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Wave height estimates from pressure and velocity data at an intermediate depth in the presence of uniform currents

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Bounds on estimates of wave heights (valid for large amplitudes) from pressure and flow measurements at an arbitrary intermediate depth have been provided. Two-dimensional irrotational steady water waves over a flat bed with a finite depth in the presence of underlying uniform currents have been considered in the analysis. Five different upper bounds based on a combination of pressure and velocity field measurements have been derived, though there is only one available lower bound on the wave height in the case of the speed of current greater than or less than the wave speed.

This article is part of the theme issue 'Nonlinear water waves'.

1. Introduction

The importance of wave height estimates in the fields of ocean energy, coastal engineering and ocean science is well known. The problem of recovering wave heights from measured pressure data at the bed is a practical way to estimate wave heights which can also be used for ground truth verification of indirect remote observations. The use of pressure sensors at the bed for this so-called recovery problem is popular due to low cost and convenience in installations [1]. Several researchers have addressed this problem in the past, such as [2,3].

As most waves relevant in ocean engineering do not fall into the category of small-amplitude theory and, in fact, linear theory may overestimate wave heights by over 10% even for waves of moderate amplitude when compared with observed data [4], investigations have been carried out to relax the approximations of linearity and small amplitudes by researchers such as

[3,5–10]. Even though there have been investigations to consider large-amplitude waves, exact results are rare in the literature. It has been shown in [11] that it is possible to have exact recovery of wave heights from the pressure data at the bed even for flows with vorticity, and [12] developed an approach to provide simple, explicit but accurate estimates of wave heights for the fully nonlinear large-amplitude two-dimensional travelling gravity waves. This was followed by Basu [13], who provided bounds on estimated wave heights from pressure measurements on a flat bed for water waves with uniform current using some results on the properties of pressure and flow fields [14].

The aim of this paper is to address the recovery problem, i.e. provide estimates of wave height from pressure and flow velocity measurements at an intermediate arbitrary (known) depth instead of measurements at the bed, without any limitation on the amplitude of wave height. The driver for this is mainly due to the convenience of implementation and flexibility in operation. The other aspect we also consider is a uniform underlying current, which still retains the irrotational structure of the two-dimensional problem.

2. Governing equations

The governing equations and the boundary conditions, including the description of current, are similar to what have been considered and presented in [13]. However, for the sake of completeness, we re-present these in this section.

We consider a two-dimensional flow represented in the Cartesian co-ordinate (X,Y) with the direction of wave propagation along the X-axis and the Y-axis pointing vertically upwards. The depth of water at rest is denoted by $d>0$, which leads to the equation of the flat bed as $Y=-d$. The free surface is given by $Y=\eta(X-ct)$, where $c>0$ is the speed of propagation of the periodic travelling waves. The surface waves are assumed to propagate over a uniform underlying current of speed k. The flow is irrotational and it may be noted that the uniform underlying current does not introduce any vorticity to the flow.

We investigate periodic travelling waves of wavelength L, with profile η moving at constant speed c. The wave crest is assumed to be located at $X=0$. Also, we assume that the flow field (u,v) has a space–time dependence of the form $(X-ct)$, i.e. $u:=u(X-ct,Y)$ $v:=v(X-ct,Y)$. To analyse this problem, we introduce a moving frame co-ordinate transformation such that $x=X-ct, y=Y$. Flows without stagnation points are considered, so that, by regularity results discussed in [15], all functions are real analytic. The only waves that are thus excluded are the so-called Stokes waves of greatest height (see the discussions in [16–18]).

The governing equations for the water waves under inviscid, incompressible conditions are given by Euler's equation

$$\left.\begin{aligned}(u-c)u_x+vu_y&=-\frac{1}{\rho}P_x \qquad \text{for } -d\le y\le\eta(x)\\ \text{and}\qquad (u-c)v_x+vv_y&=-\frac{1}{\rho}P_y-g \quad \text{for } -d\le y\le\eta(x),\end{aligned}\right\} \tag{2.1}$$

where $P(x,y)$ is the hydrodynamic pressure, ρ is the density and g is the acceleration due to gravity.

The mass conservation equation for a homogeneous, incompressible fluid is

$$u_x+v_y=0 \quad \text{for } -d\le y\le\eta(x) \tag{2.2}$$

and, due to irrotationality, we have

$$u_y=v_x \quad \text{for } -d\le y\le\eta(x). \tag{2.3}$$

The kinematic boundary conditions at the flat bed and at the free surface are, respectively, given by

$$v=0 \quad \text{on} \quad y=-d \tag{2.4}$$

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and

$$v = (u - c)\eta(x) \quad \text{on} \quad y = \eta(x). \tag{2.5}$$

The dynamic boundary condition of constant atmospheric pressure at the surface is

$$P = P_{\text{atm}} \quad \text{on} \quad y = \eta(x). \tag{2.6}$$

Mass flux (relative to the uniform flow speed c) is a constant and is expressed as

$$M = \rho \int_{-d}^{\eta(x)} [u(x, y) - c]\,\mathrm{d}y. \tag{2.7}$$

Conservation of energy leads to Bernoulli's law, which states that

$$\frac{(u - c)^2 + v^2}{2} + g(y + d) + \frac{1}{\rho}P = Q, \tag{2.8}$$

where Q is a constant throughout the fluid domain.

Let us define stream functions

$$\psi_x = -v \quad \text{and} \quad \psi_y = u - c$$

and set (see [14,19])

$$\psi(x, -d) = -\frac{M}{\rho} \quad \text{and} \quad \psi(x, \eta(x)) = 0 \quad \forall\, x \in \mathbb{R}.$$

The equivalent form of governing equations (equation (2.1)) can be expressed in terms of stream functions

$$\left.\begin{aligned} &\psi_{xx} + \psi_{yy} = 0 && \text{for } -d \le y \le \eta(x),\\ &\psi = 0 && \text{on } y = \eta(x),\\ &\psi = -\frac{M}{\rho} && \text{on } y = -d\\ \text{and}\quad &\psi_x^2 + \psi_y^2 + 2g(y + d) = 2Q - \frac{2}{\rho}P_{\text{atm}} && \text{on } y = \eta(x). \end{aligned}\right\} \tag{2.9}$$

The expression for uniform underlying current is

$$k = \frac{1}{L}\int_{-L/2}^{L/2} u(x, -d)\,\mathrm{d}x = \frac{1}{L}\int_{-L/2}^{L/2} u(x, y_0)\,\mathrm{d}x, \quad -d \le y_0 \le \eta(x), \tag{2.10}$$

which can be written as [14]

$$k = c + \frac{1}{L}\int_{-L/2}^{L/2} \psi_y(x, -d)\,\mathrm{d}x. \tag{2.11}$$

The expression in equation (2.11) may be compared with the two definitions of the Stokes wave speed (see the discussion in [20]). In the absence of current (i.e. $k = 0$), c is equal to the mean of $-\psi_y$ along the flat bed, so that the wave speed can be recovered from the wave pattern in the reference frame in which the wave is at rest (this is Stokes' first definition of the wave speed (see [20])). Three cases may arise consequently from equation (2.11) corresponding to $k > c$, $k = c$ and $k < c$; these are the strength (average speed) of the current greater than, equal to and less than the wave speed, respectively, as shown and discussed in [14]. In the following sections, these three cases will be considered separately as they lead to a different nature of flow fields, as has been shown by Basu [14].

3. Estimates on wave heights: $k > c$

We first consider the case where the speed of the current is greater than the wave velocity, i.e. $k > c$. The other two cases, i.e. $k = c$ and $k < c$, will be dealt with subsequently.

The case with $k > c$ results in flow fields (see [14] for details) different from what has been known previously about flow fields when there is no underlying current (e.g. see [21–24]). The results are summarized as follows for completeness.

Let $\bar{c} = k - c > 0$, leading to

$$\bar{c} = \frac{1}{L}\int_{-L/2}^{L/2} \psi_y(x, -d)\,\mathrm{d}x > 0$$

and from Basu [14]

$$u > c \quad \text{and} \quad M > 0, \tag{3.1}$$

throughout the fluid domain.

Furthermore, the use of properties of harmonic functions and maximum principles results in (see [14])

$$\left.\begin{aligned} &u_y(0, y) < 0 \quad &&\text{for } -d < y < \eta(0),\\ &u_y\left(\frac{L}{2}, y\right) > 0 \quad &&\text{for } -d < y < \eta\left(\frac{L}{2}\right)\\ \text{and} \qquad &u_x(x, -d) > 0 \quad &&\text{for } 0 < x < \frac{L}{2}. \end{aligned}\right\} \tag{3.2}$$

(a) Lower bounds for wave height

Evaluating Bernoulli's law (equation (2.8)) at the wave crest ($x = 0$), at the wave trough ($x = L/2$) and at a depth (d_{int} from the still water level) below the wave crest and the wave trough, respectively, we have

$$\left.\begin{aligned} &\frac{(u_{\min} - c)^2}{2} + g(\eta(0) + d) + \frac{1}{\rho}P_{\text{atm}} = Q,\\ &\frac{(u_{\max} - c)^2}{2} + g\left(\eta\left(\frac{L}{2}\right) + d\right) + \frac{1}{\rho}P_{\text{atm}} = Q,\\ &\frac{(u(0, -d_{\text{int}}) - c)^2}{2} + g(d - d_{\text{int}}) + \frac{1}{\rho}P(0, -d_{\text{int}}) = Q\\ \text{and} \qquad &\frac{(u(L/2, -d_{\text{int}}) - c)^2}{2} + g(d - d_{\text{int}}) + \frac{1}{\rho}P\left(\frac{L}{2}, -d_{\text{int}}\right) = Q, \end{aligned}\right\} \tag{3.3}$$

where $u_{\min} = u(0, \eta(0))$ and $u_{\max} = u(L/2, \eta(L/2))$ are the minimum and maximum horizontal fluid velocities at $(0, \eta(0))$ and $(L/2, \eta(L/2))$, respectively. The first and third relations in equation (3.3) lead to

$$\begin{aligned} \eta(0) + d_{\text{int}} &= \frac{P(0, -d_{\text{int}}) - P_{\text{atm}}}{\rho g} + \frac{(u(0, -d_{\text{int}}) - c)^2 - (u_{\min} - c)^2}{2}\\ &> \frac{P(0, -d_{\text{int}}) - P_{\text{atm}}}{\rho g}, \end{aligned} \tag{3.4}$$

and the second and fourth relations in equation (3.3) lead to

$$\begin{aligned} \eta\left(\frac{L}{2}\right) + d_{\text{int}} &= \frac{P(L/2, -d_{\text{int}}) - P_{\text{atm}}}{\rho g} + \frac{(u(L/2, -d_{\text{int}}) - c)^2 - (u_{\max} - c)^2}{2}\\ &< \frac{P(L/2, -d_{\text{int}}) - P_{\text{atm}}}{\rho g}, \end{aligned} \tag{3.5}$$

on using equations (3.1) and (3.2).

From equations (3.4) and (3.5), the lower bound for the estimate of wave height $H = \eta(0) - \eta(L/2)$ is obtained as

$$H = \eta(0) - \eta\left(\frac{L}{2}\right) > \frac{P(0, -d_{\text{int}}) - P(L/2, -d_{\text{int}})}{\rho g}. \tag{3.6}$$

It is interesting to observe that the lower bound is independent of the current speed. Furthermore, the lower bounds arrived at by [12,13] in the case of surface waves propagating without any underlying currents can be recovered by substituting $d_{\text{int}} = d$. It may also be noted that the right-hand side of the inequality in equation (3.4) exceeds d_{int} while the right-hand side of the inequality in equation (3.5) is less than d_{int}. This is because fluid pressure is maximal at $(0, -d_{\text{int}})$, while $P(L/2, -d_{\text{int}})$ is the minimum value of the pressure along the depth d_{int} with the average pressure at a depth d_{int} from the still water level equal to $P_{\text{atm}} + \rho g d_{\text{int}}$ (see [23] for the detailed discussion on pressure on the flat bed); this nature of pressure remains unchanged in the presence of uniform currents [14]. This was also observed by Basu [13] in the context of pressure data on the flat data.

Another point to note from equation (3.6) is that, as the depth of the points at which pressures are measured decreases, the bound becomes less sharp and is trivially true when pressures are measured at the surface. The sharpest bound is obtained for the pressure measured at the flat bed.

(b) Upper bounds for wave height

To obtain the upper bounds on the wave heights, it is convenient to formulate the problem and carry out the derivations in a transformed domain. Transforming by means of conformal change of variables (as the problem is irrotational)

$$q = -\phi(x, y) \quad \text{and} \quad p = -\psi(x, y),$$

the free boundary problem given by equation (2.9) has an equivalent boundary value problem

$$\left.\begin{aligned} &\Delta_{q,p} h = 0 && \text{for } \frac{M}{\rho} > p > 0, \\ &h = 0 && \text{on } p = \frac{M}{\rho} \\ \text{and} \quad &2\left(Q - \frac{1}{\rho}P_{\text{atm}} - gh\right)(h_q^2 + h_p^2) = 1 && \text{on } p = 0, \end{aligned}\right\} \tag{3.7}$$

for the function

$$h(q, p) = y + d \tag{3.8}$$

(see [12,23]).

Let us also define $P_*(q, p) = P(x, y)$, $u_*(q, p) = u(x, y)$.

As $q = cL/2$ corresponds to $x = L/2$, we define

$$\alpha = \frac{P_*(cL/2, M/\rho) - P_{\text{atm}}}{M/\rho} = \frac{P(cL/2, -d) - P_{\text{atm}}}{M/\rho} \tag{3.9}$$

and

$$\alpha_0 = \frac{P_*(cL/2, M_0/\rho) + \rho(u_*(cL/2, M_0/\rho) - c)^2 - P_{\text{atm}}}{M/\rho}, \tag{3.10}$$

where

$$M_0 = \rho \int_{-d_{\text{int}}}^{\eta(x)} [u(x, y) - c] < M. \tag{3.11}$$

From equations (3.9) and (3.10),

$$\alpha > 0, \alpha_0 > 0,$$

and choose a depth such that the condition $\alpha \geq \alpha_0$ is satisfied.

Equations (3.1), (3.2) and (3.7) lead to

$$\begin{aligned} P_*(q, M/\rho) &= \rho Q - \frac{\rho(u_*(q, M/\rho) - c)^2}{2} \\ &\geq \rho Q - \frac{\rho(u_*(cL/2, M/\rho) - c)^2}{2} \\ &= P_*\left(\frac{cL}{2}, \frac{M}{\rho}\right) \\ &= P_{\text{atm}} + \frac{\alpha M}{\rho}, \quad \forall\, q \in \left[0, \frac{cL}{2}\right]. \end{aligned}$$

From equations (2.1) and (3.8), we can deduce

$$\frac{1}{\rho}\Delta(q,p)P_* = -2(u_{*q}^2 + u_{*p}^2), \quad \frac{M}{\rho} > p > 0,$$

and the function

$$f(q,p) = \frac{P_*(q,p) - \alpha_0 p}{\rho} \tag{3.12}$$

is superharmonic in the strip $M/\rho > p > 0$. Also, we note that

$$\begin{aligned} f(q, M/\rho) &= \frac{P_*(q, M/\rho) - \alpha_0 M/\rho}{\rho} \\ &\geq \frac{P_{\text{atm}} + \alpha M/\rho - \alpha_0 M/\rho}{\rho} \\ &> \frac{P_{\text{atm}}}{\rho} \quad (\because\ \alpha,\ \alpha_0,\ M > 0). \end{aligned} \tag{3.13}$$

It may be noted that f equals P_{atm}/ρ on the upper surface $p=0$, and from equation (3.13) it is observed that f always attains higher values than P_{atm}/ρ on the lower boundary $p=M/\rho$. Hence, from the strong maximum principle we infer that the minimum of f is attained on the upper boundary $p=0$ and Hopf's principle ensures $f_p(q,0)>0\ \forall\ q\in R$. Hence, from equations (2.1), (2.2), (2.5) and (3.8), we have

$$f_p(q,0) = \frac{1}{\rho}P_{*p}(q,0) - \frac{\alpha_0}{\rho} > 0,$$

leading to

$$-\frac{\alpha_0}{\rho} > \frac{2(u-c)^2 u_x \eta_x + (u-c)^2 u_y(\eta_x^2 - 1) - g(u-c)}{(u-c)^2(1+\eta_x^2)} \tag{3.14}$$

on $y=\eta(x)$.

On differentiating the fourth relation in equation (2.9) along the free surface w.r.t. the x-variable, we also obtain

$$0 = (u-c)u_x + (u-c)u_y\eta_x + vv_x + vv_y\eta_x + g\eta_x \quad \text{on} \quad y=\eta(x).$$

Furthermore, from the last equation and equations (2.2), (2.3) and (2.5) we obtain

$$(u-c)u_x(1-\eta_x^2) + 2(u-c)u_y\eta_x + g\eta_x = 0 \quad \text{on} \quad y=\eta(x). \tag{3.15}$$

Using equations (3.1), (3.14), (3.15) and the fact that $\eta_x(x)<0$ for $x\in(0,L/2)$, we get

$$-\frac{\alpha_0}{\rho}\eta_x < u_x(x,\eta(x)) + u_y(x,\eta(x))\eta_x(x), \quad 0<x<\frac{L}{2},$$

which on integration on $[0, L/2]$ leads to

$$\frac{\alpha_0}{\rho}\left[\eta\left(\frac{L}{2}\right) - \eta(0)\right] > u(0,\eta(0)) - u\left(\frac{L}{2},\eta\left(\frac{L}{2}\right)\right). \tag{3.16}$$

Evaluating equation (2.8) at the wave crest and the wave trough, and from equation (3.16), we can deduce

$$\frac{\alpha_0}{2\rho g}\left[\{u(0,\eta(0))-c\}^2 - \left\{u\left(\frac{L}{2},\eta\left(\frac{L}{2}\right)\right) - c\right\}^2\right] > u(0,\eta(0)) - u\left(\frac{L}{2},\eta\left(\frac{L}{2}\right)\right), \tag{3.17}$$

which further yields

$$u(0,\eta(0)) + u\left(\frac{L}{2},\eta\left(\frac{L}{2}\right)\right) - 2c < \frac{2\rho g}{\alpha_0}. \tag{3.18}$$

From equation (3.2), we have $u(L/2, \eta(L/2)) > u(0, \eta(0)) > c$. Furthermore, as $\eta(L/2) \leq 0$, we have

$$ML = \int_{-L/2}^{L/2} M\,\mathrm{d}x = \rho \int_{-L/2}^{L/2} \int_{-d}^{\eta(L/2)} (u-c)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leq \rho \int_{-L/2}^{L/2} \int_{-d}^{0} (u-c)\,\mathrm{d}x\,\mathrm{d}y = \rho(k-c)Ld.$$

Thus

$$M \leq \rho(k-c)d. \tag{3.19}$$

Combining equations (3.10), (3.18) and (3.19), we obtain

$$u(0,\eta(0)) + u\left(\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right) - 2c < \frac{2\rho g(k-c)d}{P(L/2, -d_{\mathrm{int}}) + \rho\{u(L/2, -d_{\mathrm{int}}) - c\}^2 - P_{\mathrm{atm}}}. \tag{3.20}$$

On evaluating equation (2.8) at the wave crest/trough and using equation (2.6), we obtain

$$H = \eta(0) - \eta\left(\frac{L}{2}\right) = \frac{[u(L/2), \eta(L/2) - c]^2 - [u(0), \eta(0) - c]^2}{2g}$$
$$< \frac{[u(0), \eta(0) + u(L/2), \eta(L/2) - 2c]^2}{2g}. \tag{3.21}$$

Hence, from equations (3.20) and (3.21), we obtain the upper bound

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(L/2, -d_{\mathrm{int}}) + \rho\{u(L/2, -d_{\mathrm{int}}) - c\}^2 - P_{\mathrm{atm}}]^2} \tag{3.22}$$

on the wave height. An alternative bound is obtained on using Bernoulli's law (equation (2.8)) at a depth $y = -d_{\mathrm{int}}$ as follows:

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(0, -d_{\mathrm{int}}) + \rho\{u(0, -d_{\mathrm{int}}) - c\}^2 - P_{\mathrm{atm}}]^2}. \tag{3.23}$$

If velocity measurements are not available, then the bound in equation (3.22) can be modified based on the pressure measurement alone, and can be given as

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(L/2, -d_{\mathrm{int}}) - P_{\mathrm{atm}}]^2}. \tag{3.24}$$

Equation (3.24) can also be obtained by defining α_0 in equation (3.10) by

$$\alpha_0 = \frac{P_*(cL/2, M_0/\rho) - P_{\mathrm{atm}}}{M/\rho} > 0 \tag{3.25}$$

and proceeding following a similar approach as before, exploiting maximum principles and the governing equations of the water waves.

Equation (3.24) provides a bound less sharper than the one given by equation (3.23). In fact, a sharper bound is obtained from equation (3.23) and is given by

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(0, -d_{\mathrm{int}}) - P_{\mathrm{atm}}]^2}, \tag{3.26}$$

if velocity measurements are not available and cannot be used.

If pressure measurements are not available and only velocity measurements are available instead, then the following bound is obtained from equation (3.22) (since $P(L/2, -d_{\mathrm{int}}) > P_{\mathrm{atm}}$; see [14])

$$H < \frac{2g(k-c)^2 d^2}{[u(L/2, -d_{\mathrm{int}}) - c]^4}. \tag{3.27}$$

It may be noted that if measurements at the flat bed are available, then bounds based on pressure measurements become sharper than those based on velocity measurements, while velocity measurements should be used to arrive at sharper bounds when measurements are taken closer to the free surface.

Another interesting point to note, as was also observed in [14], is that, when $k = c$, the bounds in equations (3.22)–(3.24) and equations (3.26)–(3.27) all lead to a Stokes wave of trivial amplitude. This is equivalent to the fact that a Stokes wave of the same speed as an underlying uniform current can only coexist if the top surface is flat. Hence, the case $k = c$ needs no further perusal.

4. Estimates on wave heights: $k < c$

To deal with this case, let us define $\bar{c} = c - k > 0$. The case can be treated in a similar way as in [12] with a wave speed of $\bar{c}$ and no current. However, we need to make appropriate modifications to account for measurements at an arbitrary depth. We proceed to obtain the bounds on wave heights.

The case with $k < c$ results in flow fields (see [14] for details) similar to the flow fields when there is no underlying current (e.g. see [21–24]). The results are summarized as follows.

Let $\bar{c} = c - k > 0$, leading to

$$\bar{c} = \frac{1}{L}\int_{-L/2}^{L/2} \psi_y(x, -d)\,\mathrm{d}x > 0$$

and, from Basu [14],

$$u < c \quad \text{and} \quad M < 0 \tag{4.1}$$

throughout the fluid domain.

Furthermore, the use of properties of harmonic functions and maximum principles results in (see [14])

$$\left.\begin{aligned} &u_y(0, y) > 0 && \text{for } -d < y < \eta(0),\\ &u_y\left(\frac{L}{2}, y\right) < 0 && \text{for } -d < y < \eta\left(\frac{L}{2}\right)\\ \text{and}\quad &u_x(x, -d) < 0 && \text{for } 0 < x < \frac{L}{2}. \end{aligned}\right\} \tag{4.2}$$

(a) Lower bounds for wave height

Evaluating Bernoulli's law (equation (2.8)) at the wave crest ($x = 0$), at the wave trough ($x = L/2$) and at a depth (d_{int} from the still water level) below the wave crest and the wave trough, respectively, we have

$$\left.\begin{aligned} &\frac{(u_{\max} - c)^2}{2} + g(\eta(0) + d) + \frac{1}{\rho}P_{\mathrm{atm}} = Q,\\ &\frac{(u_{\min} - c)^2}{2} + g\left(\eta\left(\frac{L}{2}\right) + d\right) + \frac{1}{\rho}P_{\mathrm{atm}} = Q,\\ &\frac{(u(0, -d_{\mathrm{int}}) - c)^2}{2} + g(d - d_{\mathrm{int}}) + \frac{1}{\rho}P(0, -d_{\mathrm{int}}) = Q\\ \text{and}\quad &\frac{(u(L/2, -d_{\mathrm{int}}) - c)^2}{2} + g(d - d_{\mathrm{int}}) + \frac{1}{\rho}P\left(\frac{L}{2}, -d_{\mathrm{int}}\right) = Q, \end{aligned}\right\} \tag{4.3}$$

where $u_{\max} = u(0, \eta(0))$ and $u_{\min} = u(L/2, \eta(L/2))$ are the maximum and minimum horizontal fluid velocities, respectively. The first and third relations in equation (4.3) lead to

$$\eta(0) + d_{\mathrm{int}} = \frac{P(0, -d_{\mathrm{int}}) - P_{\mathrm{atm}}}{\rho g} + \frac{(u(0, -d_{\mathrm{int}}) - c)^2 - (u_{\max} - c)^2}{2} > \frac{P(0, -d_{\mathrm{int}}) - P_{\mathrm{atm}}}{\rho g}, \tag{4.4}$$

and the second and fourth relations in equation (4.3) lead to

$$\eta\left(\frac{L}{2}\right)+d_{\text{int}}=\frac{P(L/2,-d_{\text{int}})-P_{\text{atm}}}{\rho g}+\frac{(u(L/2,-d_{\text{int}})-c)^2-(u_{\min}-c)^2}{2}<\frac{P(L/2,-d_{\text{int}})-P_{\text{atm}}}{\rho g}, \tag{4.5}$$

on using equations (4.1) and (4.2).

From equations (4.4) and (4.5), the lower bound for the estimation of wave height $H=\eta(0)-\eta(L/2)$ is obtained as

$$H=\eta(0)-\eta\left(\frac{L}{2}\right)>\frac{P(0,-d_{\text{int}})-P(L/2,-d_{\text{int}})}{\rho g}, \tag{4.6}$$

arriving at the same lower bound as in equation (3.6). In fact, this is exactly the same lower bound as in the case of wave heights for surface waves without any underlying current [12] if the depth of observation 'd' is replaced by '$-d_{\text{int}}$'.

As also observed for the case $k>c$, the bound becomes less sharp and is trivially true when pressures are measured at the surface, and the sharpest bound is obtained for the pressure measured at the flat bed.

(b) Upper bounds for wave height

Using the same transformations as in §3, we obtain

$$\left.\begin{aligned}&\Delta_{q,p}h=0 && \text{for } \frac{M}{\rho}<p<0,\\ &h=0 && \text{on } p=\frac{M}{\rho}\\ \text{and}\quad &2\left(Q-\frac{1}{\rho}P_{\text{atm}}-gh\right)(h_q^2+h_p^2)=1 && \text{on } p=0,\end{aligned}\right\} \tag{4.7}$$

for the function

$$h(q,p)=y+d \tag{4.8}$$

(see [12,23]).

Let us also define $P_*(q,p)=P(x,y)$, $u_*(q,p)=u(x,y)$.

As $q=cL/2$ corresponds to $x=L/2$, we define

$$\alpha=\frac{P_*(cL/2,M/\rho)-P_{\text{atm}}}{-M/\rho}=\frac{P(L/2,-d)-P_{\text{atm}}}{-M/\rho} \tag{4.9}$$

and

$$\alpha_0=\frac{P_*(cL/2,M_0/\rho)+\rho(u_*(cL/2,M_0/\rho)-c)^2-P_{\text{atm}}}{-M/\rho}, \tag{4.10}$$

where M_0 has been defined in §3. Again, from equations (4.9) and (4.10), we infer

$$\alpha>0, \alpha_0>0,$$

and choose a depth such that the condition $\alpha>\alpha_0$ is satisfied.

Equations (4.1), (4.2) and (4.7) lead to

$$\begin{aligned}P_*\left(q,\frac{M}{\rho}\right)&=\rho Q-\frac{\rho(u_*(q,M/\rho)-c)^2}{2}\\ &\geq \rho Q-\frac{\rho(u_*(cL/2,M/\rho)-c)^2}{2}\\ &=P_*\left(\frac{cL}{2},\frac{M}{\rho}\right)\\ &=P_{\text{atm}}-\frac{\alpha M}{\rho},\quad \forall\, q\in\left[0,\frac{cL}{2}\right].\end{aligned}$$

Consider the function

$$f(q,p) = \frac{P_*(q,p) + \alpha_0 p}{\rho}. \tag{4.11}$$

Then, we have

$$\begin{aligned} f(q, M/\rho) &= \frac{P_*(q, M/\rho) + \alpha_0 M/\rho}{\rho} \\ &\geq \frac{P_{\text{atm}} - \alpha M/\rho + \alpha_0 M/\rho}{\rho} \\ &> \frac{P_{\text{atm}}}{\rho}, \end{aligned} \tag{4.12}$$

as $\alpha > 0$, $\alpha_0 > 0$ and $M < 0$. From equations (2.1) and (4.8), we can deduce

$$\frac{1}{\rho}\Delta(q,p)P_* = -2(u_{*q}^2 + u_{*p}^2), \quad \frac{M}{\rho} < p < 0.$$

The function $f(q,p)$ is superharmonic in the strip $M/\rho < p < 0$.

It may be noted that f equals P_{atm}/ρ on the upper surface $p = 0$, and from equation (4.12) it is observed that f always attains higher values than P_{atm}/ρ on the lower boundary $p = M/\rho$. Hence, from the strong maximum principle we infer that the minimum of f is attained on the upper boundary $p = 0$ and Hopf's principle ensures $f_p(q,0) < 0 \ \forall \ q \in R$. Hence from equations (2.1), (2.2), (2.5) and (4.8), we have

$$f_p(q,0) = \frac{1}{\rho}P_{*p}(q,0) + \frac{\alpha_0}{\rho} < 0,$$

leading to

$$\frac{\alpha_0}{\rho} < \frac{2(u-c)^2 u_x \eta_x + (u-c)^2 u_y(\eta_x^2 - 1) - g(u-c)}{(u-c)^2(1+\eta_x^2)}, \tag{4.13}$$

on $y = \eta(x)$. A similar inequality was derived in [12]. Following a similar approach as in [12], we have

$$|u(0,\eta(0)) - c| + \left|u\left(\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right) - c\right| < \frac{2\rho g}{\alpha_0}. \tag{4.14}$$

Furthermore, as $\eta(L/2) \leq 0$, we have

$$\begin{aligned} ML &= \int_{-L/2}^{L/2} M\,\mathrm{d}x = \rho \int_{-L/2}^{L/2}\int_{-d}^{\eta(L/2)} (u-c)\,\mathrm{d}x\,\mathrm{d}y \\ &\geq \rho \int_{-L/2}^{L/2}\int_{-d}^{0} (u-c)\,\mathrm{d}x\,\mathrm{d}y = \rho(k-c)Ld. \end{aligned}$$

Thus,

$$M \geq \rho(k-c)d. \tag{4.15}$$

Combining equations (4.10), (4.14) and (4.15), we obtain

$$|u(0,\eta(0)) - c| + \left|u\left(\frac{L}{2}, \eta\left(\frac{L}{2}\right)\right) - c\right| < \frac{|2\rho g(k-c)d|}{P(L/2, -d_{\text{int}}) + \rho\{u(L/2, -d_{\text{int}}) - c\}^2 - P_{\text{atm}}}. \tag{4.16}$$

On evaluating equation (2.8) at the wave crest/trough and using equation (2.6), we obtain

$$\begin{aligned} H = \eta(0) - \eta(L/2) &= \frac{[u(L/2), \eta(L/2) - c]^2 - [u(0), \eta(0) - c]^2}{2g} \\ &< \frac{[|u(0), \eta(0) - c| + |u(L/2), \eta(L/2) - c|]^2}{2g}. \end{aligned} \tag{4.17}$$

Hence, from equations (4.16) and (4.17), we obtain the upper bound

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(L/2, -d_{\text{int}}) + \rho\{u(L/2, -d_{\text{int}}) - c\}^2 - P_{\text{atm}}]^2}, \tag{4.18}$$

on the wave height. Similar approaches, followed as in §3, lead to four other upper bounds,

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(0,-d_{\text{int}}) + \rho\{u(0,-d_{\text{int}}) - c\}^2 - P_{\text{atm}}]^2}, \tag{4.19}$$

$$H < \frac{2g(k-c)^2 d^2}{[u(0,-d_{\text{int}}) - c]^4}, \tag{4.20}$$

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(L/2,-d_{\text{int}}) - P_{\text{atm}}]^2} \tag{4.21}$$

and

$$H < \frac{2\rho^2 g(k-c)^2 d^2}{[P(0,-d_{\text{int}}) - P_{\text{atm}}]^2}. \tag{4.22}$$

Again, if velocity measurements are not available, then the last two bounds based on pressure measurements are useful, and out of the two bounds the last one is sharper.

5. Remarks

The following remarks pertain to the analysis in this paper and results obtained.

1. The derived bounds on wave heights in this paper hold for all levels of amplitudes and thus can be applied to water waves of large amplitudes.
2. The lower bounds in this paper are independent of the speed of the underlying uniform current, while the upper bounds depend on the current speed. This was also the case in Basu [14] and is consistent with the bounds provided by Constantin [12] in the case without any current. Hence, this seems to be generally true for two-dimensional irrotational flow with uniform underlying current or without any current.
3. The lower bounds without current approach the lower bounds for the case without any current when the pressure measurement at the flat bed is used to obtain the bounds.
4. Use of a combination of pressure and velocity measurements can make the bounds on the wave heights sharper.
5. It is worth pointing out that the considerations in this paper are of great interest also in the case of non-uniform currents. This setting corresponds to waves with non-zero vorticity and one cannot rely on a harmonic velocity potential, so that the approach used in the present paper does not apply. However, the existence theory for travelling periodic water waves with vorticity of small and large amplitude is quite well developed theoretically for a general vorticity in flows without critical layers (see [25]), for constant vorticity in flows with critical layers (see [26]), and there are quite accurate numerical simulations (see [27]). Some considerations dealing with the pressure recovery for linear and weakly nonlinear waves were recently made in [28]. Further, existence and bounds on the amplitude of water waves over finite depth have been also derived by Kogelbauer [29] in terms of a number of flow parameters.

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References

1. Jones NL, Monismith SG. 2007 Measuring short-period wind waves in a tidally forced environment with a subsurface pressure gauge. *Limnol. Oceanogr. Methods* **5**, 317–327. (doi:10.4319/lom.2007.5.317)
2. Escher J, Schlurmann T. 2008 On the recovery of the free surface from the pressure within periodic traveling water waves. *J. Nonlinear Math. Phys.* **15**, 50–57. (doi:10.2991/jnmp.2008.15.s2.4)

3. Oliveras KL, Vasan V, Deconinck B, Henderson D. 2012 Recovering the water-wave profile from pressure measurements. *SIAM J. Appl. Math.* **72**, 897–918. (doi:10.1137/110853285)
4. Tsai CH, Huang MC, Young FJ, Lin YC, Li HW. 2005 On the recovery of surface wave by pressure transfer function. *Ocean Eng.* **32**, 1247–1259. (doi:10.1016/j.oceaneng.2004.10.020)
5. Clamond D. 2013 New exact relations for easy recovery of steady wave profiles from bottom pressure measurements. *J. Fluid Mech.* **726**, 547–558. (doi:10.1017/jfm.2013.253)
6. Clamond D, Constantin A. 2013 Recovery of steady periodic wave profiles from pressure measurements at the bed. *J. Fluid Mech.* **714**, 463–475. (doi:10.1017/jfm.2012.490)
7. Constantin A. 2012 On the recovery of solitary wave profiles from pressure measurements. *J. Fluid Mech.* **699**, 376–384. (doi:10.1017/jfm.2012.114)
8. Constantin A. 2014 Stokes waves in water with a non-flat bed. *J. Fluid Mech.* **740**, 17–27. (doi:10.1017/jfm.2013.635)
9. Constantin A. 2016 Extrema of the dynamic pressure in an irrotational regular wave train. *Phys. Fluids* **28**, 113604. (doi:10.1063/1.4967362)
10. Deconinck B, Oliveras K, Vasan V. 2012 Relating the bottom pressure and the surface elevation in the water wave problem. *J. Nonlinear Math. Phys.* **19**, 179–189. (doi:10.1142/S1402925112400141)
11. Henry D. 2013 On the pressure transfer function for solitary water waves with vorticity. *Math. Ann.* **357**, 23–30. (doi:10.1007/s00208-013-0899-0)
12. Constantin A. 2014 Estimating wave heights from pressure data at the bed. *J. Fluid Mech.* **743**, R2. (doi:10.1017/jfm.2014.81)
13. Basu B. 2017 Estimation of wave heights from pressure data at the bed in the presence of uniform underlying currents. *Nonlinear Anal.* **156**, 82–89. (doi:10.1016/j.na.2017.01.016)
14. Basu B. 2016 Irrotational two-dimensional free-surface steady water flows over a flat bed with underlying currents. *Nonlinear Anal. Theory Methods Appl.* **147**, 110–124. (doi:10.1016/j.na.2016.08.016)
15. Constantin A, Escher J. 2011 Analyticity of periodic traveling free surface water waves with vorticity. *Ann. Math.* **173**, 559–568. (doi:10.4007/annals.2011.173.1.12)
16. Constantin A. 2012 Particle trajectories in extreme Stokes waves. *IMA J. Appl. Math.* **77**, 293–307. (doi:10.1093/imamat/hxs033)
17. Lyons T. 2016 The pressure in a deep-water Stokes wave of greatest height. *J. Math. Fluid Mech.* **18**, 209–218. (doi:10.1007/s00021-016-0249-6)
18. Toland JF. 1996 Stokes waves. *Topol. Methods Nonlinear Anal.* **7**, 1–48. (doi:10.12775/TMNA.1996.001)
19. Constantin A 2011 *Nonlinear water waves with applications to wave–current interactions and tsunamis*. CBMS- NSF Reg. Conf. Ser. Appl. Math., no. 81, Philadelphia, PA: SIAM.
20. Constantin A. 2013 Mean velocities in a Stokes wave. *Arch. Ration. Mech. Anal.* **207**, 907–917. (doi:10.1007/s00205-012-0584-6)
21. Clamond D. 2012 Note on the velocity and related fields of steady irrotational two-dimensional surface gravity waves. *Phil. Trans. R. Soc. A* **370**, 1572–1586. (doi:10.1098/rsta.2011.0470)
22. Constantin A. 2006 The trajectories of particles in Stokes waves. *Invent. Math.* **166**, 523–535. (doi:10.1007/s00222-006-0002-5)
23. Constantin A, Strauss W. 2010 Pressure beneath a Stokes wave. *Commun. Pure Appl. Math.* **63**, 533–557. (doi:10.1002/cpa.20299)
24. Umeyama M. 2012 Eulerian–Lagrangian analysis for particle velocities and trajectories in a pure wave motion using particle image velocimetry. *Phil. Trans. R. Soc. A* **370**, 1687–1702. (doi:10.1098/rsta.2011.0450)
25. Constantin A, Strauss W. 2004 Exact steady periodic water waves with vorticity. *Commun. Pure Appl. Math.* **57**, 481–527. (doi:10.1002/cpa.3046)
26. Constantin A, Strauss W, Vărvărucă E. 2016 Global bifurcation of steady gravity water waves with critical layers. *Acta. Math.* **217**, 195–262. (doi:10.1007/s11511-017-0144-x)
27. Ribeiro R, Milewski PA, Nachbin A. 2017 Flow structure beneath rotational water waves with stagnation points. *J. Fluid. Mech.* **812**, 792–814. (doi:10.1017/jfm.2016.820)
28. Henry D, Thomas GP. 2017 Prediction of the free-surface elevation for rotational water waves using the recovery of pressure at the bed. *Phil. Trans. R. Soc. A* **376**, 20170102. (doi:10.1098/rsta.2017.0102)
29. Kogelbauer F. 2017 Existence and amplitude bounds for irrotational water waves in finite depth. *Phil. Trans. R. Soc. A* **376**, 20170094. (doi:10.1098/rsta.2017.0094)