOP correspondently are evolving due to the polarization hole burning in angular distribution of population inversion in active medium $n(\theta)$ caused by lasing at two cross-polarized SOPs. As follows from equations (2.4) and (2.6), for the case of equal powers of cross-polarized SOPs, i.e. $|v|^2 = |u|^2$, the depth of this hole $h(\theta)$ can be found as follows:

$$h(\theta) \sim n_{22} \sin(2\theta).$$

(4.1)

Thus, equation (4.1) along with equation (2.7) demonstrates that polarization hole burning contributes to the gain matrix $G$ through coefficients $D_{xy}$.

For high birefringence strength ($L_{b1} \sim L$) and $|v| \approx |u|$, condition $2\pi > \text{Re}(D_{xy})$ holds and so only solution with the rotating phase difference and SOP correspondently can exist. As follows from equation (3.11), reproducibility of SOP in $n$-round trips results in condition $L_{b1} = nL$ and an additional phase difference drift caused by polarization hole burning is proportional to $2\text{Re}(D_{xy})\sin(\Delta\phi) \neq 0$.

To illustrate interplay between birefringence and polarization hole burning, we solve equation (3.9) numerically by varying pump SOP ellipticity $\delta$ and beat length $L_{b1}$, and using parameters values quite close to the experimental ones [9], viz. $L = 10 \text{m}$, $a_1L = \ln(10)6.4$, $a_2L = 0.136, a_3 = 10^{-4}, a_4L = \ln(10)0.5, \chi = 3/2, \Delta = 0.1, l_p = 30, \gamma L_I_{ss} = 2 \times 10^{-6}$ and $\epsilon = 10^{-4}$. The results for $\delta = 1$ (circularly polarized pump) are shown in figure 2.
As follows from figure 2a–f, weak birefringence can distort spiral attractor and results in SOP localization close to the circle $s_2^2 + s_3^2 = 1$. In line with equation (3.11), SOP is reproduced in $n$-round trips with a drift caused by polarization hole burning.

By changing ellipticity from $\delta = 1$ to $\delta = 0.5$, we find that SOP localization is changing to the circle $s_2^2 + s_3^2 = a$ ($a < 1$, $s_1 \neq 0$) with slightly supressed SOP drift (figure 3a,b). With decreasing pump power from $I_p = 30$ to $I_p = 20$, we find that drift is completely supressed (figure 3c). To clarify the origin of drift, we have found output power signal $S_0$ as a function of number of round trips (figure 3d–f). As follows from figure 3d,e, $S_0$ is oscillating fast that causes changes in pulse-to-pulse rotation matrix associated with gain, viz. $G$ in equation (3.9). Light-induced anisotropy caused by elliptically polarized pump with ellipticity of $\delta = 0.5$ supresses oscillations and drift is slightly less (figure 3e). As follows from figure 3e, Stokes parameter $s_1$ and correspondently output power difference for two cross-polarized lasing SOPs $|v|^2 - |u|^2$ increase with increased light-induced anisotropy that, according to equation (3.11), results in decreased contribution of the active medium to the SOP drift. By decreasing the pump power from $I_p = 30$ to $I_p = 20$, we reduce the lasing powers $|v|^2, |u|^2$ that results in suppression of $S_0$ (normalized output power) oscillations and so, according to equation (3.11), in suppression of SOP drift as shown in figure 3c,f.

The obtained theoretical results are in a good agreement with our experimental data obtained previously for fundamental, bound state (BS) and multipulse (MP) soliton operations [7–10]. For example, application of ansatz (2.8) is justified by our experimental study where pulse width of the fundamental solion has been found fixed [7–10]. It is likely that ansatz which separates fast and slow-time variables can be used for different pulse shapes with fixed pulse widths also, viz. for Gaussian in the case of normal dispersion. If SOPs for adjacent pulses in MP or BS soliton regimes are the same and phase shift and pulse separation for BS are fixed, equation (3.9) can be still valid to describe SOP evolution for such MP and BS regimes as shown in [7–10]. To
Figure 3. Interplay between different types of birefringence caused by in-cavity polarization controller and polarization hole burning, and light-induced anisotropy caused by elliptically polarized pump; (a–c) fast and slow evolution of VSs in terms of Stokes parameters in the Poincaré sphere, (d,e) output power signal $S_0$ as a function of number of round trips. Parameters: (a–f) $L_b = 3L$, (a,d) $l_b = 30$, $\delta = 1$, (b,e) $l_b = 20$, $\delta = 0.5$ and (c,f) $l_b = 20$, $\delta = 1$.

describe SOP evolution for the soliton regimes with evolving pulse parameters, the more general equation (2.7) has to be used after minor revisions to account for gain spectral bandwidth.

5. Conclusion

We report a theoretical study of VSs with slowly and fast-evolving states of polarization based on a new vector model of an erbium-doped fibre laser with CNT mode locker. This model accounts for the vector nature of the interaction between an optical field and an erbium-doped active medium, slow relaxation dynamics of erbium ions, linear birefringence in a fibre, linear and circular birefringence of a laser cavity caused by in-cavity polarization controller and light-induced anisotropy caused by elliptically polarized pump light. Thus, the model goes beyond the limitations of the previously used models based on either coupled nonlinear Schrödinger or Ginzburg–Landau equations. We demonstrate that with increased combined fibre-POC birefringence strength, fast SOP rotation becomes a dominant factor and so a slowly evolving double scroll polarization attractor is transformed to the fast periodic attractors in the Poincaré sphere slightly distorted by pulse-to-pulse SOP drift caused by polarization hole burning. Though our previous experimental results on slow and fast SOP evolution are related to the fundamental, multipulse and BS soliton operations, an application of fundamental soliton ansatz (2.8) in the model is in good correspondence with our previous experiments. This gives insight that SOPs of adjacent pulses in multipulse and BS operations are the same and the phase shift and pulse separation for BS are fixed. In more general form as equation (2.7), this model can be also used for study of SOP evolution for the other soliton regimes with evolving pulse parameters.

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References


