Phase-bistable pattern formation in oscillatory systems via rocking: application to nonlinear optical systems

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We present a review, together with new results, of a universal forcing of oscillatory systems, termed ‘rocking’, which leads to the emergence of a phase bistability and to the kind of pattern formation associated with it, characterized by the presence of phase domains, phase spatial solitons and phase-bistable extended patterns. The effects of rocking are thus similar to those observed in the classic 2:1 resonance (the parametric resonance) of spatially extended systems of oscillators, which occurs under a spatially uniform, time-periodic forcing at twice the oscillations’ frequency. The rocking, however, has a frequency close to that of the oscillations (it is a 1:1 resonant forcing) and hence is a good alternative to the parametric forcing when the latter is inefficient (e.g. in optics). The key ingredient is that the rocking amplitude is modulated either in time or in space, such that its sign alternates (exhibits π-phase jumps). We present new results concerning a paradigmatic nonlinear optical system (the two-level laser) and show that phase domains and dark-ring (phase) solitons replace the ubiquitous vortices that characterize the emission of free-running, broad area lasers.

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1. Introduction

The phase symmetries of systems of nonlinear oscillators deeply affect the nature of spatial patterns that form therein. For instance, the continuous phase symmetry characterizing the free oscillations is responsible for the spontaneous emergence of vortices and spiral waves in reaction–diffusion, optical and other kinds of oscillatory, spatially extended nonlinear systems [1–8]. In order to control the turbulent-like behaviour associated with the presence of those phase defects different types of forcing have been considered [1,2,4,9–14], the most popular being the classic time-periodic one. Periodic forcing acts as an external clock, which is ‘seen’ by the system oscillations, hence the equivalence of all the oscillation phases disappears, and the original continuous phase symmetry is broken down to a discrete one. The nature of the discrete symmetry depends on the ratio of the forcing frequency, $\omega_f$, to the free oscillations frequency, $\omega_0$: a simple and elegant symmetry argument [11] allows to conclude that, when $\omega_f/\omega_0$ is close to a rational number, say $n:m$, $n$ equivalent phases, given by $\phi_k = \phi_1 + 2\pi(k - 1)/n$, $k = 1, \ldots, n$ [11], are preferred by the oscillations and the system becomes $n$-phase multi-stable. Once the continuous phase symmetry is replaced by the discrete one, the vortices are no more stable solutions. They are then substituted by other kind of structures, such as periodic patterns and localized structures (dissipative spatial solitons). Thus periodic forcing of spatially extended systems of oscillators is a powerful tool to control the nonlinear waves existing therein [4,9–28].

According to our previous discussion, when the forcing frequency is approximately twice the system’s natural frequency ($n:m = 2:1$), so-called 2:1 resonance or parametric resonance, two dynamically equivalent oscillation phases exist, which differ by $\pi$ [9–11]. If the system is large enough, adjacent spatial regions can oscillate with opposite phases thus forming so-called phase domains, which are unstable in the free-running case. Between adjacent domains a $\pi$ phase jump occurs and the curve containing those jumps is known as a wall or front. When the wall is abrupt, a one-dimensional phase singularity appears and the amplitude of the oscillations becomes null (one speaks then of a non-equilibrium Ising wall [10]), which manifests as a dark line of the light field in the optical case. The wall can also be smoother if the minimum oscillations amplitude is small, not null when crossing the wall (one speaks then of a non-equilibrium Bloch wall [10]), leading to a grey line in the light field in the optical case. As a parameter is varied, Ising walls can bifurcate into Bloch walls through a so-called non-equilibrium Ising–Bloch transition [10,17,29]. These phenomena have been predicted and observed in many different systems [4,10,11,13–18,25–27,29]. Apart from domain walls, pattern formation in a 2:1 resonance includes extended patterns, like rolls or labyrinths, and localized patterns, known as dissipative spatial solitons (cavity solitons in optics for obvious reasons).

Parametric forcing is, however, not always efficient, as it requires that the nonlinearity in the system be strong enough (or the natural resonance be wide enough) so that an excitation at a high frequency (twice the natural one) leads to appreciable response in the system. Optical systems, like lasers, belong to this class (they are insensitive to parametric forcing), and hence a pertinent question arises: is there an alternative to the classic 2:1 resonant forcing leading to similar effects? The positive answer is given by what is known as ‘rocking’ [30,31]. Rocking is a type of forcing in which the external perturbation frequency is close to the system’s natural frequency (it is a kind of 1:1 resonance, to which all systems respond) but, unlike the classic 1:1 periodic forcing, the amplitude of rocking is modulated in time (temporal rocking) or in space (spatial rocking). This modulation must be such that the sign of the forcing amplitude alternates in time or in space. As we will show along the rest of this article rocking leads to a behaviour analogous to that of the 2:1 resonant forcing: the oscillatory system is ‘converted’ into a phase-bistable pattern-forming system.

The rest of this article is organized as follows. In §2, we provide a simple explanation of why rocking induces a phase bistability in an otherwise phase invariant oscillatory system. In §3, we
give a quantitative description of rocking in terms of universal models. In §4, we review previous work on temporal and spatial rocking in specific systems. In §5, we study the effect of rocking in a particular, very relevant case: the laser. In §6, we consider the case when oscillations are weakly damped (below a Hopf bifurcation) and show that rocking is efficient in that case. Finally, §7 contains the main conclusion.

2. The rationale behind the phase bistability via rocking (and why ‘rocking’?)

According to the discussion in Introduction, the rocking (forcing) field can be written as

\[ R(r, t) = F(r, t)\exp(-i\omega_R t) + c.c., \]  

(2.1)

where \( \omega_R \) denotes the carrier frequency of the forcing, which is close to the natural frequency of oscillations \( \omega_0 \) (we remind that rocking is a generalized 1:1 resonant forcing), and \( F(r, t) \) is the rocking amplitude. Importantly, the sign of \( F \) must alternate in space or in time (when \( F \) is a constant, one gets the classic 1:1 resonant forcing, which, in optical terms, can result in injection locking). If the sign alternation occurs on a fast time scale or on a short spatial scale, the system’s oscillations will ‘see’ both phases (differing by \( \pi \), corresponding to sign changes), but will not be able to accommodate locally to that rapidly varying drive. To which phase of the perturbation will the system lock then? Clearly, if, on average, both phases are equally distributed, the system will tend to lock to any of the driving phases (or to some set of two opposite phases, as explained below) and, as a result, will display phase bistability.

On a more formal basis, rocking can be explained in terms of a mechanical analogy, which historically was the line of reasoning that led to the discovery of rocking and led to its name. A simple way to visualize a (single) nonlinear oscillator is to use a mechanical analogy, in which the real and imaginary parts of the complex oscillation amplitude are interpreted as the two Cartesian coordinates \( q = (q_1, q_2) \) of a fictitious massless particle affected by viscous damping and under the action of a potential \( V \) having the form of a Mexican sombrero (see figure 1). The maximum of \( V \) at the origin corresponds to the unstable off state (of null oscillation amplitude) and its degenerate minimum (the ‘valley’) to the self-oscillating state of finite amplitude (figure 1). The degeneracy of the minimum signals the phase invariance of the free oscillations, as no angle is preferred. Now, imagine that we rock\(^2\) that potential in a periodic way around some axis, say \( q_2 \), and that rocking is sufficiently fast. Where would the fictitious particle tend to rest (remind that the particle motion is damped)? It is evident that this would happen at the quietest regions, where the rocking of the potential produces less perturbation, i.e. around any of the ‘poles’ symmetrically located along the \( q_2 \)-axis (figure 1). Then, an initially phase invariant oscillator would end up being a phase-bistable one! Note that this phase bistability requires, in this picture, a potential with a maximum (an unstable point) at the origin, as otherwise there are not two separated, quiet regions. The simplest ‘rocked’ potential displaying the above features reads

\[ V(q) = -\frac{\mu}{2}q^2 + \frac{1}{4}q^4 - q_1 F_0 \cos \Omega t, \]

(2.2)

where \( q = \sqrt{q_1^2 + q_2^2} \) is the radial coordinate. The parameter \( \mu \) controls whether the potential has a local maximum at the origin (\( \mu > 0 \)), and then a degenerate circular minimum happens at the radius \( q = \sqrt{\mu} \) or the potential single extremum (minimum) occurs at the origin (\( \mu < 0 \)).

\(^2\)By ‘rock’, we mean ‘to move or cause to move from side to side or backwards and forwards’ (Collins dictionary, first definition of ‘rock’ as a verb: http://www.collinsdictionary.com/dictionary/english/rock) or ‘to move back and forth in or as if in a cradle’ (Merriam-Webster dictionary, first definition of ‘rock’ as a transitive verb: http://www.merriam-webster.com/dictionary/rock).
The dynamical equations following from (2.2) read $m \ddot{q} + \dot{q} = -\nabla V$, or $d q_i / dt = -\partial V / \partial q_i$ (remind that $m = 0$), and can be written compactly in terms of the complex amplitude of oscillations $A = q_1 + iq_2$ as

$$\frac{dA}{dt} = \mu A - |A|^2 A + F_0 \cos(\Omega t).$$

Without its last term, we recognize this equation as the simplest normal form of a Hopf bifurcation. The inhomogeneous, last term accounts for a forcing at the frequency of oscillations (1:1 resonant forcing) and amplitude proportional to $F_0 \cos(\Omega t)$ [9]. Hence, physically, rocking of an oscillator is accomplished by an almost periodic forcing at the oscillations frequency, whose amplitude is harmonically modulated in time. As explained in the Introduction, this initial definition became broader as other types of rocking (random versus periodic, and spatial versus temporal) are considered. In all cases, a sign alternation (a $\pi$ phase jump sequence) is necessary.

We note that equation (2.3) describes a single-mode, two-level laser with injected signal ($A$ is proportional to the complex amplitude of the electric field of the radiation and the signal, injected into the cavity, has amplitude proportional to $F_0 \cos(\Omega t)$, in the limit where the material variables are fast (and can be adiabatically eliminated) and the cavity is perfectly resonant with the atomic line. When $\Omega = 0$ (constant injection), there is no phase symmetry in (2.3) and the oscillations phase locks to that of the injection, corresponding to a tilted, static potential (2.2) displaying a single, isolated minimum (one of the extreme positions in figure 1). Note that the dynamical phase bistability under rocking happens at phases in quadrature with respect to the injection: if the injection amplitude is real, as in figure 1, phase locking occurs at $\pm \pi/2$ (on the $q_2$-axis).

3. A universal description of rocking

A lot of physical insight can be gained by studying universal models, which, on the other hand, allow giving a broad applicability to the results. As rocking is a modified 1:1 resonant forcing of oscillatory systems, these universal models are complex Ginzburg–Landau (CGL) equations, containing an inhomogeneous term [1,2,10–12,32,33], which are valid close to a spatially uniform Hopf bifurcation. If we express the oscillations field (e.g. the electric field in the optical case) as $\text{Re}\{A(r, t) \exp(-i\omega_R t)\}$, where $A$ is a complex field amplitude and $\omega_R$ is the angular frequency of rocking (close to the natural frequency of oscillations $\omega_0$), the dynamics of the system is universally governed by the following CGL equation [1,2,10–12,30–33]:

$$\partial_t A(r, t) = \mu(1 + iv)A + i(\omega_R - \omega_0)A + \alpha \nabla^2 A - \beta |A|^2 A + \eta F(r, t).$$

**Figure 1.** Qualitative three-dimensional plot of the potential $V$ associated with equation (2.1), which describes a rocked system. (b) Without injection ($F = 0$), the potential is radially symmetric in agreement with the phase invariance of the free-running laser. (a,c) With constant injection ($\Omega = 0$), the potential tilts along the direction $\text{Re}\{A\}$ proportionally to the forcing amplitude $F$ and a single isolated minimum (marked with a black dot) appears, corresponding to the phase-locked state of the usual laser with injected signal. Under rocking ($\Omega \neq 0$), the potential oscillates back and forth between the two cases (a), and (c), through (b) one. Under such forcing, a particle would tend to remain close to the imaginary axis $\text{Re}\{A\} = 0$, around either of the two regions separated by the local maximum around the origin. (Online version in colour.)
where $r = (x, y)$, $\nabla^2 = (\partial_x^2 + \partial_y^2)$, $\mu$ is real and measures the distance from the bifurcation, $v$ measures the linear variation of the oscillation frequency around the bifurcation, the complex coefficients $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ account for diffusion/diffraction and saturation/nonlinear frequency shift, respectively, and $\eta$ is a parameter that measures how effective forcing is. After a simple rescaling of time, space and amplitudes $A$ and $F$, equation (3.1) can be made dimensionless and the coefficients $\alpha$ and $\beta$ and $\kappa$ can be chosen at will (just the ratios $\alpha_2/\alpha_1$ and $\beta_2/\beta_1$ are relevant). In particular, we take $|\alpha| = |\beta| = \eta = 1$ without loss of generality. As well, if the system is above threshold ($\mu > 0$), which is the situation we consider for most of this work, one can take $\mu = 1$, and the CGL equation becomes

$$
\partial_t A(t, r) = (1 + i\theta)A + \alpha|A|^2 A - \beta|A|^2 A + F(r, t),
$$

(3.2)

where we keep the same symbols as in (3.1) and we defined the normalized detuning $\theta = \nu + (\omega_R - \omega_0)/\mu$. Note that, in the following, space, time, the fields $A$ and $F$, and the three parameters ($\theta$, $\alpha$ and $\beta$), in equation (3.2) are dimensionless. Concerning parameters $\alpha$ and $\beta$, one can use $\alpha = 1$ ($\alpha = i$ if diffusion (diffraction) dominates, and $\beta = 1$ ($\beta = \pm i$) if saturation (nonlinear dispersion) dominates, without any loss of generality. In the optical context, equation (3.2) represents the simplest description of broad area lasers with injected signal, whose complex amplitude is proportional to $F(r, t)$.

Equation (3.2) is the one we will consider throughout this section. As for the rocking amplitude, we will consider both

the temporal case: $F(t, r) = F_0 \cos \Omega t$

(3.3a)

and the spatial case: $F(t, r) = F_0 \cos Kx$,

(3.3b)

where $F_0$ is taken as a real without loss of generality as it just sets the reference phase at $x = 0$.

In order to capture analytically the main effects of rocking, and following our discussion in §2, the limit where the rocking frequency $\Omega$ (in the temporal case) or $K$ (in the spatial case) are large, is especially interesting. In this case, the driving term in equation (3.2) is highly oscillating and solutions to the problem should be well approximated by [34,35]

$$
A(t, r) = A_f(t, r) + A_s(r, t),
$$

(3.4)

where the subscripts ‘$f$’ and ‘$s$’ refer to fast and slow components, either in time (if rocking is temporal) or in space (if rocking is spatial). Substituting (3.4) and (3.3) into (3.2) and equating like terms with respect to the fast or slow frequencies in the system, and considering only the leading-order terms, we get

$$
\partial_t A_f = F_0 \cos \Omega t 
$$

and $\nabla^2 A_f = -\frac{F_0}{\alpha} \cos Kx,$

$$
\partial_t A_s = (1 + i\theta)A_s + i\alpha|A_s|^2 A_s - \beta \left( A_f^2 A_s^* + 2 |A_f|^2 A_s + |A_s|^2 A_f + |A_f|^2 A_f^* + 2 |A_s|^2 \langle A_f \rangle \right),
$$

(3.5)

(3.7)

where the angular brackets denote an averaging over the fast scale (temporal or spatial). Using (3.6), we get

$$
\partial_t A_s = (1 + i\theta - 2\beta \gamma)A_s - \gamma A_s^* + \alpha|A_s|^2 A_s - \beta|A_s|^2 A_s,
$$

(3.8)

Equation (3.1) is valid only if $\text{Re} \alpha$, $\text{Re} \beta \geq 0$, hence $\text{arg} \alpha$, $\text{arg} \beta \in [-\pi/2, \pi/2]$ in equation (3.4).
where the ‘effective rocking parameter’

\[
\gamma = \frac{1}{2} \left(\frac{F_0}{\Omega}\right)^2 \quad \text{and} \quad \gamma = \frac{1}{2} \left(\frac{F_0}{K^2}\right)^2,
\]

(3.9)
in the temporal and in the spatial cases, respectively.\(^4\) Equation (3.8) is a central result in rocking theory as it governs the evolution of the ‘slow’ component, which is the one having its own dynamics (the fast component is just ‘slaved’ to the rocking injection; see (3.6)). Alternative derivations of equation (3.8), based on rigorous multiple scale methods [36], can be found in [30–33]. Our derivation has considered the simplest case of harmonic rocking, equation (3.3), but in general any periodic form of rocking, and even certain types of random rocking, have been demonstrated to be effective as well [31–33,37]. In these cases, equation (3.8) still holds, the only difference being in the definition of the effective parameter \(\gamma\), which follows from the averages \(\langle A_s^2 \rangle\) and \(\langle |A_t|^2 \rangle\) in equation (3.7), and depends on the specific form of forcing.

Equation (3.8) is a so-called parametric CGL equation and captures the main effects rocking. In the absence of forcing \((F_0 = 0)\), \(\gamma = 0\) and equation (3.8) becomes the classic CGL equation with the continuous phase symmetry \(A_s \rightarrow A_s e^{i\phi}\) for arbitrary \(\phi\), corresponding to free oscillations. Once rocking in on, \(\gamma \neq 0\) and the presence of the term proportional to \(A_s^2\) breaks the continuous phase symmetry down to the discrete one \(A_s \rightarrow -A_s\), so that any two solutions connected by this symmetry are dynamically equivalent. Hence, the system becomes phase bistable and these two phases differ by \(\pi\). Equation (3.8) also shows that both types of rocking (spatial and temporal) are equivalent, the only difference being in the definition of the effective rocking parameter \(\gamma\) in (3.9). Finally, the effects of rocking are seen not to depend separately on its amplitude \((F_0)\) and frequency \((\Omega\) or \(K\)), but on the effective rocking parameter \(\gamma\).

Concerning the conditions for an efficient rocking, we observe that the phase symmetry breaking term, \(-\gamma A_s^2\) in equation (3.8), exists only if the average \(\langle A_s^2 \rangle \neq 0\) (see equation (3.7)). Note as well that, in passing from equation (3.7) to equation (3.8), the terms proportional to \(\langle A_t \rangle\) and \(\langle |A_t|^2 A_s \rangle\) have disappeared. This is because of the form of rocking we have considered, leading to (3.6). Should those averages not vanish, no phase symmetry would exist in the end, hence weakening the phase bistability when \(\langle A_t \rangle\) is relatively small, or completely destroying phase bistability when \(\langle A_t \rangle\) becomes dominant. Hence, the necessary conditions for efficient rocking are \(\langle A_t \rangle = \langle |A_t|^2 A_s \rangle = 0\) and \(\langle A_s^2 \rangle \neq 0\), which point to a forcing with two opposite phases, equally distributed on average, as we guessed in §2.

Equation (3.8) allows two kinds of spatially homogeneous solutions: the off state \(A_s = 0\) (hence \(A = A_t\)) and the on (or ‘rocked’) state \(A_s \neq 0\), the latter of which is phase-bistable. The off state exists always, whereas the rocked states exist in a closed region of the parameter space \(\gamma - \theta\) (the ‘rocking balloon’), whose form depends on the structural parameters \(\alpha\) and \(\beta\). Next, we concentrate on the case \(\alpha = i\) and \(\beta = 1\), which is typical in nonlinear optics. Figure 2 summarizes the bifurcation diagram of both solutions as obtained by a standard linear stability analysis against perturbations with wavenumber \(k\). Increasing \(\gamma\) inside the balloon leads to a decrease of the oscillations amplitude, till \(A_s\) becomes null by crossing the balloon upper boundary (continuous line); hence, the bifurcation from the trivial state \(A_s = 0\), which exists always, to the rocked states is supercritical by entering the balloon from above. On the contrary, the (V-shaped) lateral boundaries of the balloon correspond to a saddle-node bifurcation. The left one (for negative detuning) gives rise to oscillations of the slow component, meaning a loss of the phase locking. These oscillations have an infinite period at the bifurcation and are the continuation of free-running orbits that exist in the system at \(\gamma = 0\): for null injection, equation (3.8) admits the limit cycles \(A_s = A_{LC} \exp[-i(\omega_{LC} + \phi)]\), with \(A_{LC} = 1/\sqrt{|\text{Re}\beta|}, \omega_{LC} = -\theta + \text{Im}\beta/\text{Re}\beta\) and arbitrary \(\phi\), which extend (perturbed) towards non-null values of \(\gamma\). These orbits can terminate

\(^4\)A phase factor \(e^{2\pi i}\) multiplying \(A_s^2\) in (3.8) has been removed, as it disappears after the simple rotation \(A_s \rightarrow A_s e^{i\phi}\). This phase factor reads \(e^{2\pi i} = \beta\) in the temporal case and \(e^{2\pi i} = \beta \alpha^{-2}\) in the spatial case (remind that we have chosen \(|\alpha| = |\beta| = 1\).
at a saddle-node (with diverging period) or can disappear at a Hopf bifurcation of the off state, similarly to the case of the laser with injected signal [38–40]. For positive detuning, the generic scenario is of pattern formation where the most unstable wavenumber reads $k_c = \sqrt{\theta}$.

4. Rocking in specific systems

The universal description of rocking has been given in [30] (temporal periodic rocking), [31] (spatial rocking, both periodic and random) and [37] (temporal random rocking). Clearly, the predictions from universal nonlinear dynamical models are only quantitative in the very limit where such descriptions hold, which in our case requires operating the system very close to a spatially uniform Hopf bifurcation. Clearly, such predictions must be contrasted with experiments and with theoretical studies of specific models.

So far rocking has been investigated experimentally in photorefractive oscillators (PROs) [41,42], which are a kind of nonlinear optical systems, and in nonlinear electronic circuits [37,43], finding good agreement with the theoretical predictions. Both types of experiments have an oscillatory nature and exhibit phase invariance in the absence of external perturbations. In [41], a PRO was submitted to temporal periodic rocking, and the transmutation of vortices into phase domains and the stable excitation of phase domain walls were demonstrated. A similar setup, but now with a small aspect ratio (small Fresnel number in the optical terminology) was considered in [42], now under spatial rocking. That experiment could evidence the phase-bistable nature of the rocked emission, but clearly not the pattern formation predicted by the theory; an experiment addressing pattern formation in a spatially rocked PRO is under progress at present.

The nonlinear electronic circuits considered in [37,43] were Chua circuits, which are highly versatile and controllable, in particular they can be tuned to a Hopf bifurcation giving rise to self-oscillations. Clearly, these systems are zero-dimensional in space; hence, in those experiments, only the phase-bistable response associated to rocking was demonstrated, both under periodic rocking [43] and under random rocking [37]. The fact rocking shows its effectiveness in systems as diverse as optical and electronic evidences its universality.

From the theoretical side, rocking has been investigated in PRO models under temporal [41] and (small aspect ratio) spatial [42] rocking. Two-level laser models have been studied as

![Bifurcation diagram](http://rsta.royalsocietypublishing.org/)

**Figure 2.** Bifurcation diagram of equation (3.8) corresponding to the case $\alpha = i$, $\beta = 1$. The off solution is stable for large enough $\gamma$, whereas the ‘rocked states’ (the homogeneous steady states) are within the balloon. For positive detuning $\theta$ patterns form, either from the off state by crossing the horizontal line at $\gamma = 1$ or by leaving the rocking balloon through its right boundary, which is a pattern-forming bifurcation as well. For negative detuning $\theta$ oscillations happen, either by leaving the balloon through its left boundary (saddle-node bifurcation) or by crossing the Hopf bifurcation of the off state downwards. The dashed line represents the right boundary of the existing region of the rocked states, which become unstable before reaching it as explained above. (Online version in colour.)
well: temporal rocking in zero-dimensional class B lasers was considered in [44], spatial rocking in small aspect ratio lasers (with just two transverse modes) was studied in [45], and zero-dimensional bidirectional lasers under temporal rocking were investigated in [46]. In all cases, the basic phenomenon, phase bistability, was demonstrated, which in its turn yields new types of solutions, especially in the bidirectional laser case [46]. The spatial rocking has also been numerically investigated in semiconductor-based lasers, namely in broad area semiconductor lasers (so-called BAS lasers), where evidences of rocking patterns were shown [47], also in vertical-cavity surface-emitting lasers (so-called VCSELs), where more rich rocking patterns were demonstrated [48]. Next, we present the first investigation of rocking in a large Fresnel number two-level laser.

5. Pattern formation via rocking in two-level lasers

Lasers are paradigmatic, self-oscillatory nonlinear optical systems. Pattern formation therein is relevant both from the basic science viewpoint and from the applied science one, because of their potential applicability in information storage and processing [49–53]. Here, we investigate pattern formation in two-level lasers submitted to temporal rocking.

The starting point of the study is the classic set of dimensionless Maxwell–Bloch equations for a two-level laser with injected signal

\[
\begin{align*}
\partial_t E(r, t) &= \sigma [-(1 + i\Delta)E + P] + i\nabla^2 E + E_{in}, \\
\partial_t P(r, t) &= -(1 - i\Delta)P + (r - N)E \\
\partial_t N(r, t) &= b[-N + \text{Re}(EP^*)],
\end{align*}
\]

(5.1)

for the complex envelopes of the electric field \(E\), and the medium polarization \(P\), and for the (real) population inversion \(N\). Time \(t\) is normalized to the polarization decay rate \(\gamma_\perp\), \(\sigma = \kappa/\gamma_\perp\) and \(b = \gamma_\parallel/\gamma_\perp\) are normalized decay rates (\(\kappa\) is the one for the cavity and \(\gamma_\parallel\) is the one for the population inversion), \(\Delta = (\omega_c - \omega_a)/(\gamma_\perp + \kappa)\) is the normalized cavity detuning (\(\omega_a\) is the atomic resonance frequency and \(\omega_c\) is the closest longitudinal cavity mode frequency), and \(r\) is the pump parameter: free-running lasing threshold happens at \(r = 1\) for a perfectly tuned cavity (\(\Delta = 0\)).

Finally, \(E_{in}\) is the complex envelope of an injected signal, which is the essential element of rocking. Equation (5.1) are written in the frequency frame of the on-axis lasing solution; i.e. the actual light electric field and the injected field are proportional to \(E e^{-i\omega_c t + \text{c.c.}}\) and \(E_{in} e^{-i\omega_a t + \text{c.c.}}\), respectively (note that \(\gamma_\perp t\) is the actual time), where the lasing emission frequency is given by the mode pulling formula \(\omega_0 = (\gamma_\parallel \omega_c + \kappa \omega_a)/(\gamma_\perp + \kappa)\). As we are considering temporal rocking, the injected signal has the form

\[E_{in} = E_0 \cos(\omega t)e^{-i\delta t},\]

(5.2)

which corresponds to the injection of two plane waves into the laser cavity with equal amplitudes \((E_0/2)\), which we take as real (without loss of generality) and with frequencies \(\omega_0 + \gamma_\parallel \delta \pm \gamma_\perp \omega\); the carrier frequency \(\omega_0 + \gamma_\parallel \delta\) plays the role of \(\omega_R\) in equation (2.1) and the rocking frequency \(\gamma_\perp \omega\) plays the role of the modulation frequency \(\Omega\) in equation (2.3). Parameter \(\delta\) controls the detuning of the rocking mid-frequency from the free-running laser frequency \(\omega_0\). In order that this kind of injection is compatible with the uniform field and single longitudinal mode approximations subjacent to (5.1), both the rocking detuning \(\gamma_\parallel \delta\) and the modulation frequency \(\gamma_\perp \omega\) must be much smaller than the cavity-free spectral range, which can be expressed as \(4\pi \kappa/T\); where \(T\) is the (very small) transmission factor of the laser output coupler: \(|\delta|, |\omega| < 4\pi \sigma /T\). As \(0 < T < 1\), \(\delta\) and \(\omega\) can be large as compared with the normalized cavity damping rate \(\sigma\) without violating the hypothesis leading to (5.1).

In the following, we concentrate on the detuning side \(\Delta > 0(\omega_c > \omega_a)\), in which, in the absence of injection, the off state \(E = 0\) destabilizes to the on-axis (spatially uniform) lasing solution at \(r = r_0 = 1 + \Delta^2\).
Figure 3. Transformation of vortices into phase domains in a rocked two-level laser as given by a simulation of equations (5.1). The light intensity of the slow component is shown on the left, and its phase is on the right. Rocking is applied from the second frame \((t = 40)\) on. Frames are given for \(t = 20, t = 40, t = 50,\) and \(t = 80\) from top to bottom. The size of the window is \(300 \times 300\) and the grid of points is \(128 \times 128\). Parameters are \(\sigma = b = 1\) (class C laser), \(r = 6, \Delta = 2, \delta = 0, E_0 = 20\) and \(\omega = 2\pi\).

In order to gain analytical insight into the problem, we consider first a limit where an equation isomorphic to (3.8) can be obtained. This occurs close to the free-running lasing threshold, \(r - r_0 = O(\varepsilon^2)\), where \(\varepsilon\) is a smallness parameter, and when injection amplitude and frequencies are small enough: \(E_0, \theta = O(\varepsilon^2)\) and \(\omega = O(\varepsilon)\). Further introducing slow time scales \(T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t, \ldots\), and spatial scales \((X, Y) = \varepsilon(x, y)\), motivated by the linear stability analysis of the off solution in the absence of rocking, and making use of standard multiple scale analysis [30–33,36,54–62], we are able to express the complex electric field amplitude as the sum of a fast part and of a slow part, as in (3.4), namely \(E = (A_f + A_s)e^{-i\delta t}\), where we moved to the reference
Figure 4. Bloch-type domain walls. Parameters are as in figure 3, but with a smaller rocking amplitude, $E_0 = 13$. (a) Intensity, (b) phase, (c) intensity profile and (d) phase profile (across the vertical line in (a)) are plotted.

Frame of the rocking mid-frequency. Here, $A_t = D(E_0/\omega) \sin \omega t$, and the slow envelope verifies the equation

$$\partial_t A_s = \frac{\sigma D}{1 - i \Delta} \left[ (r - r_0 - 2\gamma)A_s - \gamma A_s^* - |A_s|^2 A_s \right] + i\delta A_s + iD\nabla^2 A_s, \quad \text{(5.3)}$$

where $D = [1 + \sigma(1 + i\Delta)/(1 - i\Delta)]^{-1}$, and $\gamma = 1/2(E_0/\omega)^2$ is the effective rocking parameter, to be compared with (3.9). Equation (5.3) is a CGLE with broken phase invariance, similar to equation (3.8), as expected, which evidences the main effect of rocking: the emergence of a phase bistability. Equation (5.3) has been derived under the assumptions $\Delta > 0$ and $\sigma, b = O(\varepsilon^0)$. (The latter defines so-called class C lasers.) The detuning condition follows from analysing the diffusion coefficient, given by $\text{Re}(iD) = 2\sigma \Delta / [(1 + \sigma)^2 + (1 - \sigma)^2 \Delta^2]$, which becomes negative (and then equation (5.3) loses its validity) when $\Delta < 0$ [55,56]. On physical grounds, for $\Delta < 0$, lasing involves off axis emission (tilted waves, having a spatial dependence), which equation (5.3) cannot account for, and a different treatment is necessary. In fact, for small $\Delta$ (either positive or negative), a complex Swift–Hohenberg equation, not a CGL equation, governs the dynamics of the free-running laser (and of other nonlinear optical cavities) [41,56,58,59,61–63]. Nevertheless, the specific description (CGL or Swift–Hohenberg) is not essential regarding the effects of rocking, as discussed in [41]. Concerning the size of the normalized decay rates $\sigma$ and $b$, equation (5.3) is valid for any value of $\sigma$; in particular, in the usual case $\sigma \ll 1$ (so-called class A laser), $D \to 1$ simply. However, if $b \ll 1$ (a situation characterizing the so-called class B lasers, like solid-state and semiconductor lasers), not a single equation, but two, are needed to capture the system dynamics: one for $E$ and one for $N$ [44,62,63]. In such case, few analytical insights can be gained; nevertheless, rocking is efficient as well in class B lasers.
Figure 5. Ising-type domain walls. Parameters are as in figure 3. (a) Intensity, (b) phase, (c) intensity profile and (d) phase profile (across the vertical line in (a)) are plotted.

Figure 6. (a,b) Ising-type labyrinths obtained in a rocked two-level laser, for the same parameters as in figures 3–5, but for a slightly positive rocking detuning, $\delta = 0.1$.

We have performed extensive numerical simulations of the original Maxwell–Bloch equations (5.1) under different conditions (class C and class B lasers), finding that the effect of rocking is robust, even extremely far from the conditions used to derive equation (5.3), e.g. for $r - r_0 = 1$. In figure 3, we show an example of the ‘conversion’ of vortices (the basic spatial structures in free-running lasers) into phase domains, which is typical for null or negative detuning $\delta$. These phase domains are of Bloch type for small injection (figure 4), or of Ising type for larger injection (figure 5) [10]. We note that Ising and Bloch walls can coexist in two-dimensional systems because of curvature effects [64,65].

For positive rocking detuning $\delta$, the typical scenario involves pattern formation via labyrinths (figure 6). In a small detuning region close to the labyrinth formation, phase (dark-ring) cavity
Figure 7. Dark-ring cavity soliton. Parameters are as in figures 5 and 6, but with an intermediate detuning $\delta = 0.055$ and size of the window (150 $\times$ 150). (a) Intensity, (b) phase, (c) intensity profile and (d) phase profile are plotted.

Figure 8. Intensity (a,b) and phase (c,d) of several structures found in class B lasers. From left to right, phase domains ($\delta = 0$) and labyrinths ($\delta = 0.0040$). Rest of parameters: $\Delta = 0$, $r = 1.5$, $b = 0.01$, $\sigma = 0.1$, $F = 0.04$ and $\omega = 0.042$. The size of the integration window is 600 $\times$ 600. Simulations were performed on a 64 $\times$ 64 grid. Time-step $dt = 0.3$. 
Figure 9. Cavity solitons for class B laser in the case of temporal rocking. Parameters are $\sigma = 0.1$, $b = 0.01$, $r = 1.5$, $\Delta = 0$, $\delta = 0.0026$, $E_0 = 0.04$, $\omega = 0.042$, (a) shows the intensity, and (b) shows the intensity profile across the vertical cut marked in (a). Rest of parameters as in figure 8. (Online version in colour.)
6. Rocking in weakly damped oscillatory systems

So far, we have considered self-oscillatory systems, i.e. systems in which nonlinear oscillations emerge spontaneously. In other words, we have considered systems above a spatially uniform Hopf bifurcation. As explained in §2, the initial idea leading to rocking led to the conclusion that such above threshold condition was necessary for rocking to be effective. Nevertheless, a recent study [70] has demonstrated that it is not the case: everything depends on the nature of the nonlinearity (real—saturating—or imaginary—dispersive) and on the non-locality (real—diffusive—or imaginary—diffractive). In [70], the effects of rocking in a passive optical cavity containing a Kerr nonlinear medium (the classic Lugiato–Lefever model [71]) were studied. It was demonstrated, analytically and numerically, that the same type of phenomena occurring in lasers (above threshold) happen in such a Kerr cavity. The (dimensionless) model reads

\[ \partial_t A = F - (1 + i\eta \theta)A + i\nabla^2 A + i\eta |A|^2 A, \]

(6.1)

where \( A \) is the intracavity field complex amplitude, \( F \) is the amplitude of injected field, \( \eta \theta \) is the normalized detuning between the injection frequency and the cavity resonance frequency (the single longitudinal mode case is considered), and \( \eta \) is +1 for a self-focusing nonlinearity and −1 for a self-defocusing one. Equation (6.1) is as equation (3.2), with \( \alpha = i, \beta = -i\eta \), but with damping (the linear term \(-A\)) instead of gain (\(+A\) in (3.2)). As shown in [70], when the injection is of rocking type, as in equation (3.3), the decomposition (3.4), with (3.5) and (3.6), holds and the slow part is governed by the following damped nonlinear Schrödinger equation with parametric gain,

\[ \partial_t A_s = -[1 + i\eta(\theta - 2\gamma)]A_s + i\eta\gamma A_s^* + i\nabla^2 A_s + i\eta |A_s|^2 A_s. \]

(6.2)

This equation predicts the existence of phase-bistable homogeneous states and patterns, and these predictions hold even far from the conditions leading to (6.2), which are fast rocking.

Note that this result concerning the effectiveness of Rocking below a Hopf bifurcation (i.e. for weakly damped waves) is not at odds with the reasoning in §2 because in the case of the Kerr cavity, equation (6.1), no mechanical analogue (in terms of a potential) can be drawn. In fact, if we consider a variational case (deriving from a potential) below the oscillation threshold (\( \mu = -1 \)), it is easy to check, following the steps in §3, that equation (2.3) for a single oscillator admits solutions of the form ((3.4) and (3.6)), where the slow part verifies

\[ \frac{dA_s}{df} = -(1 + 2\gamma)A_s - \gamma A_s^* - |A_s|^2 A_s, \]

(6.3)

and this equation does not hold but the trivial solution \( A_s = 0 \), i.e. no phase bistability below the Hopf bifurcation if the nonlinearity is real. It is the imaginary character of the Kerr nonlinearity that makes rocking effective below the oscillation threshold.

Figures 10 and 11 illustrate typical results of the numerical integration of a Kerr nonlinear system below the threshold, equation (6.1). We note that both the temporal and the spatial rocking is possible, as already shown in [70]. Here, we concentrate on spatial rocking by injecting a periodic pattern in the form of parallel stripes (figure 10), which can be called one-dimensional rocking, and by injecting a pattern of square symmetry (figure 11)—two-dimensional rocking. In both cases, we show the small-scale pattern, its amplitude (figure 11a) and phase (figure 11b), and also the large-scale patterns (figure 11c,d) corresponding to the slow component. The large-scale patterns were obtained by spatial filtering, i.e. by removing in spatial Fourier domain the field components with sufficiently large transverse wavevector. All variety of patterns (rolls, phase domains and phase solitons) can be obtained, depending mainly on the detuning parameter and on the sign of the nonlinearity: here, in figures 10 and 11, we show patterns occurring at moderate detuning values—the metastable phase domains.
Figure 10. Field domains of opposite phases obtained for the passive resonator with Kerr nonlinearity, for \( \eta = -1 \) (defocusing nonlinearity) in the cases of spatial rocking, by injecting one-dimensional-periodic function (stripes). (a,b) The amplitude and the phase of small-scale pattern, respectively; (c,d) The intensity and the phase of the large-scale pattern (after low-\( k \) pass filtering). Parameters: space window \((70 \times 70)\), detuning \( \theta = 2.5 \), \( F_0 = 9.5 \) and \( k_0^2 = 7 \).

Figure 11. Field domains of opposite phases obtained for the passive resonator with Kerr nonlinearity, for \( \eta = -1 \) (defocusing nonlinearity), in the cases of spatial rocking, by injecting two-dimensional-periodic function of square symmetry. (a,b) The amplitude and the phase of the small-scale pattern, respectively; (c,d) The intensity and the phase of the large-scale pattern (after low-\( k \) pass filtering). Parameters: space window size \((70 \times 70)\), detuning \( \theta = 7 \), \( F_0 = 16 \) and \( k_0^2 = 2 \).
7. Conclusion

We have reviewed, and given new results of, a universal forcing method of self-oscillatory systems known as ‘rocking’. Rocking is a forcing around the oscillators natural frequency (it is a 1 : 1 resonance), with forcing amplitude whose sign alternates (in time or in space). This alternation creates conditions for phase bistability and for the spatial patterns associated with it as we explained in §§2 and 3 in the context of universal order parameter equations. The method represents an alternative to the classical 2 : 1 resonant forcing and allows phase-bistable patterns in systems where such forcing is ineffective (e.g. in lasers). The new results presented here demonstrate that rocking is effective for the excitation of phase patterns and phase cavity solitons (dark-ring solitons) in two-level lasers, particularly in class B lasers.

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References


