Transient motions of an oscillating system caused by forcing terms proportional to the velocity of the structural motion

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Damping limits the motions of an oscillator, which is a dynamic system. The selection of formulations for damping is discussed. If the forcing of the dynamic system contains terms that are proportional to the velocity of motion of the oscillator (drag-type forcing functions), these effects will additionally contribute to dampening the oscillations. Should the total damping under certain conditions become apparently negative, the oscillations will grow until the damping has again become positive. Investigations into damping effects that apparently are negative, and discussions where apparent negative damping might appear in practical applications are of great interest.

Keywords: apparent negative damping; large vibrations; galloping

1. Introduction

In this paper, we consider initially a one degree-of-freedom forced oscillator with viscous damping, described by the well-known equation of motion

\[ m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = f_e(t). \]  

(1.1)

Here,

— \( f_e(t) \) is the external, time-varying excitation (for example, a hydrodynamic wave force);
— \( m \) is the mass of the system (including the added mass when relevant);
— \( c_0 \) is the damping constant (viscous damping); and
— the stiffness \( k \), which represents a linear restoring spring force, is assumed to be constant.

Structural damping is associated with a constant \( c \), and critical damping is obtained for the case \( \xi = c/2m\omega_0 = 1 \), while the natural frequency of the undamped motion is given by \( \omega_0 = \sqrt{k/m} \). As is well known in structural analysis, the damping term changes the natural frequency of motion to \( \omega' = \omega_0 \sqrt{1 - \xi^2} \).

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If the forcing term is zero, the displacement $y(t)$ is given as freely damped oscillating motion. Should the value of the damping $\xi$ take on a negative value, however, the oscillations will have increasingly large amplitudes (figure 1).

Obviously, increasing amplitudes resulting from equation (1.1) would be non-physical, representing energy generation in the system. However, in the case of resonant external loading, the oscillations could grow to large values. Let us, furthermore, investigate the case when the external forcing term $f_e(t)$ of the dynamic system contains terms that are proportional to the velocity of motion of the oscillator. The effects of these will contribute to dampening the oscillations. The following one degree-of-freedom oscillator described by the equation of motion is then

$$\frac{d^2y(t)}{dt^2} + c(t)\frac{dy(t)}{dt} + ky(t) = g_0(t). \tag{1.2}$$

Here

— $g_0(t)$ is the ‘remaining’ external, time-varying excitation load, reduced by the load proportional to the velocity term, $\frac{dy(t)}{dt}$, and
— $c(t) \equiv c_0 + \Delta c(t)$ is the damping term, consisting of a constant term $c_0$ (that, for example, would represent the structural damping) and a time-varying term $\Delta c(t)$.

2. Effect of damping

In order to study the effects of the varying damping term, $\Delta c(t)$, we will follow an approach reported by Jonassen [1] and by Gudmestad et al. [2]. We will define three regions $U_1$, $U_2$ and $U_3$ in $\mathbb{R}^2$ (the two-dimensional plane) where $U_1$ is a disc centred in the origin with radius $r_1$, the set $U_2$ is an annulus centred on the origin...
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with inner radius \( r_1 \) and outer radius \( r_2 \) and the set \( U_3 \) is an annulus centred on the origin with inner radius \( r_2 \) and outer radius \( r_3 \). Here, we have \( 0 < r_1 < r_2 < r_3 \). We will call \( U_1 \) the fixed-point region, \( U_2 \) the gluing region, where ‘bump functions’ are used to smoothly transform an oscillator in \( U_1 \) to the oscillator in \( U_3 \), and where \( U_3 \) is the outer region. Our aim is to construct a vector field in \( U_1 \) with the following (geometrical) properties:

— the vector field should model an oscillator;
— the non-forced field should have the origin as a fixed point;
— the fixed point should periodically or almost periodically change stability, from an attracting fixed point to a repelling fixed point; and
— the model should be as simple as possible.

There are a large number of candidates for such models, both among linear and nonlinear oscillators. To meet our last aim, we start with a linear oscillator of the form

\[
\frac{d^2 y(t)}{dt^2} + a(t) \frac{dy(t)}{dt} + \omega^2 y(t) = b(t),
\]  

(2.1)

where \( a(t) \) is a periodic or almost periodic function with \(-1 \leq a(t) \leq 1\). The minimum value of \( a(t) \) is \(-1\) and the maximum of \( a(t) \) is 1. This choice will clearly fulfil our other aims. For simplicity, we may choose \( a(t) \) to be periodic, for example,

\[
a(t) = \sin\left( \frac{t}{10} \right),
\]  

(2.2)

implying that the stability of the origin is slowly changing compared with a time scale \( t \) of the order of 1. We remark that even with this simple equation, one cannot find the exact solution in a closed form, that is, one has to give the solution as a power series in \( t \). In order to mimic a drag force loading term, see Morison et al. [3], we will choose

\[
b(t) = \sin(kt)|\sin(kt)|.
\]  

(2.3)

In a numerical simulation presented in figure 2a,b, we have used \( k = 5 \) and \( \omega = 1 \). Figure 2a shows the local behaviour of the model in the extended region of \( U_1 \). Only the \( y(t) \) coordinate is shown here for \( 0 \leq t \leq 200 \).

Note that the numbers \( r_i, i = 1,2,3, \) can be chosen such that this model has ‘bursts’ of the trajectory into the regions \( U_2 \) and \( U_3 \). In region \( U_3 \), the full model could be a Duffing-type oscillator, and the bump functions are polynomials. Figure 2b shows the full orbit in the phase space. For further discussion on the details of the modelling, reference is made to Jonassen [1].

We have thus demonstrated that an ‘apparent negative damping’ gives rise to ‘burst types’ of displacements (an oscillatory instability) and we will look into different physical examples of apparent negative damping terms. The phenomenon is well known in a number of fields; it should, however, be emphasized that the phenomenon has to be linked to the physical conditions, as no system can physically sustain increased growth of the amplitudes unless energy is fed into the system.

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3. Case 1: galloping of a one degree-of-freedom system in steady flow with magnitude $U$, e.g. [4]

When a non-circular cross-sectional structure experiences fluid forces that change unfavourably with the orientation in the flow, it may cause the structure to start vibrating. This vibration changes the orientation of the structure, which oscillates with the fluid force. If this force tends to increase the vibration, then the structure will become unstable. This phenomenon is known as ‘galloping’. Examples of galloping phenomenon can be seen during the vibration of ice-coated power line cables or twin pipelines. The amplitude of the vibration on galloping can reach 10 times or even more the body dimension of the structure [5].

The analysis of galloping is based on the assumption of quasi-steady fluid dynamics. This means that it is assumed that the body changes its velocity much slower than the time required for formation and shedding of vortices. Thus, the forces at every instant will be the same as in a steady flow having the instantaneous magnitude and direction of the relative flow velocity.

The quasi-steady assumption was based on a previous study of an aerodynamically instability by Den Hartog in 1927 [6], known as the Den Hartog stability criterion. The quasi-steady condition is valid only if the frequency of the shedding ($f_s$) is higher than the natural frequency of the structure ($f_n$), $f_s \gg f_n$ [4]. This condition is often met at higher reduced velocities, such that

$$U_{\text{red}} = \frac{U}{f_n D} > 20,$$

(3.1)

where $U$ is the free-stream velocity, $f_n$ is the natural frequency and $D$ is the depth of the cross-section normal to the free stream. For many cases, it is found that a galloping-like instability happens in a reduced velocity range $1 < (U/(f_n D)) < 20$, where the quasi-steady assumption is questionable and vortex-induced vibrations are more likely to occur [4].
Unstable galloping vibrations of the structure occur when the flow velocity is higher than the critical velocity for the onset galloping [4], i.e. the velocity at which the energy input of the flow to the structure is higher than the dissipated energy by the structural damping. The amplitude of the unstable galloping vibration is limited by nonlinearities in the fluid force or by nonlinearities in the structures [4]. A model of galloping for a one degree-of-freedom system in a steady flow with velocity $U$ can be seen in figure 3.

From figure 3, the drag force $F_D$ is directed along the relative velocity vector, while the lift force $F_L$ is similarly acting in a direction $\alpha$ relative to the vertical with positive direction upwards,

$$
F_L = \frac{1}{2} \rho D U_{rel}^2 C_L \tag{3.2}
$$

and

$$
F_D = \frac{1}{2} \rho D U_{rel}^2 C_D. \tag{3.3}
$$

The global lift force in the $y$-direction, $F_y$ (positive downward) is expressed through the velocity $U$ as

$$
F_y = \frac{1}{2} \rho D U^2 C_y. \tag{3.4}
$$

By transforming $F_L$ and $F_D$ to the $y$-direction $F_y$, we find

$$
F_y = -F_L \cos \alpha - F_D \sin \alpha. \tag{3.5}
$$

Substituting equations (3.2) and (3.3) into equation (3.4), we find

$$
C_y = \frac{F_y}{(1/2) \rho DU^2} = \frac{U_{rel}^2}{U^2} (-C_L \cos \alpha - C_D \sin \alpha). \tag{3.6}
$$

A positive $C_y$ implies a destabilizing force, while a positive $C_L$ represents a stabilizing force. By assuming $\alpha$ to be small so that $\dot{y}/U \approx \alpha$, and by expanding in the MacLaurin series, the lift force is

$$
F_L = \frac{1}{2} \rho DU^2 \left\{ C_L|_{\alpha=0} - \left( \frac{\partial C_L}{\partial \alpha} + C_D \right)|_{\alpha=0} \frac{\dot{y}}{D} \right\}. \tag{3.7}
$$

The dynamic equation for a structure with viscous damping is

$$
\ddot{y} + 2\omega_y \left\{ \dot{y} - \frac{\rho DU}{4m\omega_y} \frac{\partial C_y}{\partial \alpha} \bigg|_{\alpha=0} \right\} \ddot{\alpha} + \omega_y^2 y = -\frac{\rho DU^2}{2m} C_L|_{\alpha=0}. \tag{3.8}
$$
Figure 4. ‘Piggyback’ pipeline configuration with two pipes of diameters $D$ and $(1/2)D$. The flow is from left to right.

Figure 5. Amplitude of the response of a piggyback pipeline geometry with pipe diameters $D$ and $(1/2)D$ plotted against $U_{\text{red}}$ [7].

The total damping ratio (including structural and flow damping effects) can now be written as

\[ \zeta_T = \zeta_y - \frac{\rho DU \partial C_y}{4m\omega_y} \frac{\partial C_y}{\partial \alpha} \bigg|_{\alpha=0}. \] (3.8)

For positive $\partial C_y/\partial \alpha$, i.e. destabilizing, the total damping $\zeta_T$ may apparently become negative, and structural failure could therefore be expected.

Hang [7] and Lubis [8] studied the response of a ‘piggyback’ pipeline configuration (figure 4) in a constant flow. Galloping-like instabilities were observed for a piggyback configuration with pipes of diameters $D$ and $(1/2)D$, respectively, where the flow is from left to right (figure 4). They found that for this geometry, a large response starts at $U_{\text{red}} = 4$ and increases with increasing $U_{\text{red}}$ (see equation (3.1)). Reference is made to figure 5. Further tests should be carried out to verify the onset of galloping for these low values of $U_{\text{red}}$. It is possible that vortex-induced vibrations onset at $U_{\text{red}} = 4$ and that galloping takes over at a higher value of $U_{\text{red}}$.

As for the special pipeline configuration discussed above, galloping on ice-coated cables occurs when certain amounts of ice develop on the cables. It can be caused by glaze ice and rime ice or wet snow on the conductor.

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The ice or snow develop a certain shape on the cables and make the structure become a non-circular body. The non-circularity of the structure makes it susceptible to galloping for a wide variation of ice-coated cables.

4. Case 2: motion of a slender offshore structure taking the relative velocity motion term into account \[2\]

The general second-order ordinary differential equation for the horizontal response \(y(t)\) of a one degree-of-freedom slender offshore structure when subjected to constant nonlinear drag loading (that is, the loading generated by the velocity \(U\) of a current), according to experiments, is given by the term \((1/2)\rho C_d' DU|U|\) per unit length of the structure; see, for example, Sarpkaya & Issacsson \[9\],

\[
m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = \frac{1}{2} \rho C_d' DU|U|. \tag{4.1}
\]

Here,

- \(m\) is the mass of the structure;
- \(k\) is the linear stiffness of the structure;
- \(D\) is the diameter of the slender structure;
- \(\rho\) is the density of the fluid (water);
- \(C_d'\) is the drag coefficient;
- \(U\) is the velocity of the constant flow (current) upstream the structure; and
- \(|U|\) represents the absolute value of the velocity \(U\).

This drag type of loading is, in general, attributed to the shedding of vortices in the downstream flow direction of the current. In the case of a combined wave and current loading, the nonlinear drag loading is, according to Morison’s postulate \[3\] for slender structures \((D/L \leq 0.2, \text{ where } L\) is the wavelength), given by

\[
m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = \frac{1}{2} \rho C_d D \left\{ \frac{du(t)}{dt} + U \right\} \left| \frac{du(t)}{dt} + U \right|. \tag{4.2}
\]

Here,

- \(u(t)\) is the displacement of the oscillating flow;
- \(du(t)/dt\) is the velocity of the oscillating flow; and
- \(C_d\) is the modified drag coefficient for the combined flow. \(C_d\) exhibits a significant variation with Reynolds number (\(Re\)), Keulegan–Carpenter number (\(KC\)) and relative roughness (\(k_r/D\)). In this paper, we will treat \(C_d\) as a constant.

For the selection of values for \(C_d\), in accordance with international recommendations, see, for example, Gudmestad & Moe \[10\]. It should be noted that the influence of the displacement of the structure on the flow is not accounted for in this analysis; see, for example, Gudmestad & Connor \[11\] for a discussion of the ‘relative velocity effects’.

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In addition, we have to include the mass force term (also denoted the inertia term) $\rho C_m (\pi/4) D^2 (d^2 u(t)/dt^2)$, which, for slender structures, may be omitted as the drag term dominates; see, for example, Sarpkaya & Isaacson [9] for criteria to be fulfilled to omit the mass forcing term from the analysis. It should be noted that the mass term is a linear term proportional to $du(t)/dt$ and that this term would cause traditional dynamic amplification of the response at the resonance frequency, i.e. when the natural frequency of the system $\omega = \Omega$, the frequency of the loading. Further resonance effects will be triggered when integrating the force contributions into the free surface of the wave(s).

Additional nonlinear loading terms have been suggested by, for example, Newman [12] and Faltinsen et al. [13]; the Faltinsen, Newman and Vinje (FNV) method. These terms are thought to account for an occasional ‘ringing’-like response of structures, and the FNV method represents today’s state of art with respect to understanding ringing response.

In addition to including current in the loading term of the equation of motion, we should also include the effect of the motion of the structure itself on the forcing term, that is, we should consider the relative acceleration and velocity terms in equation (1.2), as was suggested by Gudmestad & Connor [11]. Equation (4.2) (without current $U$) would then read

$$m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = \frac{1}{2} \rho C_d D \left\{ \frac{du(t)}{dt} - \frac{dy(t)}{dt} \right\} \left| \frac{du(t)}{dt} - \frac{dy(t)}{dt} \right|$$

$$+ \rho (C_m - 1) \frac{\pi}{4} D^2 \left( \frac{d^2 u(t)}{dt^2} - \frac{d^2 y(t)}{dt^2} \right) + \rho \frac{\pi}{4} D^2 \frac{d^2 u(t)}{dt^2}.$$  (4.3)

Note that $u(t)$ here refers to the water particle displacement, while $y(t)$ refers to the displacement of the structure. The right-hand side of equation (4.3) represents the forcing term, taking the relative motion of the structure into account.

Let us consider the nonlinear relative velocity drag forcing term of equation (4.3),

$$\frac{1}{2} \rho C_d D \left\{ \frac{du(t)}{dt} - \frac{dy(t)}{dt} \right\} \left| \frac{du(t)}{dt} - \frac{dy(t)}{dt} \right|.$$  (4.4)

Let us then denote $du(t)/dt = \dot{u}$ and $dy(t)/dt = \dot{y}$, then

$$\{ \ddot{u} - \ddot{y} \} | \dot{u} - \dot{y} | = | \dot{u} - \dot{y} | | \ddot{u} - \ddot{y} | \dot{y}.$$  (4.5)

Here, $dy(t)/dt = \dot{y}$ is the velocity of the structural response that, in general, is an oscillatory function.
There are two cases

\[ \{ \dot{u} - \dot{y} \} \geq 0, \text{ then } |\dot{u} - \dot{y}| \dot{u} - |\dot{u} - \dot{y}| \dot{y} = \dot{u}^2 - \dot{u} \dot{y} - \dot{u} \dot{y} + \dot{v}^2 \approx \dot{u}^2 - 2\dot{u} \dot{y} \]  

(4.6)

and

\[ \{ \dot{u} - \dot{y} \} \leq 0, \text{ then } |\dot{u} - \dot{y}| \dot{u} - |\dot{u} - \dot{y}| \dot{y} = -\dot{u}^2 + \dot{u} \dot{y} + \dot{u} \dot{y} - \dot{v}^2 \approx -\dot{u}^2 + 2\dot{u} \dot{y}. \]  

(4.7)

For both cases, we neglect the velocity square of the structural response. From equations (4.6) and (4.7), the terms proportional to \( 2\dot{u} \dot{y} \) represent oscillatory damping and can be transferred to the left-hand side of the equation of motion. Thus, the damping term becomes of the form \( c(t) \equiv c_0 + \Delta c(t) \) consisting of one linear term \( c_0 \) and one time-varying term \( \Delta c(t) \), as was discussed previously. The damping term may apparently become negative as follows:

Case (1) \( \{ \dot{u} - \dot{y} \} \geq 0 \) and \( \dot{u} \leq 0 \), which will be satisfied in cases \( \dot{u} \leq 0 \) and \( \dot{y} \leq \dot{u} \leq 0 \).

Such a situation could arise on the downward-sloping side (back) of the wave (where \( \dot{u} \leq 0 \)), when the structural motion has a negative velocity \( \dot{y} \leq \dot{u} \leq 0 \).

Case (2) \( \{ \dot{u} - \dot{y} \} \leq 0 \) and \( \dot{u} \geq 0 \), which will be satisfied in cases \( \dot{u} \geq 0 \) and \( \dot{y} \geq \dot{u} \geq 0 \).

Such a situation could arise on the upward-sloping side (front) of the wave (where \( \dot{u} \geq 0 \)) when the structural motion has a positive velocity \( \dot{y} \geq \dot{u} \geq 0 \).

In case we consider the relative velocity terms of the Morison equation, we see from this that it would be possible to obtain situations where apparently negative damping could occur. The burst type of response would be initiated for relatively small values of the water particle velocity, i.e. for small values of \( \dot{u} \), and such situations would occur when the wave crosses the waterline. In this position, the water particle acceleration is at a maximum, whereby the effect could be interpreted as being caused by the inertia term of the loading.

For an example case (the Draugen offshore oil and gas production platform in the Norwegian Sea), Gudmestad et al. [2] have reported ‘burst-type displacements’ that could explain the ringing-like response of offshore structures under special load situations.

An interesting reference is that of Martin et al. [14]. They discuss stream-wise vibrations and explain these as drag instability.

5. Case 3: ice engineering

Theoretical work has been conducted to increase an understanding of the dynamic response of offshore structures in ice. Määttänen [15] proposed that steady-state vibration of a narrow vertical offshore structure is a self-excited process where nonlinear forces owing to ice crushing provide an apparent negative damping effect to the structure.
Kärnä et al. [16] present a model of dynamic ice forces on vertical offshore structure, and their model makes use of the concept of apparent negative damping.

6. Nonlinear mechanics

Nonlinear differential equations, as in the van der Pol oscillator (6.1) and the Duffing oscillator (6.2), are useful for explanation of large motions occurring in the case of apparent negative damping. For general references regarding nonlinear dynamics, see, for example, Bogoliubov & Mitropolsky [17], Nayfeh & Balachandran [18] and Nayfeh [19].

The van der Pol oscillator is represented by the equation

$$\frac{d^2 y(t)}{dt^2} - \mu (1 - y^2) \frac{dy(t)}{dt} + y(t) = 0. \quad (6.1)$$

The Duffing oscillator is expressed by

$$\frac{d^2 y(t)}{dt^2} + \delta \frac{dy(t)}{dt} + \beta y(t) = \gamma \cos \omega t. \quad (6.2)$$

Here, \(\mu, \delta, \beta\) and \(\gamma\) are constants, while \(\omega\) represents the frequency of an external harmonic loading.

Ogink & Metrekine [20] study a wake oscillator for the modelling of vortex-induced vibration, in which the coupling term between the oscillator and the structural equation is written in the form of a convolution integral. The equation of motion has similar features to the van der Pol equation and can model self-exited and self-limiting oscillations.

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